UNIVERSITY OF CAMBRIDGE

NATURAL SCIENCES TRIPOS Part IA

Wednesday, 9 June, 2010 9:00 am to 12:00 pm

MATHEMATICS (2)

Before you begin read these instructions carefully:

The paper has two sections, A and B. Section A contains short questions and carries 20 marks in total. Section B contains ten questions, each carrying 20 marks.

You may submit answers to ***all*** *of section A, and to no more than* ***five*** *questions from section B.*

The approximate number of marks allocated to a part of a question is indicated in the right hand margin.

Write on one side of the paper only and begin each answer on a separate sheet. *(For this purpose, your section A attempts should be considered as one single answer.)*

Questions marked with an asterisk (*) require a knowledge of B course material.

At the end of the examination:

Tie up ***all of your section A answer*** *in a single bundle, with a completed blue cover sheet.*

Each section B question has a number and a letter (for example, ***11S****). Answers to these questions must be tied up in* ***separate*** *bundles, marked* ***R, S, T, X, Y*** *or* ***Z*** *according to the letter affixed to each question.* ***Do not join the bundles together****. For each bundle, a blue cover sheet* ***must*** *be completed and attached to the bundle, with the correct letter* ***R, S, T, X, Y*** *or* ***Z*** *written in the section box.*

A ***separate*** *yellow master cover sheet listing all the questions attempted* ***must*** *also be completed. (Your section A answer may be recorded just as A: there is no need to list each individual short question.)*

Every cover sheet must bear your examination number and desk number.

STATIONERY REQUIREMENTS
6 blue cover sheets and treasury tags
Yellow master cover sheet
Script paper

SPECIAL REQUIREMENTS
Approved calculators allowed.

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

SECTION A

1 Determine the angle in the range $(0, \pi)$ between the vectors $\mathbf{a} = (1, 1, 1)$ and $\mathbf{b} = (-1, 2, 3)$. [2]

2 Evaluate the following definite integral

$$\int_{-\pi/2}^{+\pi/2} x \sin x \, \mathrm{d}x \, .$$

[2]

3 Consider the function $f(x, y) = x^4 + 4\,x^2y^2 - 2\,x^2 + 2\,y^2 - 1$. Verify that the point $(x, y) = (0, 0)$ is a stationary point. [1]

Find one of the other two stationary points. [1]

4 Find the eigenvalues of the matrix

$$\begin{pmatrix} 1/3 & \sqrt{3} \\ \sqrt{3} & -1/3 \end{pmatrix} .$$

[2]

5 Find the real and imaginary parts of the complex number

$$\frac{3 + 5i}{5 - 4i} \, .$$

[2]

6 Give the first three terms of the Taylor expansion of the function $f(x) = \cos x$ about the point $x = \pi/3$. [2]

7 Solve the differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{y+1}{x-1}$$

given the boundary condition $y = 1$ at $x = 0$. [2]

8 You roll two 6-sided dice. What is the probability that *neither* die rolls a 1? [1]

What is the probability that at least one die rolls a 1? [1]

9 Determine the equation of the straight line that is tangent to the curve $y = x\sin(x^2) + 1$ at $x = 0$. [2]

10 Calculate the divergence of

$$\mathbf{F}(x, y, z) = xy^2z^2\mathbf{i} + (3\,xz^2 + y^4)\mathbf{j} + x^3y^2z\mathbf{k}\,.$$

[2]

SECTION B

11S

(a) A point P lying in the plane has cartesian coordinates (x, y) and the origin O has cartesian coordinates $(0, 0)$. Express (x, y) in terms of plane polar coordinates (r, θ). [3]

(b) Consider an ellipse defined by the equation

$$r = \frac{1}{A - B\cos\theta}$$

where $A > B > 0$. By rewriting the equation of this curve in terms of cartesian coordinates (x, y), or otherwise, show that the centre is at

$$(x, y) = \left(BC^{-2}, 0\right) ,$$

where $C^2 = A^2 - B^2$. [12]

(c) Find the lengths of the semi-axes of the ellipse in terms of A and C. [5]

12T

The equations of motion of a particle of mass m that moves on the XY plane under the action of the force $\mathbf{F} = -k\mathbf{r}$, where k is a positive constant and $\mathbf{r} = (x, y)$ is its position, are

$$m\,\frac{d^2x}{dt^2} = -kx\,,$$

$$m\,\frac{d^2y}{dt^2} = -ky\,.$$

(a) Show that the energy of the particle, given by

$$E = \tfrac{1}{2}\,m\left|\frac{d\mathbf{r}}{dt}\right|^2 + \tfrac{1}{2}\,k|\mathbf{r}|^2\,,$$

does not change with time. [8]

(b) Write the general solutions of the equations of motion and their specific solutions for the initial conditions $x = X_0$, $y = 0$, $dx/dt = 0\,, dy/dt = V_0$ when $t = 0$. [8]

(c) Show that the orbit of the particle is a circle if $kX_0^2 = mV_0^2$. [4]

13X

(a) A function f of two variables x and y is defined as

$$f(x,y) = \exp\left[\frac{-1}{x^2+y^2}\right] + 3\ .$$

Find the position(s) of the stationary point(s) of f and determine the character (maximum, minimum or inflexion) of each. [5]

(b) A function g of two variables x and y is defined as follows:

$$g(x,y) \;=\; \sinh\left(y\sqrt{x^2+y^2} - 4y\right)\ .$$

Sketch contours of g in the (x,y)-plane, making sure to indicate on the sketch the positions and character of all the stationary points, and making sure to label the heights of any important contours or features. [15]

14Y

(a) (i) Evaluate the determinant of the matrix A where

$$A \;=\; \frac{1}{\sqrt{8}}\begin{pmatrix} \sqrt{3} & -\sqrt{2} & -\sqrt{3} \\ 1 & \sqrt{6} & -1 \\ 2 & 0 & 2 \end{pmatrix}\ .$$

[5]

(ii) Show that the matrix A is orthogonal. [5]

(b) B and C are real non-zero 3×3 matrices and satisfy the equation

$$(BC)^T + \left(C^{-1}B\right) \;=\; 0\,.$$

(i) Prove that if C is orthogonal then B is antisymmetric. [4]

(ii) Without assuming that C is orthogonal, prove that B is singular. [6]

15R

Find the Fourier series for the function $f(x)$ which is defined to be equal to $\cos \mu x$ on the interval $-\pi \leqslant x \leqslant \pi$, and which is defined by periodicity 2π outside that interval. Here, μ is not an integer. [10]

Simplify the solution for $\mu = \frac{1}{3}$. Hence show that

$$\frac{\pi}{\sqrt{3}} = \frac{A}{2} - \sum_{n=1}^{\infty} \frac{(-1)^n}{Bn^2 - \frac{1}{3}},$$

where A and B are integers that you must determine. [10]

16S

Find the curl of each of the vector fields

$$\mathbf{F}_1 = \mathbf{a} \wedge \mathbf{r},$$

$$\mathbf{F}_2 = \frac{\mathbf{r}}{(r^2+1)^{3/2}},$$

where $\mathbf{r} = (x, y, z)$ and $\mathbf{a} = (0, 1, 1)$. [8]

State whether each is conservative or not. [2]

For each field check your answer by evaluating

$$\int \mathbf{F}_n \cdot dr$$

for $n = 1, 2$ along

(a) the straight line from $(1, 0, 0)$ to $(0, 1, 0)$, [5]

(b) the shortest circular arc from $(1, 0, 0)$ to $(0, 1, 0)$, centred on the origin. [5]

17X

(a) In the Grand Arcade are two jars and a bishop. Jar X contains 20 red and 30 green snakes. Jar Y contains 180 red and 170 green snakes. Outwardly the jars are identical. The bishop selects a jar at random with equal probability of choosing X or Y. He shakes this jar vigorously to mix the contents and then reaches in and draws out a snake.

(i) Calculate the probability that the snake which was pulled out was red. [3]

(ii) Calculate the probability that the bishop chose Jar X, *given that the snake was red.* [3]

(b) A standard English "Scrabble"™ set contains 100 tiles, of which 2 are blank, 34 are vowels, and 64 are consonants. If all the tiles are initially in a bag, and seven are then drawn at random without replacement, calculate the probability of drawing 3 vowels and 4 consonants.
[Note: You may get full marks by leaving the answer as a function of one or more binomial cofficients of the form $\binom{n}{m}$, and are encouraged to do so!] [6]

(c) Imagine you are a Tripos Examiner who is setting a question on conditional probability containing the usual sorts of expressions ($P(A)$, $P(B|A)$, $P(A \cap B)$, $P(\bar{A})$, *etc.*) relating two not-necessarily independent events A and B. Imagine further that you want to keep the question you are setting "physical" (*i.e.* with all probabilities safely in the range $0 \leqslant p \leqslant 1$).

(i) Suppose that you started by fixing the values of $P(A)$ and $P(B|A)$. *In terms of these values*, calculate the minimum and maximum values you will be able to use for $P(B)$ in your question. [Leave your answer in the form $\alpha \leqslant P(B) \leqslant \beta$ where α and β are to be determined.] [4]

(ii) Suppose instead that you started by fixing the values of $P(B|A)$ and $P(B|\bar{A})$. *In terms of these values*, calculate the minimum and maximum values you will be able to use for $P(B)$ in your question. [Again, leave your answer in the form $\alpha \leqslant P(B) \leqslant \beta$ where α and β are to be determined.] [4]

18Y

(a) Evaluate the volume integral

$$I = \int_{x=-1}^{1} \int_{y=-2}^{2} \int_{z=-3}^{3} \left(x^2 + y^2 + z^2\right)\, dz\, dy\, dx\,.$$

[5]

(b) Evaluate the integral

$$\int_{y=0}^{1} \int_{x=y}^{1} e^{x^2}\, dx\, dy\,.$$

[Hint: consider reversing the order of integration.] [5]

(c) Find using cylindrical polar coordinates the volume of the closed solid bounded by the paraboloids $z = 9 - x^2 - y^2$ and $z = x^2 + y^2$. [10]

19R*

A right circular cylinder of radius a and length l has volume $V = \pi a^2 l$ and surface area $A = 2\pi a(a + l)$. Use Lagrange multipliers to do the following:

(a) Show that, for a given area, the maximum volume is

$$V = \frac{1}{3}\sqrt{\frac{A^3}{C\pi}}\,,$$

determining the integer C in the process. [10]

(b) For a cylinder inscribed in the unit sphere, i.e. $a^2 + l^2/4 = 1$, show that the value of l/a which maximises the area of the cylinder is

$$D + \sqrt{E}\,,$$

determining the integers D and E as you do so. [10]

[Hint: you need not show that suitable extrema you find are actually maxima.]

20T*

The Hermite polynomials are defined by

$$H_n(x) \;=\; (-1)^n e^{x^2} \, \frac{d^n}{dx^n} \, e^{-x^2} \, ,$$

for integer n.

(a) Calculate the explicit expressions for H_0, H_1 and H_2. [6]

(b) Show that

$$\left(2x + \frac{d}{dx}\right) \frac{d^n}{dx^n} \, e^{-x^2} \;=\; -2n \, \frac{d^{n-1}}{dx^{n-1}} \, e^{-x^2} \, .$$

[7]

(c) Use the previous result to derive the recurrence relation

$$\frac{dH_n}{dx} \;=\; 2n \, H_{n-1} \, .$$

[7]

END OF PAPER