MATHEMATICS (2)

Before you begin read these instructions carefully:

You may submit answers to no more than **six** questions. All questions carry the same number of marks.

The approximate number of marks allocated to a part of a question is indicated in the right hand margin.

Write on **one** side of the paper only and begin each answer on a separate sheet.

At the end of the examination:

Each question has a number and a letter (for example, **6A**).

Answers must be tied up in **separate** bundles, marked **A**, **B** or **C** according to the letter affixed to each question.

**Do not join the bundles together.**

For each bundle, a blue cover sheet must be completed and attached to the bundle.

A **separate** yellow master cover sheet listing all the questions attempted must also be completed.

**Every cover sheet must bear your examination number and desk number.**

**STATIONERY REQUIREMENTS**

- 6 blue cover sheets and treasury tags
- Yellow master cover sheet
- Script paper

**SPECIAL REQUIREMENTS**

- None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.
1B Axisymmetric solutions $\Phi(r, \theta)$ of Laplace’s equation satisfy

$$\frac{1}{r^2} \frac{\partial}{\partial r} (r^2 \frac{\partial \Phi}{\partial r}) + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} (\sin \theta \frac{\partial \Phi}{\partial \theta}) = 0,$$

where $(r, \theta, \phi)$ are the standard spherical polar co-ordinates. Show, by using the method of separation of variables, that the solution to this equation can be written as

$$\Phi(r, \theta) = \sum_{n=0}^{\infty} (A_n r^n + B_n r^{-n-1}) P_n(\cos \theta)$$

for constants $A_n, B_n$. Derive the equation which the functions $P_n$ must satisfy.

A spherically symmetric shell of material lies between $a < r < b$. The temperature on the boundaries of the shell is

$$T(a, \theta) = T_0 + T_1 \cos \theta, \quad T(b, \theta) = T_2 + T_3 \cos \theta,$$

for constants $T_0, T_1, T_2, T_3$. Determine the steady-state axisymmetric temperature within the shell.

[You may assume that $P_0(\cos \theta) = 1$ and $P_1(\cos \theta) = \cos \theta$.]
2B Prove that the Green’s function
\[ G(x; x_0) = -\frac{1}{4\pi|x - x_0|} \]
is the fundamental solution in three dimensions satisfying \( G(x; x_0) \to 0 \) as \( |x| \to \infty \) and
\[ \nabla^2 G(x; x_0) = \delta(x - x_0) . \] \((*)\)

Using the method of images find the Green’s function \( G(x; x_0) \) satisfying \((*)\) for \( x \in V \) and \((\text{fixed}) \) \( x_0 \in V \) when:

(i) \( V \) is the half space of \( \mathbb{R}^3 \) with \( z > 0 \), \( G = 0 \) on \( z = 0 \), and \( G \to 0 \) as \( |x| \to \infty \) for \( x \in V \).

(ii) \( V \) is the interior of the sphere \( r < a \), and \( G = 0 \) on \( r = a \).

A point charge \( e \) is placed at \( x_0 \in V \), where \( V \) is a hollow hemisphere of radius \( a \):
\[ V = \{(x, y, z) : x^2 + y^2 + z^2 < a^2 \text{ and } z > 0\} . \]
The boundary of \( V \) is earthed. Derive the electrostatic potential in \( V \).

3B

Let \( f(z) = u + iv \) for real \( u, v \) be an analytic function of \( z = x + iy \) for real \( x, y \). Prove that \( u \) and \( v \) satisfy the Cauchy-Riemann equations
\[ \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} . \]

Determine \( f(z) \) when \( u(x, y) = e^x((x^2 - y^2) \cos y - 2xy \sin y) \).

Determine the singularities (poles or branch points) of the following functions:

(i) \( f(z) = z^4 \log \left(\frac{z - 1}{z + 1}\right) \), \( \quad \) (ii) \( f(z) = \frac{z}{z^2 + 4z + 5} \).

[You may assume that \( \int y^2 \cos y \, dy = y^2 \sin y - 2 \sin y + 2y \cos y \) and \( \int y \sin y \, dy = \sin y - y \cos y \).]
4B

State the Residue Theorem. [4]

Let $n$ be a positive integer with $n \geq 3$. Identify the poles of

$$f(z) = \frac{1}{1 + z^n}.$$ 

By integrating $f(z)$ along a contour enclosing only one pole, prove that

$$\int_{0}^{\infty} \frac{1}{1 + x^n} \, dx = \frac{\pi}{n \sin \left( \frac{\pi}{n} \right)}.$$ 

Furthermore, compute the integral  

$$I = \int_{0}^{\infty} \frac{x^{\frac{1}{n}}}{1 + x^n} \, dx.$$ 

[8]

5B

(i) Define the Laplace transform $\tilde{f}(p)$ of a function $f(t)$ (where $f(t) = 0$ for $t < 0$). State the Bromwich inverse integral formula for the inverse Laplace transform, and describe how the Bromwich contour should be chosen in the integral.

Assuming that $f(t) \to f(0)$ as $t \to 0$ from above, and also that $f(t)e^{-pt} \to 0$ as $t \to +\infty$, show that the Laplace transform of $f'(t)$ is $pf(p) - f(0)$. [8]

(ii) The function $S_n(t)$ is defined by

$$S_n(t) = \begin{cases} 
  n & \text{if } 1 - \frac{1}{2n} < t < 1 + \frac{1}{2n} \\
  0 & \text{otherwise}
\end{cases}$$

where $n$ is a positive integer. The function $f(t)$ satisfies $f(0) = 0$ and

$$f'(t) - f(t) = S_n(t) \quad (*)$$

for $t > 0$, with $f(t) = 0$ for $t < 0$. Using the Laplace transform, find $f(t)$ for $t < 1 - \frac{1}{2n}$ and $t > 1 + \frac{1}{2n}$. [12]

[For this question, Jordan’s Lemma may be used without proof, provided it is stated carefully].
Define a tensor $T$ of rank two in $\mathbb{R}^3$, and demonstrate that every such tensor can be decomposed as

$$T_{ij} = Y \delta_{ij} + \Omega_{ij} + S_{ij},$$

where $\Omega_{ij}$ is antisymmetric, $S_{ij}$ is symmetric and traceless and $Y$ is a scalar. What are the numbers of independent components of $\Omega_{ij}$ and $S_{ij}$? \[6\]

Let $A_i$ be a non-zero rank one tensor and let $T_{ij}$ be a symmetric rank two tensor. Find scalars $\alpha$, $\beta$, a rank one tensor $B_i$, and a symmetric traceless rank two tensor $C_{ij}$ such that

$$T_{ij} = \alpha \delta_{ij} + \beta A_i A_j + (B_i A_j + B_j A_i) + C_{ij},$$

$$A_i B_i = 0, \quad C_{ij} A_i = 0.$$ \[14\]

[The summation convention is assumed. It is not necessary to derive the transformation laws for $Y$, $\Omega_{ij}$, $S_{ij}$, $\alpha$, $\beta$, $B_i$ and $C_{ij}$.]
Write down a general Lagrangian of a system with \( n \) degrees of freedom undergoing small oscillations, and state the polynomial equation for the normal frequencies. [4]

Let \( A, B, C \) be three identical particles of unit mass, and let \( O \) be a fixed point. Assume that \( A \) is suspended from \( O \) by a string of length \( 3a \), \( B \) is suspended from \( A \) by a string of length \( 2b \) and \( C \) is suspended from \( B \) by a string of length \( c \). The system moves in a plane containing \( A, B, C \) and \( O \) under the influence of gravity.

Let \( x, y, z \) denote the horizontal displacements of \( A, B, C \) from their equilibria. Show that the approximate Lagrangian for small oscillations is

\[
L = \frac{1}{2} (\dot{x}^2 + \dot{y}^2 + \dot{z}^2) - g \left( \frac{x^2}{a} + \frac{(x-y)^2}{b} + \frac{(y-z)^2}{c} \right),
\]

where \( g \) is the acceleration due to gravity. [8]

Show that the necessary and sufficient condition for the existence of a normal mode \( y(t) = 0 \) is

\[
\frac{1}{a} + \frac{1}{b} - \frac{1}{c} = 0.
\]

[8]
8C

State the Lagrange theorem relating the order of a group to orders of its subgroups. [2]

Let $G$ be a group in which every element other than the identity has order 2. Show that this group is Abelian. Let $a, b$ be distinct elements of $G$ different from the identity element $I$. Show that $\{I, a, b, ab\}$ is a subgroup of $G$ of order 4. [10]

Deduce that any group of order $2p$, where $p > 2$ is prime, must contain an element of order $p$. [8]

9C

Let $G$ and $H$ be two groups. Define the terms isomorphism, homomorphism and kernel, and show that if $\phi : G \rightarrow H$ is a homomorphism then the kernel of $\phi$ is a normal subgroup of $G$. [6]

Show that matrices of the form

$$B = \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix},$$

where $n$ is an integer, form a subgroup $H$ of the multiplicative group of invertible $2 \times 2$ matrices. Is $H$ cyclic? Does it have any proper subgroups? [8]

Demonstrate that $H$ is isomorphic to the additive group of all integers. [6]
10C

Let $G$ be a group, and let $D$ be a map $D : G \to GL(2, \mathbb{R})$, where $GL(2, \mathbb{R})$ is the group of real $2 \times 2$ invertible matrices. What does it mean for $D$ to be a representation of $G$? \[4\]

Let $G$ be a cyclic subgroup of $GL(2, \mathbb{R})$ of order 4 given by

$$G = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \right\}.$$  

Suppose that real $x, y, \tilde{x}, \tilde{y}$ are related by

$$\begin{pmatrix} \tilde{x} \\ \tilde{y} \end{pmatrix} = A \begin{pmatrix} x \\ y \end{pmatrix}$$  for real $x, y$, where $A = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in G$.

Suppose in addition that $x, y, \tilde{x}, \tilde{y}$ satisfy the relation

$$\hat{a}\tilde{x}^2 + \hat{b}\tilde{y}^2 = ax^2 + by^2 \quad (\star)$$

for real $\hat{a}, \hat{b}, a, b$. Find the real $2 \times 2$ matrix $M$, whose components do not depend on $\hat{a}, \hat{b}, a, b$, such that the relation $(\star)$ holds for all real $x, y$, and $M$ satisfies

$$\begin{pmatrix} a \\ b \end{pmatrix} = M \begin{pmatrix} \hat{a} \\ \hat{b} \end{pmatrix}.$$  

[7]

Show that the map

$$A \mapsto D(A) = M^{-1}$$

is a representation of $G$. Is this representation faithful? \[9\]

END OF PAPER