UNIVERSITY OF CAMBRIDGE

NATURAL SCIENCES TRIPOS Part IB & II (General)

Friday, 29 May, 2009 9:00 am to 12:00 pm

MATHEMATICS (2)

Before you begin read these instructions carefully:

*You may submit answers to no more than **six** questions. All questions carry the same number of marks.*

The approximate number of marks allocated to a part of a question is indicated in the left hand margin.

*Write on **one** side of the paper only and begin each answer on a separate sheet.*

At the end of the examination:

*Each question has a number and a letter (for example, **6A**).*

*Answers must be tied up in **separate** bundles, marked **A, B or C** according to the letter affixed to each question.*

Do not join the bundles together.

For each bundle, a blue cover sheet must be completed and attached to the bundle.

*A **separate** yellow master cover sheet listing all the questions attempted **must** also be completed.*

Every cover sheet must bear your examination number and desk number.

STATIONERY REQUIREMENTS
6 blue cover sheets and treasury tags
Yellow master cover shett
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1C A Sturm-Liouville operator $\mathcal{L}$ acts on a function $y(x)$ as

$$\mathcal{L}\, y \;=\; \frac{1}{w(x)}\left(-\frac{d}{dx}\left[p(x)\frac{dy}{dx}\right] \;+\; q(x)y\right),$$

where $y(x)$ is defined on a closed interval $a \leqslant x \leqslant b$ with boundary conditions $y(a) = y(b) = 0$. The normalised eigenfunctions $\mathcal{L}$ are denoted by $y_n(x)$ for $n = 1, 2, \ldots,$ with corresponding eigenvalues λ_n.

State without proof the properties of *orthogonality* and *completeness* obeyed by the eigenfunctions of $\mathcal{L}$ and derive the relation [8]

$$w(\xi)\sum_{n=1}^{\infty} y_n(\xi)^* y_n(x) \;=\; \delta(x-\xi).$$

Using this relation, derive a formula for the Green's function $G(x;\xi)$ of the operator $\mathcal{L}$ which obeys [6]

$$\mathcal{L}\, G(x;\xi) \;=\; \delta(x-\xi).$$

Show that the Green's function for the operator,

$$\mathcal{L} \;=\; -\frac{d^2}{dx^2} \;+\; \omega^2,$$

defined on the interval $0 \leqslant x \leqslant 1$ acting on functions $y(x)$ with boundary conditions $y(0) = y(1) = 0$, can be written as [6]

$$G(x;\xi) \;=\; \sum_{n=1}^{\infty} \frac{2\sin(n\pi\xi)\sin(n\pi x)}{\omega^2 + n^2\pi^2}.$$

2C The Laplace equation obeyed by an axisymmetric function $\Phi(r,\theta)$ in spherical polar coordinates reads

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial \Phi}{\partial r}\right) + \frac{1}{r^2\sin(\theta)}\frac{\partial}{\partial \theta}\left(\sin(\theta)\frac{\partial \Phi}{\partial \theta}\right) = 0. \qquad (*)$$

Derive the general solution

$$\Phi(r,\theta) = \sum_{n=0}^{\infty}\left(A_n r^n + B_n r^{-n-1}\right) P_n\left(\cos(\theta)\right),$$

where $P_n\left(\cos(\theta)\right)$ is the nth Legendre polynomial defined as the solution of the differential equation

$$\frac{d}{d\xi}\left((1-\xi^2)\frac{dP}{d\xi}\right) + \lambda P = 0,$$

[8] where $\xi = \cos\theta$ and $\lambda = n(n+1)$, which is non-singular at $\xi = \pm 1$.

Find the solution of $(*)$ defined in the region between two spheres of radii a and $b > a$, with boundary conditions

$$\begin{aligned} \Phi(r,\theta) &= a^2, & r &= a, \\ &= b^2\cos^2(\theta), & r &= b. \end{aligned}$$

[12] You may use the formulae $P_0\left(\cos(\theta)\right) = 1$, $P_2\left(\cos(\theta)\right) = (3\cos^2(\theta) - 1)/2$.

3C Derive the fundamental solution

$$G(\mathbf{x}, \mathbf{x}_0) \;=\; -\frac{1}{4\pi|\mathbf{x}-\mathbf{x}_0|},$$

to the Poisson equation

$$\nabla^2 G \;=\; \delta\left(\mathbf{x}-\mathbf{x}_0\right). \qquad (*)$$

in three-dimensional space with boundary condition $G \to 0$ as $|\mathbf{x}| \to \infty$. [4]

Use the method of images to find solutions of $(*)$ of the form

$$G(\mathbf{x}, \mathbf{x}_0) \;=\; -\sum_{n=0}^{N} \frac{s_i}{4\pi|\mathbf{x}-\mathbf{x}_i|},$$

where G is defined in each of the following regions of three-dimensional space and vanishes on the boundary of the region in each case:

i) The octant $x > 0$, $y > 0$, $z > 0$.

ii) The interior of a sphere of radius a centered at the origin.

iii) The region $-1 \leqslant x \leqslant +1$ between two parallel plates in the case where $\mathbf{x}_0 = \mathbf{0}$.

In each case you should determine the number N of images (which may be infinite) and specify the points $\mathbf{x}_i$ and signs $s_i = \pm 1$ for $i = 1, 2, \ldots, N$ $(s_0 = +1)$. [16]

[6] **4C** State and prove *Cauchy's Theorem.*

By considering the integral of the function $f(z) = \exp(-az^2)$, with $a > 0$, around an appropriate contour, show that the integral

$$I(\rho, a) \;=\; \int_{-\infty+i\rho}^{+\infty+i\rho} \exp(-az^2)\; dz,$$

[8] which is evaluated along the line $\mathrm{Im}(z) = \rho$, is independent of the real number ρ.

Hence prove that

$$\int_{-\infty}^{+\infty} \exp(-ax^2 + i\nu x)\; dx \;=\; \sqrt{\frac{\pi}{a}}\, \exp\left(-\frac{\nu^2}{4a}\right),$$

[6] for real a and ν with $a > 0$. You may use without proof the standard formula

$$\int_{-\infty}^{+\infty} \exp(-ax^2)\; dx \;=\; \sqrt{\frac{\pi}{a}}.$$

5A It can be shown that the Laplace transform of the convolution integral

$$\int_0^t f_1(\tau) f_2(t-\tau) d\tau, \quad t > 0,$$

equals the product of the Laplace transforms of the functions $f_1(t)$ and $f_2(t)$.

Use the above result to show the following: Let $\Phi(t)$ satisfy Abel's integral equation, i.e.

$$\int_0^t \Phi(\tau)(t-\tau)^{-\alpha} d\tau = f(t), \quad t > 0,$$

where α is a constant satisfying $0 < \alpha < 1$ and $f(t)$ is a given function satisfying $f(0) = 0$.
[12] Then $\Phi(t)$ is given by

$$\Phi(t) = \frac{1}{\Gamma(\alpha)\Gamma(1-\alpha)} \int_0^t \frac{df(\tau)}{d\tau} (t-\tau)^{\alpha-1} d\tau.$$

[You may use the fact that the Laplace transform of $t^{-\alpha}$ equals $\Gamma(1-\alpha)/s^{1-\alpha}$, $0 < \alpha < 1$.]

[8] Find the explicit solution in the case that $f(t) = t$.

6B What does it mean for a tensor to be isotropic? Give examples of isotropic tensors
[5] of rank 0 and 2. Do there exist isotropic tensors of rank 1? Explain your answer.

The moment of inertia with respect to an arbitrary point in space $\mathbf{P} = (P_1, P_2, P_3)$, of a body B of mass density $\rho(x)$, is

$$I_{ij}(\mathbf{P}) = \int_B \Big(|\mathbf{x} - \mathbf{P}|^2 \delta_{ij} - (\mathbf{x} - \mathbf{P})_i (\mathbf{x} - \mathbf{P})_j \Big) \rho(x) d^3x.$$

Let $\mathbf{G} = (G_1, G_2, G_3)$ be the centre of mass of B. Derive a formula for $I_{ij}(\mathbf{P}) - I_{ij}(\mathbf{G})$ in
[8] terms of $X_j = (P_j - G_j)$ and the total mass $M = \int_B \rho(x) d^3x$.

In the case that B is a ball of unit radius and uniform density $\rho = \rho_0$, find $I_{ij}(\mathbf{P})$ in terms of M and $\mathbf{X}$. For which $\mathbf{P}$ is $I_{ij}(\mathbf{P})$ isotropic? Calculate the eigenvalues and a
[7] set of eigenvectors of $I_{ij}(\mathbf{P})$ in terms of $\mathbf{X}$.

7B The Lagrangian for a system with N degrees of freedom is of the form

$$L = \tfrac{1}{2} \sum_{i,j} T_{ij} \dot{q}_i \dot{q}_j - \tfrac{1}{2} \sum_{i,j} V_{ij} q_i q_j$$

where $q_1, \ldots, q_N$ are co-ordinates, and it may be assumed the matrices T_{ij}, V_{ij} are symmetric, real matrices and T_{ij} is positive. Write down the Euler-Lagrange equations of
[3] motion, and define the normal modes and normal frequencies of the system.

Consider the system comprising one particle of mass M located at position x_2 on a line (taken to be the x axis), which is coupled to each of two other particles of unit mass located at positions x_1 and x_3. The coupling is via springs of unit coupling constant. (The two particles at locations x_1 and x_3 are not coupled to each other, only to the one located
[5] at x_2.) Show that the system is described by a Lagrangian as above with $N = 3$ and

$$T = \begin{pmatrix} 1 & 0 & 0 \\ 0 & M & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad V = \begin{pmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{pmatrix}.$$

[6] Compute the normal frequencies and corresponding normal modes.

What is the behaviour of the normal modes as $M \to +\infty$? Explain this behaviour
[6] in relation to the expected physical behaviour of the system in this limit.

8B Define a homomorphism $\Phi : G \to H$ between two finite groups G and H. Show that K, the kernel of Φ, is a normal subgroup of G. Assuming that the image of Φ contains all of H, show that the quotient group G/K is isomorphic to H. [6]

Let $SL(2, \mathbb{C})$ be the set of 2×2 complex matrices with unit determinant. Show that it is a group, under matrix multiplication. [2]

Let $\mathcal{M}$ be the set of fractional linear transformations of the complex plane, i.e. functions of the form

$$f(z) = \frac{az + b}{cz + d}\ , \qquad ad - bc = 1.$$

Show that $\mathcal{M}$ is a group, where the group composition law $f_1 \circ f_2$ is composition of functions i.e, $f_1 \circ f_2(z) = f_1(f_2(z))$. Consider the map $\Phi : SL(2, \mathbb{C}) \to \mathcal{M}$ which associates to the matrix

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

the function $f(z) = \frac{az+b}{cz+d}$. Show that Φ is a homomorphism. [6]

Is Φ an isomorphism? If your answer is yes, prove it. If your answer is no, find its kernel K and verify that it is a normal subgroup, and deduce that $\mathcal{M}$ is isomorphic to a quotient group of $SL(2, \mathbb{C})$. [6]

9B State and prove Lagrange's theorem on the order $|H|$ of subgroups H of a finite group G of order $|G|$. [4]

What is a cyclic group? Show that there are only two distinct order four groups, one of which is cyclic. [5]

Let D_4 be the group of symmetries of the square. Give a minimal set of generators for D_4, explaining their geometrical action on the square. List the elements of D_4 and display its multiplication table. [6]

Find the largest subgroup of D_4 which is cyclic. Does D_4 have any other subgroups of the same order as this largest subgroup? Either prove it does not, or state and prove which subgroups of the same order do occur. [5]

10B Let V be the multiplicative Abelian group of order four with elements $\{1, a, b, c\}$, where 1 is the identity, $a^2 = b^2 = c^2 = 1$ and $ab = c$. Display the group multiplication table of V and determine its conjugacy classes. [5]

What is an irreducible representation of V? State the relation between the number of conjugacy classes and the number of inequivalent irreducible representations of a finite group, and deduce how many inequivalent irreducible representations V has. Give all the one dimensional representations of V. [5]

Show that the map $D : V \to GL(3, \mathbb{R})$ given by

$$D(1) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad D(a) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

$$D(b) = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} D(c) = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

is a representation of V. Decompose D into irreducible representations. [5]

Write down the character table for V and state and verify the orthogonality relation for characters. [5]

END OF PAPER