UNIVERSITY OF CAMBRIDGE

NATURAL SCIENCES TRIPOS Part IA

Wednesday 11 June 2008 9 to 12

MATHEMATICS (2)

Before you begin read these instructions carefully:

The paper starts with section A, comprised of short questions carrying 20 marks in total, and is followed by ten further questions each carrying 20 marks.

You may submit answers to ***all*** *of section A, and to no more than* ***five*** *other questions.*

The approximate number of marks allocated to a part of a question is indicated in the right hand margin.

Write on ***one*** *side of the paper only and begin each answer on a separate sheet. (For this purpose, your section A attempts should be considered as one single answer.)*

Questions marked with an asterisk (*) require a knowledge of B course material.

At the end of the examination:

Tie up all of your section A answer in a single bundle, with a completed blue cover sheet.

Each further question has a number and a letter (for example, ***3Z****). Answers to these questions should be tied up in* ***separate*** *bundles, marked* ***R, S, T, X, Y*** *or* ***Z*** *according to the letter affixed to each question.* ***Do not join the bundles together****. For each bundle, a blue cover sheet* ***must*** *be completed and attached to each bundle, with the appropriate letter* ***R, S, T, X, Y*** *or* ***Z*** *written in the section box.*

A ***separate*** *yellow master cover sheet listing all the questions attempted* ***must*** *also be completed. (Your section A answer may be recorded just as A: there is no need to list each individual short question.)*

Every cover sheet must bear your examination number and desk number.

STATIONERY REQUIREMENTS
6 blue cover sheets and treasury tags
Yellow master cover sheet
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

SECTION A

1 Determine the angle in the range $(0, \pi)$ between the vectors $\mathbf{a} = (2, -1, -1)$ and $\mathbf{b} = (1, 1, -2)$. [1]

2 Evaluate the following integrals:

(i) $\displaystyle\int_0^{\pi/4} \frac{1}{\cos^2 x}\, \mathrm{d}x\,;$ [1]

(ii) $\displaystyle\int_{-\pi/2}^{+\pi/2} x \cos x \, \mathrm{d}x\,.$ [1]

3 Find the lines on which the function

$$f(x, y) = \frac{xy}{x^2 + xy + y^2}$$

is stationary away from the origin $(x, y) = (0, 0)$. [2]

4 The position vector $\mathbf{r}$ of a point with Cartesian coordinates (x, y, z) is given by $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$, where $\{\mathbf{i}, \mathbf{j}, \mathbf{k}\}$ is an orthonormal basis. Its length is $r = \sqrt{x^2 + y^2 + z^2}$. Determine:

(i) $\boldsymbol{\nabla}(r^2)$; [1]

(ii) $\boldsymbol{\nabla} \cdot \left(r^2 \mathbf{r}\right)$. [2]

5 Find the real and imaginary parts of the complex number

$$\frac{1+i}{1-i}\,.$$ [2]

6 Find the eigenvalues of the matrix

$$\begin{pmatrix} 1/2 & \sqrt{2} \\ \sqrt{2} & -1/2 \end{pmatrix}.$$ [2]

7 Give the first two non-zero terms of the Taylor expansion at $x = 0$ of $e^{\cos x}$. [2]

8 Consider the partial differential equation

$$\frac{\partial u}{\partial t} = \frac{1}{2}\frac{\partial^2 u}{\partial x^2}\,.$$

(i) Find the value for the parameter α such that the function

$$u(x,t) = \alpha x^2 + \beta x + t$$

is a solution, where β is a constant. [1]

(ii) Hence find a solution $u(x,t)$ satisfying the conditions $u(0,0) = 0$ and $u(1,0) = 0$. [2]

9 The inscribed circle of a square is centred on the middle of the square and has a radius half the length of the side of the square. An experiment is performed selecting points randomly from the interior of a square with side of unit length (so that all points inside the square have equal probability).

(i) What is the probability that a point lies within the inscribed circle? [1]

(ii) What is the probability for the points inside the inscribed circle to be found between a distance r and $r + \mathrm{d}r$ from the centre? [2]

END OF SECTION A

10Y

The equations of two straight lines, L_1 and L_2, with respect to an origin O are

$$L_1: \qquad \mathbf{x}(\lambda) = \lambda\mathbf{a} + \mathbf{b}\,,$$

$$L_2: \qquad \mathbf{x}(\mu) = \mu\mathbf{c} + \mathbf{d}\,.$$

Show that if the two straight lines intersect then

$$\mathbf{a}\cdot(\mathbf{c}\wedge\mathbf{d}) + \mathbf{c}\cdot(\mathbf{a}\wedge\mathbf{b}) = 0\,. \qquad (\star)$$ [6]

By considering how $\mathbf{a}, \mathbf{b}, \mathbf{c}$ and $\mathbf{d}$ change under a change of origin, show that the condition $(\star)$ does not depend upon the choice of origin O. [3]

If $\mathbf{x}_1 = \mathbf{x}(\lambda_1)$ and $\mathbf{x}_2 = \mathbf{x}(\lambda_2)$ lie on L_1, and $\mathbf{x}_3 = \mathbf{x}(\mu_1)$ and $\mathbf{x}_4 = \mathbf{x}(\mu_2)$ lie on L_2, show that

$$\begin{aligned}\Big((\mathbf{x}_3-\mathbf{x}_1)\wedge(\mathbf{x}_4-\mathbf{x}_1)\Big)\cdot(\mathbf{x}_2-\mathbf{x}_1) &= (\mathbf{d}-\mathbf{b})\cdot(\mathbf{a}\wedge\mathbf{c})(\mu_2-\mu_1)(\lambda_1-\lambda_2)\\ &= (\mathbf{d}-\mathbf{b})\cdot\Big((\mathbf{x}_4-\mathbf{x}_3)\wedge(\mathbf{x}_2-\mathbf{x}_1)\Big)\,.\end{aligned}$$ [6]

Show that the volume of the tetrahedron with vertices $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3$ and $\mathbf{x}_4$ is unchanged if one slides $\mathbf{x}_1$ and $\mathbf{x}_2$ along L_1 keeping their distance apart constant and/or $\mathbf{x}_3$ and $\mathbf{x}_4$ are slid similarly along L_2. [5]

[*Hint: the volume of a tetrahedron with one vertex at the origin and the others at* $\mathbf{u}$, $\mathbf{v}$ *and* $\mathbf{w}$ *is* $\frac{1}{6}\mathbf{u}\cdot(\mathbf{v}\wedge\mathbf{w})$.]

11X

(a) The University of Carrbridge in Inverness-shire has a total of N students. The students are distributed randomly among a number of colleges (Porterhouse being the most famous) such that each contains exactly S students. In the university as a whole there are L lazy students (and therefore $N-L$ diligent students). Assume that $L \geq S$ and $N-L \geq S$.

(i) What is the probability that *all* the students in Porterhouse are lazy? [3]

(ii) What is the probability that exactly n students in Porterhouse are lazy? [4]

(b) Ancient regulations require the Porterhouse "Director of Science" (DoS) to select one of his students, at random, for a mid-term interview.

What is the probability, $P(\text{lazy})$, that the selected student is lazy? [1]

(c) The selected student tells the DoS that she is not lazy, but the DoS is not convinced. He decides to hide at the back of some of the student's subsequent lectures to observe how many she attends. Assume that this particular student attends any given lecture with a probability f, independently of her attendance at any other lecture(s).

Write down the probability, $P(k|K, f)$, that the DoS finds the selected student in k of the K lectures he visits ($0 \leq k \leq K$). Leave your answer in terms of k, K and f. [3]

(d) The DoS hides at the back of two lectures in total ($K = 2$) and finds the student to be absent on both occasions ($k = 0$).

(i) Calculate $P(\text{absent}|\text{lazy})$ assuming that a lazy student attends on average 1/2 of his/her lectures (i.e. assume that $f = 1/2$). [1]

(ii) Calculate $P(\text{absent}|\text{diligent})$ assuming that a diligent student attends on average 9/10 of his/her lectures (i.e. assume $f = 9/10$). [1]

(iii) Using these answers, and that to part (b), (or otherwise) show that

$$P(\text{absent}) = \frac{24L + N}{100N}.$$

[3]

(iv) Compute numerical values for $P(\text{lazy})$ and $P(\text{lazy}|\text{absent})$ assuming that only 10% of students are lazy. (*Hint: use Bayes' Theorem.*) Comment on your result. [4]

12Y

Given that

$$\int_{-\infty}^{\infty} e^{-\frac{1}{2}x^2}\,\mathrm{d}x = \sqrt{2\pi}\,,$$

find

$$I(a,b) = \int_{-\infty}^{\infty} e^{-\frac{1}{2}\left(ax^2 + \frac{b}{x^2}\right)}\,\mathrm{d}x\,,$$

where a and b are both positive. [15]

[*Hint: the substitution* $y = \frac{1}{2}\left(\sqrt{a}x - \sqrt{b}/x\right)$ *may prove helpful if careful attention is paid to the limits of integration.*]

By use of a suitable substitution, or otherwise, find

$$J(a,b) = \int_{-\infty}^{\infty} \frac{1}{x^2} e^{-\frac{1}{2}\left(ax^2 + \frac{b}{x^2}\right)}\,\mathrm{d}x\,.$$

[5]

13Z

Solve the following second-order differential equations subject to $y = 0$ and $\mathrm{d}y/\mathrm{d}x = 0$ for $x = 0$:

(i)

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} - 7\frac{\mathrm{d}y}{\mathrm{d}x} + 12y = 144x\,;$$

[6]

(ii)

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + 3\frac{\mathrm{d}y}{\mathrm{d}x} + 2y = 10\sin x\,;$$

[7]

(iii)

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + 2\frac{\mathrm{d}y}{\mathrm{d}x} + y = 8e^{-x}\,.$$

[7]

14T

(a) Show that

$$u(x,t) = \frac{A}{\cosh^2(x - vt)}$$

is a solution of the equation

$$\frac{\partial u}{\partial t} - 6u\frac{\partial u}{\partial x} + \frac{\partial^3 u}{\partial x^3} = 0$$

for particular values of (the constants) A and v to be determined. [12]

(b) The function $f(x,t)$ satisfies the equation

$$\frac{\partial f}{\partial t} + \frac{1}{2}\left(\frac{\partial f}{\partial x}\right)^2 = k\frac{\partial^2 f}{\partial x^2}\,,$$

where k is a constant.

(i) Show that $y(x,t) = \partial f/\partial x$ satisfies the equation

$$\frac{\partial y}{\partial t} + y\frac{\partial y}{\partial x} = k\frac{\partial^2 y}{\partial x^2}\,.$$ [4]

(ii) Show that $g(x,t) = \exp(-f/2k)$ satisfies the diffusion equation,

$$\frac{\partial g}{\partial t} = k\frac{\partial^2 g}{\partial x^2}\,.$$ [4]

15Z

(a) Which of the following vector fields, given in Cartesian coordinates, are conservative:

(i) $\mathbf{F}_1 = (3x^2y, x^3 - 2yz^2, -2y^2z)$;

(ii) $\mathbf{F}_2 = (2xy, x^2 - z^2, -3xz^2)$?

In each case, if $\mathbf{F}$ is conservative, find a scalar potential ϕ such that $\mathbf{F} = \boldsymbol{\nabla}\phi$. [6]

(b) For both $\mathbf{F}_1$ and $\mathbf{F}_2$, evaluate explicitly $\int_A^B \mathbf{F} \cdot \mathrm{d}\mathbf{r}$ between points $A = (0,0,0)$ and $B = (2,1,3)$ along:

(i) the straight lines from $(0,0,0)$ to $(0,1,0)$, then to $(2,1,0)$ and finally to $(2,1,3)$;

(ii) the curve $x = 2t$, $y = t^3$ and $z = 3t^2$.

Comment on the relation of your results to those for part (a) above. [14]

16S

(a) Consider the matrix

$$\mathbf{A} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} .$$

(i) Show that $\mathbf{A}^3 = \mathbf{I}$. [2]

(ii) For any integer $n \geqslant 0$, find $\mathbf{A}^{3n+2}$. [2]

(iii) Find the inverse of $\mathbf{A}^2$. [2]

(iv) For any integer $n \geqslant 0$, find the inverse of $\mathbf{A}^{3n+1}$. [2]

(b) Consider the matrix

$$\mathbf{B} = \begin{pmatrix} 1 & 2 & 0 \\ 2 & \nu & \mu \\ 0 & \mu & 1 \end{pmatrix} ,$$

where μ and ν are real.

(i) When $\nu = 0$ and μ is a positive integer, find the smallest value of μ such that all eigenvalues of $\mathbf{B}$ are integers, and find the corresponding eigenvectors. [8]

(ii) When $\nu = 5$, find the condition on μ such that all eigenvalues of $\mathbf{B}$ are positive. [4]

17X

(a) Consider the Fourier series

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right) \right]$$

with real Fourier coefficients (a_n and b_n) and with $a_1 \neq 0$ or $b_1 \neq 0$. What is the period T of $f(x)$? [1]

(b) Write down an expression for $[f(x)]^2$ and hence (or otherwise) prove Parseval's theorem,

$$\frac{1}{T}\int_{-T/2}^{T/2} [f(x)]^2 \mathrm{d}x = \left(\frac{a_0}{2}\right)^2 + \frac{1}{2}\sum_{n=1}^{\infty} \left(a_n^2 + b_n^2\right) ,$$

stating clearly any orthogonality relations you use. [8]

(c) Find the Fourier series for the function $f(x)$ with period 2π which satisfies $f(x) = x$ in the range $-\pi < x < \pi$. Use this result to show that:

$$1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots = \frac{\pi}{4} .$$

[11]

18R*

(a) State the divergence theorem, explaining the meaning of all symbols used. [2]

Use the divergence theorem to evaluate the surface integral $\int \mathbf{r} \cdot \mathrm{d}\mathbf{S}$, where $\mathbf{r}$ is the position vector of a point relative to the origin, over the surface of the ellipsoid

$$\frac{x^2 + y^2}{a^2} + \frac{z^2}{b^2} = 1 \, .$$

[6]

(b) For a constant vector $\mathbf{a}$ and a scalar function ϕ, show that

$$\boldsymbol{\nabla} \cdot (\mathbf{a}\phi) = \mathbf{a} \cdot (\boldsymbol{\nabla}\phi) \, .$$

[2]

By considering a vector $\mathbf{a}\phi$ in the divergence theorem, where $\mathbf{a}$ is a constant but otherwise arbitrary vector, show that

$$\int \phi \, \mathrm{d}\mathbf{S} = \int (\boldsymbol{\nabla}\phi) \, \mathrm{d}V \, ,$$

where $\mathrm{d}V$ is the volume element. [2]

Hence, or otherwise, evaluate the surface integral $\int z \, \mathrm{d}\mathbf{S}$ over the surface of the ellipsoid in part (a) above. [2]

(c) For a constant vector $\mathbf{a}$ and a vector field $\mathbf{v}$, show that

$$\boldsymbol{\nabla} \cdot (\mathbf{a} \wedge \mathbf{v}) = -\mathbf{a} \cdot (\boldsymbol{\nabla} \wedge \mathbf{v}) \, .$$

[3]

Hence show that

$$\int \mathbf{v} \wedge \mathrm{d}\mathbf{S} = -\int (\boldsymbol{\nabla} \wedge \mathbf{v}) \, \mathrm{d}V \, .$$

[3]

19T*

(a) The transverse displacement $y(x,t)$ of an elastic string obeys the wave equation

$$\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2},$$

where the wave speed c is a constant. The string has ends fixed at $x = 0$ and $x = l$. At time $t = 0$ it has zero displacement. By the method of separation of variables, or otherwise, show that the displacement at time $t > 0$ is of the form

$$y(x,t) = \sum_{n=1}^{\infty} a_n \sin\left(\frac{n\pi x}{l}\right) \sin\left(\frac{n\pi ct}{l}\right),$$

where a_n are constants. [5]

At time $t = 0$, the string is struck so that the initial transverse velocity is

$$\frac{\partial y}{\partial t}(x, t = 0) = v(x).$$

Write down an expression for the a_n in terms of an integral over $v(x)$. [1]

If $v(x)$ has the property that

$$\int_0^l v(x) f(x)\,\mathrm{d}x = f(l/2)$$

for any function $f(x)$, find the frequencies of the lowest two harmonics excited, and find the ratio of their amplitudes. [4]

(b) A thin layer of matter lying on the $y = 0$ plane, and with surface density $\Sigma(x) = \Sigma_0 \cos x$, gives rise to a gravitational potential $V(x,y)$. For $y \neq 0$, V obeys Laplace's equation

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = 0.$$

Show that the solutions to this equation which satisfy the boundary condition $V \to 0$ as $|y| \to \infty$ are of the form

$$V(x,y) = f(x)e^{-k|y|},$$

where $k > 0$ is a constant and the form of the function $f(x)$ is to be determined. [6]

To account for the matter in the $y = 0$ plane, V must also satisfy the condition

$$\lim_{\epsilon \to 0} \left(\left.\frac{\partial V}{\partial y}\right|_{y=\epsilon} - \left.\frac{\partial V}{\partial y}\right|_{y=-\epsilon} \right) = 2\pi\Sigma(x).$$

Find $V(x,y)$. [4]

END OF PAPER