UNIVERSITY OF CAMBRIDGE

NATURAL SCIENCES TRIPOS **Part IB & II (General)**

Friday 30 May 2008 9 to 12

MATHEMATICS (2)

Before you begin read these instructions carefully:

You may submit answers to no more than ***six*** *questions. All questions carry the same number of marks.*

The approximate number of marks allocated to a part of a question is indicated in the right hand margin.

Write on ***one*** *side of the paper only and begin each answer on a separate sheet.*

At the end of the examination:

Each question has a number and a letter (for example, ***6A****).*

Answers must be tied up in ***separate*** *bundles, marked* ***A, B or C*** *according to the letter affixed to each question.*

Do not join the bundles together.

For each bundle, a blue cover sheet must be completed and attached to the bundle.

A ***separate*** *yellow master cover sheet listing all the questions attempted* ***must*** *also be completed.*

Every cover sheet must bear your examination number and desk number.

STATIONERY REQUIREMENTS
6 blue cover sheets and treasury tags
Yellow master cover shett
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1C

A Sturm-Liouville operator $\mathcal{L}$ acts on a function $y(x)$ as,

$$\mathcal{L}\, y \;=\; \frac{1}{w(x)}\left(-\,\frac{d}{dx}\left[p(x)\,\frac{dy}{dx}\right] \;+\; q(x)\, y\right)$$

where $y(x)$ is defined on a closed interval.

Show that the eigenvalues of $\mathcal{L}$ are real and that eigenfunctions belonging to distinct eigenvalues are orthogonal with respect to an inner product which you should define. *You may use the fact that $\mathcal{L}$ is self-adjoint without proof.*

[6]

The Chebyshev equation is

$$(1-x^2)\,\frac{d^2y}{dx^2} - x\,\frac{dy}{dx} + n^2 y \;=\; 0\,.$$

Put this equation in the form of an eigenvalue equation for a Sturm-Liouville operator with eigenvalue n^2, defined on the interval $-1 \leqslant x \leqslant 1$, determining the corresponding functions $w(x)$, $p(x)$ and $q(x)$ and check that $y_1(x) = x$ is an eigenfunction with eigenvalue $n^2 = 1$.

[8]

Find a second eigenfunction of the form $y_3(x) \;=\; x^3 + Bx$ and determine the corresponding eigenvalue. Check explicitly that the resulting eigenfunction $y_3(x)$ is orthogonal to $y_1(x)$ with respect to the appropriate inner product.

[6]

2C

The general solution of the Laplace equation

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\,\frac{\partial T}{\partial r}\right) \;+\; \frac{1}{r^2}\frac{\partial^2 T}{\partial \theta^2} \;=\; 0 \qquad (*)$$

has the form

$$T(r,\theta) \;=\; A_0 + C_0\log(r) + \sum_{n=1}^{\infty}\left(A_n r^n + C_n r^{-n}\right)\cos(n\theta) + \sum_{n=1}^{\infty}\left(B_n r^n + D_n r^{-n}\right)\sin(n\theta)\,.$$

A steady state temperature distribution $T(r,\theta)$ obeys the Laplace equation $(*)$ in the annulus

$$a < r < b \qquad 0 \leqslant \theta < 2\pi\,.$$

The temperature distribution on the boundaries of the annulus at $r = a$ and $r = b$ is fixed so that it varies linearly with θ with a discontinuity at $\theta = \pi$. More precisely,

$$\begin{aligned} T(a,\theta) \;&=\; \mu_a\theta && 0 \leqslant \theta < \pi\,, \\ T(a,\theta) \;&=\; \mu_a(\theta - 2\pi) && \pi < \theta < 2\pi\,, \end{aligned}$$

and

$$\begin{aligned} T(b,\theta) \;&=\; \mu_b\theta && 0 \leqslant \theta < \pi\,, \\ T(b,\theta) \;&=\; \mu_b(\theta - 2\pi) && \pi < \theta < 2\pi \end{aligned}$$

for some constants μ_a and μ_b. Find the temperature distribution in the annulus by determining the coefficients in the series for the general solution given above.

[20]

3C

Use the Divergence Theorem to derive the solution

$$G(\mathbf{x}, \mathbf{x}_0) \;=\; \frac{1}{2\pi}\,\log|\,\mathbf{x} - \mathbf{x}_0|$$

to the two-dimensional Poisson equation

$$\nabla^2 G \;=\; \delta(\mathbf{x} - \mathbf{x}_0)\,.$$

[6]

A point charge e is placed at $\mathbf{x}_0$ with polar coordinates $(r, \theta) = (r_0, \theta_0)$ in the two-dimensional wedge-shaped region W where $r > 0$ and $0 < \theta < \pi/3$. The two half-lines, $r > 0$, $\theta = 0$ and $r > 0$, $\theta = \pi/3$, which form the boundary of this region are earthed (so that the electrostatic potential vanishes there) . Using the method of images show that the electrostatic potential in W can be written as

$$\Phi(\mathbf{x}, \mathbf{x}_0) \;=\; -\,\frac{e}{2\pi\varepsilon_0}\,\log|\mathbf{x} - \mathbf{x}_0| \;-\; \frac{e}{2\pi\varepsilon_0}\sum_{j=1}^{5} s_j \log|\mathbf{x} - \mathbf{x}_j|$$

where you should determine the position of the five image points $\mathbf{x}_j$ in polar coordinates, illustrate their positions in the plane and also determine the appropriate signs $s_j = \pm 1$.

[8]

In the special case where $r_0 = 1$ and $\theta_0 = \pi/6$ show that

$$\Phi(\mathbf{x}, \mathbf{x}_0) \;=\; -\,\frac{e}{4\pi\varepsilon_0}\,\log\left[\frac{r^6 - 2r^3\sin(3\theta) + 1}{r^6 + 2r^3\sin(3\theta) + 1}\right]\,.$$

Hint: You may find it useful to introduce the complex variable $z = r\exp(i\theta)$.

[6]

4C

If $f(z)$ is a complex analytic function with a simple pole at $z = z_0$ define the residue $\text{Res}_{z=z_0}[f(z)]$. If C is a circle (traversed anti-clockwise) centered at the point $z = z_0$ which encloses no other singularities of f prove that

$$\oint_C f(z)\,dz \;=\; 2\,\pi i \;\text{Res}_{z=z_0}[f(z)]\,. \qquad (*)$$

[5]

In the following you may assume that $(*)$ holds when C is replaced by any closed contour encircling the simple pole at $z = z_0$ anti-clockwise and no other singularities of f.

Consider the integral

$$\mathcal{I}(R) \;=\; \oint_{C_R} \frac{\exp(az)}{1+\exp(z)}\,dz$$

where $0 < a < 1$ and C_R is a rectangular contour with corners at the points $z = \pm R$, $\pm R + 2\pi i$, for real R, traversed in the anticlockwise direction.

By considering the integral along each side of the rectangle show that

$$\lim_{R\to\infty}[\mathcal{I}(R)] \;=\; (1-\exp(2\pi i a))\,\mathcal{J}\,,$$

where

$$\mathcal{J} \;=\; \int_{-\infty}^{+\infty} \frac{\exp(ax)}{1+\exp(x)}\,dx\,.$$

Hence show that

$$\mathcal{J} \;=\; \frac{\pi}{\sin(\pi a)}\,.$$

[15]

5C State *Jordan's Lemma* and use it to compute the inverse Fourier transforms of the following functions $\tilde{f}(k)$

$$\frac{1}{a+ik}\,, \qquad \frac{1}{a^2+k^2}\,, \qquad \frac{a^2-k^2}{(a^2+k^2)^2}$$

where $a > 0$.

[20]

6A

State the transformation rule for components $T_{ij\ldots k}$ of a general Cartesian tensor of rank n in three dimensions. What does it mean for such a tensor to be isotropic?

An isotropic fourth rank tensor must be of the form

$$c_{ijkl} \;=\; \alpha\,\delta_{ij}\,\delta_{kl} \;+\; \beta\,\delta_{ik}\,\delta_{jl} \;+\; \gamma\,\delta_{il}\,\delta_{jk}$$

where α, β, γ are scalars. Justify this claim, stating clearly any general result to which you appeal.

[6]

The stress σ_{ij} and strain e_{ij} in a linear elastic medium are tensors related by

$$\sigma_{ij} \;=\; c_{ijkl}\,e_{kl}\,.$$

If e_{ij} is symmetric and the medium is isotropic (so that c_{ijkl} has the form given above) show that this relation can be expressed

$$\sigma_{ij} \;=\; \lambda e_{kk}\,\delta_{ij} \;+\; 2\mu\,e_{ij}$$

for certain λ and μ which should be expressed in terms of α, β, γ.

Show also that the strain can be written in the form $e_{ij} = p\delta_{ij} + d_{ij}$ where d_{ij} is traceless and p is a scalar, to be determined. Hence deduce that the stored elastic energy density $E = \frac{1}{2}\,\sigma_{ij}\,e_{ij}$ is non-negative for any deformation of the solid provided that

$$\mu \geqslant 0 \qquad \text{and} \qquad \lambda \geqslant -2\mu\,/\,3\,.$$

[14]

7A

A particle X of mass $3m$ is suspended vertically downwards from a fixed point O by a light spring with spring constant $2k$. A second particle Y of mass $2m$ is suspended vertically downwards from X by a light spring of spring constant k, and a third particle Z of mass m is suspended vertically downwards from Y by an identical spring, also with constant k. When the system is in equilibrium, the lengths OX, XY, YZ are a, b, c respectively, while the unstretched length of each spring is l.

Consider vertical motion of the particles, with x, y, z being the downward displacements of X, Y, Z from their equilibrium positions (there is no horizontal motion). Write down an expression for the total potential energy V as a sum of elastic and gravitational contributions, and hence explain why

$$V = k\,x^2 + \frac{k}{2}\,(x-y)^2 + \frac{k}{2}\,(y-z)^2 + V_0$$

where V_0 is a constant (depending on a, b, c, l and g, the acceleration due to gravity) which you need not determine. Find the equations of motion for x, y, z.

[6]

Show that the normal frequencies for vertical motion of the particles are

$$\left(\frac{k}{m}\right)^{1/2}, \qquad \left(\frac{k}{m}\right)^{1/2}\left(1 \pm \sqrt{\frac{2}{3}}\right)^{1/2}$$

and find the corresponding vectors which define the normal modes.

[10]

At which normal frequency will all three particles oscillate in phase? If the particles are released from rest, how should the initial displacements of X and Z be chosen to ensure that Y remains at its equilibrium position?

[4]

8B

Define the terms 'normal subgroup' and 'coset'.

[2]

If H is a subgroup of a finite group G and G has twice as many elements as H demonstrate that H is normal in G.

[8]

Show that the order of any subgroup H of G divides the order of G.

[10]

9B

Let $\theta : G \longrightarrow H$ be a homomorphism of two groups with kernel K. Show that K is a normal subgroup of G.

[7]

What is the relationship between the quotient group G/K and H?

[3]

Let $GL(n,\mathbb{R})$ be a group of all invertible n by n matrices and let $SL(n,\mathbb{R})$ be the subset of $GL(n,\mathbb{R})$ consisting of all matrices of determinant 1. Show that $SL(n,\mathbb{R})$ is a normal subgroup of $GL(n,\mathbb{R})$ and that the quotient group $GL(n,\mathbb{R})/SL(n,\mathbb{R})$ is isomorphic to the multiplicative group of non-zero real numbers.

[10]

10B

Let $H = \{1,-1,i,-i\}$ be a multiplicative group of order 4 generated by i such that $i^2 = -1$.

Consider a map $\rho : H \longrightarrow GL(2,\mathbb{R})$ such that

$$\rho(i) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Determine $\rho(1)$, $\rho(-1)$ and $\rho(-i)$ such that ρ is a representation.

[10]

The multiplicative quaternion group Q has elements $\{\pm 1, \pm\mathbf{i}, \pm\mathbf{j}, \pm\mathbf{k}\}$, where

$$\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = \mathbf{ijk} = -1.$$

Show that

$$\mathbf{i} \rightarrow \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \quad \mathbf{j} \rightarrow \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \quad \mathbf{k} \rightarrow \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

gives rise to a representation of Q in $GL(2,\mathbb{C})$ and construct a representation

$$D : Q \rightarrow GL(4,\mathbb{R})$$

of Q by 4 by 4 real matrices.

[6]

Let S be an invertible 4 by 4 matrix. Show that the map $\widetilde{D}(q) = SD(q)S^{-1}$ where $q \in Q$ is another representation of Q and show that characters of D and $\widetilde{D}$ are the same.

[4]

END OF PAPER