A mechanical system has three degrees of freedom and is described by coordinates $q_1, q_2$ and $q_3$, where $q_1 = q_2 = q_3 = 0$ corresponds to a position of equilibrium of the system. The kinetic energy $T = \frac{1}{2} \sum_{i=1}^{3} \sum_{j=1}^{3} T_{ij}(q) \dot{q}_i \dot{q}_j = \frac{1}{2} q^T T q$, defining a matrix $T$. The potential energy is given by the function $V(q)$.

Define the Lagrangian $\mathcal{L}$ of the system and write down the corresponding Euler-Lagrange equations. What conditions must apply at the equilibrium position $q_1 = q_2 = q_3 = 0$? Calculate the leading-order non-constant terms in a Taylor expansion of $V(q)$ about this position, and hence show that these leading-order non-constant terms can be written as $\frac{1}{2} q^T V q$ for some constant matrix $V$. Deduce the form of the Lagrangian and the corresponding Euler-Lagrange equations for small disturbances from equilibrium. With reference to this set of equations define the terms **normal frequencies** and **normal modes**. [5]

Consider a system consisting of a heavy horizontal bar of mass $M$, from the ends of which two masses $m$ hang on identical vertical springs, each with spring constant $\mu$. The bar itself is suspended from a fixed point by a spring with spring constant $\lambda$. (The bar is constrained to remain horizontal.) Define $q_1, q_2$ and $q_3$ to be the vertical displacements of, respectively, the bar and the two masses away from their equilibrium positions. Show that the relevant matrices $T$ and $V$ (as defined above) take the form

$$T = \begin{pmatrix} M & 0 & 0 \\ 0 & m & 0 \\ 0 & 0 & m \end{pmatrix}, \quad V = \begin{pmatrix} \lambda + 2\mu & -\mu & -\mu \\ -\mu & \mu & 0 \\ -\mu & 0 & \mu \end{pmatrix},$$

and hence construct the Lagrangian for this system. [6]

Hence for the case $M = 2$, $m = 1$, $\lambda = 4$ and $\mu = 1$, derive the corresponding normal frequencies and normal modes. [6]

Give a brief geometrical description of each normal mode. [3]