UNIVERSITY OF CAMBRIDGE

NATURAL SCIENCES TRIPOS Part IA

Monday 9 June 2008 9 to 12

MATHEMATICS (1)

Before you begin read these instructions carefully:

The paper starts with section A, comprised of short questions carrying 20 marks in total, and is followed by ten further questions each carrying 20 marks.

You may submit answers to ***all*** *of section A, and to no more than* ***five*** *other questions.*

The approximate number of marks allocated to a part of a question is indicated in the right hand margin.

Write on ***one*** *side of the paper only and begin each answer on a separate sheet. (For this purpose, your section A attempts should be considered as one single answer.)*

Questions marked with an asterisk (*) require a knowledge of B course material.

At the end of the examination:

Tie up all of your section A answer in a single bundle, with a completed blue cover sheet.

Each further question has a number and a letter (for example, ***3Z****). Answers to these questions should be tied up in* ***separate*** *bundles, marked* ***R, S, T, X, Y*** *or* ***Z*** *according to the letter affixed to each question.* ***Do not join the bundles together****. For each bundle, a blue cover sheet* ***must*** *be completed and attached to each bundle, with the appropriate letter* ***R, S, T, X, Y*** *or* ***Z*** *written in the section box.*

A ***separate*** *yellow master cover sheet listing all the questions attempted* ***must*** *also be completed. (Your section A answer may be recorded just as A: there is no need to list each individual short question.)*

Every cover sheet must bear your examination number and desk number.

STATIONERY REQUIREMENTS
6 blue cover sheets and treasury tags
Yellow master cover sheet
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

SECTION A

1 Find all solutions of the equation

$$\sqrt{3}\cos\theta + \sin\theta = \sqrt{2}$$

in the range $0 \leqslant \theta < 2\pi$. [2]

2 Differentiate the following with respect to x:

(i) $\sin(x^2)\,;$ [1]

(ii) $\dfrac{e^{2x}}{x}\,.$ [1]

3 Evaluate the following indefinite integrals:

(i) $\displaystyle\int \frac{x}{\sqrt{x^2+1}}\,\mathrm{d}x\,;$ [1]

(ii) $\displaystyle\int x e^{x}\,\mathrm{d}x\,.$ [1]

4 Find all solutions of

$$x^4 - 3x^2 + 2 = 0\,.$$ [2]

5 Find the distance between the points of intersection of the curves $xy = 1$ and $x = 2y$. [2]

6 Determine the equation of the line that is normal to the curve $y^2 = 4x$ at $y = 2$. [2]

7 Sketch the graphs of

(i) $\dfrac{x}{x-1}$ for $-\infty < x < \infty\,;$ [1]

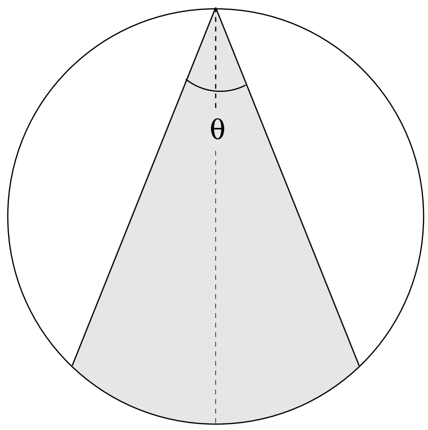
(ii) $x \ln x$ for $0 < x < \infty\,.$ [1]

8 Find the ranges of x for which the inequality

$$|x-2|+|x-1|>1$$

is satisfied. [2]

9 In the figure below, the shaded area is divided equally by the diameter of a circle of unit radius. Find the shaded area in terms of the angle θ. [2]



10 (i) Evaluate $\sum_{n=0}^{N}(1+n)^3-\sum_{n=0}^{N}n^3$. [1]

(ii) Hence, or otherwise, evaluate $\sum_{n=0}^{N}n(1+n)$. [1]

END OF SECTION A

11Y

An operator K, capable of acting on any vector $\mathbf{r}$, is defined by

$$K\mathbf{r} = \mathbf{r} - (\mathbf{n} \cdot \mathbf{r})\,\mathbf{n}\,,$$

where $\mathbf{n}$ is a unit vector. Show that for all vectors $\mathbf{r}$, $K^2\mathbf{r} = K\mathbf{r}$, while if $\mathbf{r}$ is parallel to $\mathbf{n}$ then $K\mathbf{r} = 0$, and if $\mathbf{r}$ is perpendicular to $\mathbf{n}$ then $K\mathbf{r} = \mathbf{r}$. [3]

Hence show that K acts as a projection operator, that is, it maps an arbitrary vector to its projection on the plane perpendicular to $\mathbf{n}$. [1]

Take K to be projection onto a plane P equally inclined to three orthonormal vectors $\mathbf{i}, \mathbf{j}$ and $\mathbf{k}$. Find the unit normal $\mathbf{n}$ to the plane P. [2]

If $\mathbf{e}_1$ and $\mathbf{e}_2$ are two orthonormal vectors lying in the plane P, such that $\mathbf{e}_1$ is equally inclined to $\mathbf{i}$ and $\mathbf{j}$, find $\mathbf{e}_1$ and $\mathbf{e}_2$. Hence give the coordinates with respect to the basis $\mathbf{e}_1, \mathbf{e}_2$ of the projection of the point $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ onto the plane P. [6]

Show that circles in the plane $z = 0$ project to ellipses on P and give the ratio of their minor to their major axes. [8]

12T

(a) Let the complex numbers z_1, z_2, z_3 and z_4 represent the vertices of a plane quadrilateral $ABCD$ in the complex plane. Show that $ABCD$ is a parallelogram if

$$z_1 - z_2 + z_3 - z_4 = 0\,.$$ [2]

(b) Let $z = x + iy$ with x and y real. Find the real and imaginary parts of the following in terms of x and y:

(i) $\dfrac{1-z}{1+z}\,;$ [2]

(ii) $e^{iz}\,;$ [2]

(iii) $z \sin z\,.$ [2]

(c) Find all the roots of the following equations and plot them in the Argand diagram:

(i) $z^4 + z^2 + 1 = 0\,;$ [6]

(ii) $\sin z = \cosh 2\,.$ [6]

13Z

(a) Derive the Taylor series about $x = 0$ for $\ln(1+x)$ and $\ln(1-x)$ and hence derive the Taylor series about $x = 0$ for

$$\ln\left(\frac{1+x}{1-x}\right).$$

[6]

(b) Find the first non-vanishing term in the power-series expansion of

$$(1-e^x)\left(1+\frac{x}{3}\right)^{-3} + \ln(1+x).$$

[7]

(c) Find the values of a and b if the power-series expansion of

$$\left(\frac{1+ax}{1+bx}\right)\ln(1+x)$$

contains no term in x^2 and no term in x^3, and show that the coefficient of x^4 is then $-1/36$. [7]

14R

(a) By reversing the order of integration, evaluate

$$\int_0^8 \int_{y^{1/3}}^2 \sqrt{x^4+1}\,\mathrm{d}x\,\mathrm{d}y.$$

[8]

(b) By evaluating an appropriate double integral, find the volume of the wedge lying between the planes $z = px$ and $z = qx$ $(p > q > 0)$ and the cylinder $x^2 + y^2 = 2ax$ (where $a > 0$). [8]

Find also the area of the curved surface of the wedge. [4]

15X

(a) The probability that an experiment involving T trials has S successes (T and S are non-negative integers satisfying $0 \leq S \leq T$) is

$$P(S = s) = \binom{T}{s} p^s (1-p)^{T-s}$$

for some value of the constant p satisfying $0 < p < 1$.

Prove that the mean value of S is pT. [5]

(b) Let X be a discrete random variable able to take any integer value n (so that $-\infty < n < \infty$) with corresponding probability p_n. Define the mean and variance of X. [2]

Let Y be a continuous random variable which has probability density function $f(y)$ and which is able to take real values in the range $-\infty < y < \infty$. Write down an expression for $P(\alpha < Y < \beta)$ (for $\alpha < \beta$). [1]

Let Z be a third random variable defined to be the sum of the two independent random variables X and Y defined above. Is Z a discrete random variable, a continuous random variable, or neither? [1]

Write down an expression for $P(\alpha < Z < \beta)$ (for $\alpha < \beta$). [3]

Hence, or otherwise, find a probability density function $g(z)$ for Z. [2]

Now assume that $p_0 = 0$ and $p_n = 3/(n\pi)^2$ otherwise. Furthermore, assume that

$$f(y) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-y^2/(2\sigma^2)} ,$$

where σ is a positive constant.

(i) Sketch $g(z)$ for the case $\sigma \ll 1$. (Detailed calculations are *not* required.) [3]

(ii) Sketch $g(z)$ for the case $\sigma \gg 1$. (Detailed calculations are *not* required.) [3]

16Z

(a) Solve the following first-order differential equations:

(i) $x\dfrac{\mathrm{d}y}{\mathrm{d}x} = 3(1-y)$ given that $y = 0$ for $x = 1$; [2]

(ii) $(1+x^2)^{1/2}\dfrac{\mathrm{d}y}{\mathrm{d}x} = xe^{-y}$ given that $y = 0$ for $x = 0$; [2]

(iii) $x\dfrac{\mathrm{d}y}{\mathrm{d}x} = y + (x^2 - y^2)^{1/2}$. [4]

(b) Using an integrating factor, or otherwise, solve the following first-order differential equations:

(i)

$$\cos x\frac{\mathrm{d}y}{\mathrm{d}x} + (\cos x + \sin x)y = 2 + \sin 2x$$

given that $y = 2$ for $x = 0$; [6]

(ii)

$$\frac{x(x^2-1)}{y^2}\frac{\mathrm{d}y}{\mathrm{d}x} - \frac{1}{y} = x^3$$

given that $y = -1/2$ for $x = 2$. [6]

17T

(a) Consider the function $f(x,y) = xy(1-x+y)$.

(i) Find and plot the triangle on which $f = 0$. [2]

(ii) Calculate $\boldsymbol{\nabla} f$ at the points $(1/2, 0)$, $(1/2, -1/2)$ and $(0, -1/2)$ and mark the directions of $\boldsymbol{\nabla} f$ at these points on your plot. [4]

(iii) Find the position and nature of the stationary point which lies within the triangle, and the value of f at that point. [4]

(iv) Sketch the contours of f inside the triangle. [2]

(b) Locate and identify the nature of the stationary points of the function

$$g(x,y) = y^2(1-x) - x^2(1+x)\,.$$ [8]

18S

(a) Consider the matrix

$$\mathbf{S} = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 3 & 2 \\ 1 & 2 & 1 \end{pmatrix} .$$

(i) Calculate $\det \mathbf{S}$. [2]

(ii) Find $\mathbf{S}^{-1}$. [4]

(iii) Find a 3×3 matrix $\mathbf{T}$ such that $\mathbf{TS} \neq \mathbf{ST}$. [2]

(b) $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{C}$ are 3×3 square matrices. The trace of a matrix $\mathbf{A}$ with elements a_{ij} is defined by

$$\mathrm{Tr}\mathbf{A} = \sum_{i=1}^{3} a_{ii} .$$

(i) Prove that $\det \mathbf{A} = \det(\mathbf{A}^T)$. [4]

(ii) Show that $\mathrm{Tr}(\mathbf{AB}) = \mathrm{Tr}(\mathbf{BA})$. [2]

(iii) Show further that

$$\mathrm{Tr}(\mathbf{ABC}) = \mathrm{Tr}(\mathbf{CAB}) = \mathrm{Tr}(\mathbf{BCA}) .$$ [2]

(c) Find the value of μ for which the system of linear equations

$$\begin{aligned} 2x + y &= 0 \\ x + 3y + \mu z &= 0 \\ y + 2z &= 0 \end{aligned}$$

has non-zero solutions. [4]

19X*

(a) Let $f(x)$ denote an arbitrary real function defined everywhere (except perhaps at x_0) on an open interval containing x_0. Such a function $f(x)$ is said to tend to a limit K as x tends to x_0 [denoted by "$f(x) \to K$ as $x \to x_0$" or by "$K = \lim_{x \to x_0} f(x)$"] if for any $\epsilon > 0$, there exists a $\delta > 0$ such that $|f(x) - K| < \epsilon$ for all $0 < |x - x_0| < \delta$.

What does it mean to say that $f(x)$ is continuous at x_0? [3]

What does it mean to say that $f(x)$ is differentiable at x_0? [2]

A function

$$q(x,n) = \begin{cases} x^n \sin(1/x) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \text{ and } n \neq 4 \\ 1 & \text{if } x = 0 \text{ and } n = 4 \end{cases}$$

is defined for integer $n \in \{0, 1, 2, 3, 4\}$ and for real x in the range $-1 < x < 1$.

(i) Determine the values of n for which $q(x,n)$ tends to a limit as x tends to zero. [2]

(ii) Determine the values of n for which $q(x,n)$ is continuous at $x = 0$. [2]

(iii) Determine the values of n for which $q(x,n)$ is differentiable at $x = 0$. [2]

(b) Determine the following limit in terms of the two real-valued parameters A and B:

$$\lim_{x \to 0} \frac{Ae^{A/x^2} + Be^{B/x^2}}{e^{A/x^2} + e^{B/x^2}} .$$

[3]

(c) State the ratio test for determining convergence properties of series. [2]

State what, if anything, the ratio test tells you about the convergence or divergence of the following series:

(i) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$;

(ii) $\displaystyle\sum_{n=1}^{\infty} \frac{(n!)^3 e^{3n}}{(3n)!}$.

[4]

20Y*

(a) State and prove Leibnitz's formula for $\dfrac{\mathrm{d}^n}{\mathrm{d}x^n}\big(f(x)g(x)\big)$. [5]

Hence, or otherwise, show that

$$\frac{\mathrm{d}^n}{\mathrm{d}x^n}\big(f(x)g(x)h(x)\big) = \sum_{p+q+r=n} \frac{n!}{p!q!r!} \frac{\mathrm{d}^p f}{\mathrm{d}x^p} \frac{\mathrm{d}^q g}{\mathrm{d}x^q} \frac{\mathrm{d}^r h}{\mathrm{d}x^r} \,,$$

where the sum is over non-negative integers p, q and r that sum to n. [5]

(b) Prove Schwarz's inequality

$$\int f^2 \,\mathrm{d}x \int g^2 \,\mathrm{d}x \geq \left(\int fg \,\mathrm{d}x\right)^2 .$$

[5]

A vehicle moving forward with variable speed in one dimension is at $x = x_i$ at time $t = t_i$ and $x = x_f$ at time $t = t_f$, $t_f > t_i$, $x_f > x_i$. The time average of the speed $v = \mathrm{d}x/\mathrm{d}t$ is

$$V_t = \frac{1}{t_f - t_i} \int_{t_i}^{t_f} v \,\mathrm{d}t \,,$$

while its space-averaged speed is

$$V_x = \frac{1}{x_f - x_i} \int_{x_i}^{x_f} v \,\mathrm{d}x \,.$$

Which is greater, V_t or V_x? [1]

Give a proof of your assertion. [4]

END OF PAPER