NATURAL SCIENCES TRIPOS Part IA

Monday, 8 June, 2009 9:00 am to 12:00 pm

MATHEMATICS (1)

Before you begin read these instructions carefully:

The paper has two sections, A and B. Section A contains short questions and carries 20 marks in total. Section B contains ten questions, each carrying 20 marks.

*You may submit answers to **all** of section A, and to no more than **five** questions from section B.*

The approximate number of marks allocated to a part of a question is indicated in the right hand margin.

*Write on **one** side of the paper only and begin each answer on a separate sheet. (For this purpose, your section A attempts should be considered as one single answer.)*

Questions marked with an asterisk (*) require a knowledge of B course material.

At the end of the examination:

*Tie up **all of your section A answer** in a single bundle, with a completed blue cover sheet.*

*Each section B question has a number and a letter (for example, **11S**). Answers to these questions should be tied up in **separate** bundles, marked **R, S, T, X, Y** or **Z** according to the letter affixed to each question. **Do not join the bundles together**. For each bundle, a blue cover sheet **must** be completed and attached to each bundle, with the appropriate letter **R, S, T, X, Y** or **Z** written in the section box.*

*A **separate** yellow master cover sheet listing all the questions attempted **must** also be completed. (Your section A answer may be recorded just as A: there is no need to list each individual short question.)*

Every cover sheet must bear your examination number and desk number.

STATIONERY REQUIREMENTS
6 blue cover sheets and treasury tags
Yellow master cover sheet
Script paper

SPECIAL REQUIREMENTS
Approved calculators allowed.

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

SECTION A

1 Given that $x = 3$ is a solution of

$$x^3 - x^2 - 8x + 6 = 0$$

find all the other solutions of this equation. [2]

2 Find the solution of the equation

$$\sqrt{x+16} - x + 4 = 0.$$

[2]

3 Differentiate

(a) $\ln(x^3)$;

[1]

(b) $x \cos(2x)$.

[1]

4 Integrate

(a) $\dfrac{1}{x^2 - 4x}$;

[1]

(b) $x \exp(x^2)$.

[1]

5 Find all solutions of the equation

$$1 + \cos(x) + \cos(2x) = 0$$

in the range $0 \leqslant x \leqslant \pi$. [2]

6 Find the solution of the equation in the simplest form

$$\frac{\log_{10} x}{1 - \log_{10} 2} = 2.$$

[2]

7 Evaluate

(a) $$\sum_{n=0}^{100} (3n + 2);$$

[1]

(b) $$\sum_{n=0}^{\infty} \left(-\frac{1}{4}\right)^n.$$

[1]

8 Sketch the graphs of

(a) $$y = (\ln x)^2 \quad \text{for} \quad 0 < x < \infty;$$

[1]

(b) $$y = \sin(x)\cos(x) \quad \text{for} \quad -\pi \leqslant x \leqslant \pi.$$

[1]

9 Find the (shortest) distance between the point $(4, -6)$ and the line $y = 2x + 2$. [2]

10 The position $P(x, y)$ of a particle at time t is given by the equations

$$x = \cos(t), \qquad y = 2\sin(t), \qquad 0 \leqslant t \leqslant 2\pi.$$

(a) Sketch the particle's path. Indicate the direction of motion. [1]

(b) Calculate the slope of the path dy/dx at the point $(\frac{1}{\sqrt{2}}, \sqrt{2})$. [1]

SECTION B

11S

(a) Find $\dfrac{dy}{dx}$ as a function of x for

(i)

$$y = \frac{x^3(x+1)}{(3x-2)^4};$$

[3]

(ii)

$$y = \tanh^{-1}\left(\frac{x}{a}\right).$$

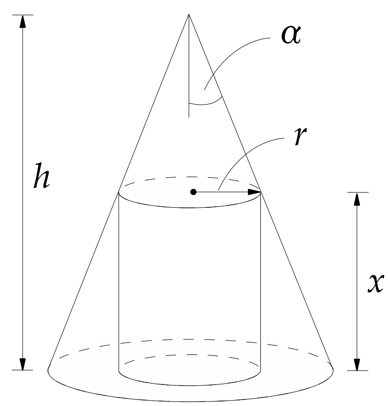
[3]

(b) If $y = [\sin^{-1}(x/a)]^2$ show that

$$(a^2 - x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx} - 2 = 0\,.$$

[4]

(c) A cylinder is inscribed in a right circular cone so that one end is on the base of the cone and the circumference of the other end is on the surface of the cone (as shown in the diagram).



The cone has height h and semi-vertical angle α. The cylinder has height x and radius r.

(i) Find an expression for the volume of the cylinder in terms of h, x and α. [4]

(ii) Show that for a given value of α and h, the value of x for which the volume of the cylinder is a maximum is $x = h/3$.

What is the ratio of the volume of the cylinder to that of the cone for this value of x? [4]

(iii) Show that for a given α and h, the curved surface area of the cylinder is a maximum for a height of $x = h/2$. [2]

12X Porterhouse, the oldest and most famous college in the University of Carrbridge, has F Fellows. It is customary that after any meeting of the Fellowship, the Butler will arrive carrying a silver tray on which C chocolate biscuits and D plain biscuits are arranged. There are always more *plain* biscuits than Fellows ($D > F$) as plain biscuits are cheap. Whether or not there are more *chocolate* biscuits than Fellows depends on the financial health of the college.

The Butler takes the tray to each of the Fellows in turn, starting with the oldest and ending with the youngest. Each Fellow takes a biscuit from the tray when it reaches him. If a Fellow is in the fortunate position of being able to make a *free choice* between chocolate and plain biscuits, he selects *chocolate* with probability p and *plain* with probability $1-p$. If all the chocolate biscuits have run out, however, he is *forced* to take one of the remaining plain biscuits.

(a) When the college finances are in good health, the Butler can provide more chocolate biscuits than Fellows ($C > F$). For this case:

(i) find the probability $P_{k,n}$ that exactly k chocolate biscuits have been taken by the time that the n-th Fellow has taken his biscuit; [3]

(ii) find the mean number $\bar{k}_n$ and variance V_n in the number of chocolate biscuits that will have been taken by the time that the n-th Fellow has taken his biscuit. [3]

(b) When the college finances are poor, the Butler has to provide fewer chocolate biscuits than Fellows ($C < F$). For this case:

(i) find the probability $P_{\text{LastChoc},n}$ that the last (i.e. the C-th) chocolate biscuit is taken by the n-th Fellow (you may assume $n \geqslant C$ and $C \geqslant 1$). [8]

(ii) Hence, or otherwise, prove that the probability P_{Disaster} that one or more Fellows is forced to take a plain biscuit is given by

$$P_{\text{Disaster}} = \sum_{f=C}^{F-1} \binom{f-1}{C-1} p^C (1-p)^{f-C}.$$

You must set out your argument clearly to obtain full credit. [6]

[*You make quote and then use, without proof, standard results regarding named distributions.*]

13Y

(a) Find the eigenvalues and eigenvectors of the matrix

$$A = \begin{pmatrix} 2 & -2 \\ 1 & 4 \end{pmatrix}.$$

[5]

Find the transformation $B^{-1}AB$ which diagonalises the matrix A. Compute the diagonal matrix $C = B^{-1}AB$. [5]

(b) Find the eigenvalues and eigenvectors of the matrix

$$M = \begin{pmatrix} 1 & 4 & 0 \\ 4 & 1 & 3 \\ 0 & 3 & 1 \end{pmatrix}.$$

[6]

Show that the eigenvectors of the matrix M are orthogonal. [4]

14Z

(a) Show that

$$\sin[-i(2n+1)\ln(-1)] = 0$$

for integer n. [6]

(b) Find all the roots of

$$z^2 = \sqrt{i}.$$

[6]

(c) Show that

$$\cosh(x-y) = \cosh(x)\cosh(y) - \sinh(x)\sinh(y).$$

[4]

(d) Show that

$$\sinh(2x) = 2\sinh(x)\cosh(x).$$

[4]

15S

(a) If $[\mathbf{a}, \mathbf{b}, \mathbf{c}]$ denotes the triple scalar product $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})$, prove the following properties

(i) $[\mathbf{a}, \mathbf{b}, \mathbf{c}] = [\mathbf{b}, \mathbf{c}, \mathbf{a}]$;

(ii) $[\mathbf{a}, \mathbf{a}, \mathbf{c}] = 0$;

(iii) $[\mathbf{a}, \mathbf{b}, \mathbf{c}] = 0$ if and only if $\mathbf{a}, \mathbf{b}$ and $\mathbf{c}$ are co-planar (excluding 'zero vectors').

[6]

[*Notation:* $\mathbf{a} \times \mathbf{b}$ *is equivalent to* $\mathbf{a} \wedge \mathbf{b}$.]

(b) For vectors $\mathbf{a}, \mathbf{b}, \mathbf{c}$ and $\mathbf{d}$, show that

$$[\mathbf{a} \times \mathbf{b},\ \mathbf{a} \times \mathbf{c},\ \mathbf{d}] = (\mathbf{a} \cdot \mathbf{d})[\mathbf{a}, \mathbf{b}, \mathbf{c}].$$

[4]

(c) Show that if two straight lines

$$\begin{aligned} \mathbf{r} &= \mathbf{a} + t\mathbf{u} \\ \mathbf{r}' &= \mathbf{b} + t'\mathbf{v} \end{aligned}$$

intersect (where $\mathbf{r}, \mathbf{r}', \mathbf{a}, \mathbf{b}, \mathbf{u}$ and $\mathbf{v}$ are vectors, and t and t' are scalars), then

$$[\mathbf{v}, \mathbf{b}, \mathbf{u}] = [\mathbf{v}, \mathbf{a}, \mathbf{u}].$$

[5]

Also show that if $[\mathbf{a}, \mathbf{u}, \mathbf{v}] \neq 0$ the point of intersection is

$$\mathbf{r} = \mathbf{a} + \frac{[\mathbf{a}, \mathbf{b}, \mathbf{v}]}{[\mathbf{a}, \mathbf{u}, \mathbf{v}]}\,\mathbf{u}.$$

[5]

16T The function $\Gamma(p)$ is defined for positive p by

$$\Gamma(p) = \int_0^\infty x^{p-1} e^{-x} dx. \tag{1}$$

(a) Derive the recurrence relation $\Gamma(p+1) = p\Gamma(p)$ and use it to show that $\Gamma(n+1) = n!$ for integer n. [5]

(b) Evaluate the integral

$$\int_0^1 \left(\ln \frac{1}{x}\right)^4 dx.$$

[5]

(c) Show by making a suitable substitution in (1) above that

$$\Gamma\left(\tfrac{1}{2}\right) = 2\int_0^\infty e^{-t^2} dt.$$

[5]

(d) Use (c) to calculate $\left[\Gamma\left(\frac{1}{2}\right)\right]^2$ and hence obtain the value of $\Gamma\left(\frac{1}{2}\right)$. [5]

17T

(a) Solve the following differential equations

(i)

$$\frac{dy}{dx} = \frac{y}{x} - \tan\frac{y}{x},$$

such that $y = 1$ at $x = 2/\pi$; [7]

(ii)

$$\frac{dy}{dx} = \cos(x+y),$$

such that $y = 0$ at $x = 0$. [7]

(b) Show that $y \sin 2x\, dx + \sin^2 x\, dy$ is an exact differential. Hence solve

$$y \sin 2x\, dx + \sin^2 x\, dy = 0,$$

such that $y = 1$ at $x = \pi/2$. [6]

18X The notation $\left(\frac{\partial a}{\partial b}\right)_c$ indicates the partial derivative of a with respect to b at constant c.

(a) If $v \equiv v(s,t)$ express $\left(\frac{\partial u}{\partial s}\right)_v$ in terms of the partial derivatives of $u(s,t)$ and $t(s,v)$. To gain full marks you must make clear which variables, if any, are held constant in any partial derivatives which feature in your answer. [3]

(b) Find the function $\mu(x)$ satisfying $\mu(1) = 1$ for which

$$df = 2\mu(x)\sin y\ dx - \mu(x)x\cos y\ dy$$

is an exact differential. For the function $\mu(x)$ just found, determine $f(x,y)$. [7]

(c) Three variables a, b and c are related by

$$a = \frac{bc(b^2 - c^2)}{(b^2 + c^2)^2}.$$

Prove that

$$\left(\frac{\partial a}{\partial b}\right)_c \left(\frac{\partial b}{\partial c}\right)_a \left(\frac{\partial c}{\partial a}\right)_b = -1.$$

[10]

19Z* Consider the one-dimensional wave equation

$$\frac{\partial^2\psi}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2\psi}{\partial t^2},$$

where c is a constant.

(a) By setting $\xi = x - ct$ and $\eta = x + ct$ show that the wave equation becomes

$$\frac{\partial^2\psi}{\partial\xi\partial\eta} = 0.$$

[8]

(b) Then show that this equation has solutions of the type

$$\psi(\xi, \eta) = f(\xi) + g(\eta)$$

given any functions f and g. [2]

(c) The wave equation can also be solved by the method of separation of variables by setting

$$\psi(x, t) = X(x)T(t).$$

Use this method to find the general solution for the boundary conditions $\psi(0, t) = \psi(L, t) = 0$, where L is a non-zero constant. [10]

20Y*

(a) Determine whether the following series are convergent

(i)

$$\sum_{n=1}^{\infty} \frac{\sqrt{n}}{\sqrt{n+1}};$$

[2]

(ii)

$$\sum_{n=1}^{\infty} \frac{n^2}{n!};$$

[2]

(iii)

$$\sum_{n=1}^{\infty} \frac{n^n}{n!}.$$

[3]

(b) Evaluate the following limits

(i)

$$\lim_{x\to 0} \frac{\sin 3x}{\sinh x};$$

[3]

(ii)

$$\lim_{x\to 0} \frac{\tan x - \tanh x}{\sinh x - x};$$

[4]

(iii)

$$\lim_{x\to 0} \left(\frac{\operatorname{cosec} x}{x^3} - \frac{\sinh x}{x^5} \right).$$

[6]

END OF PAPER