UNIVERSITY OF CAMBRIDGE

NATURAL SCIENCES TRIPOS Part IA

Monday 11 June 2007 9 to 12

MATHEMATICS (1)

Before you begin read these instructions carefully:

The paper has two sections, A and B. Section A comprises short questions and carries 20 marks in total. Section B contains ten questions, each carrying 20 marks.

You may submit answers to ***all*** *of section A, and to no more than* ***five*** *questions from section B.*

The approximate number of marks allocated to a part of a question is indicated in the right hand margin.

Write on ***one*** *side of the paper only and begin each answer on a separate sheet. (For this purpose, your section A attempts should be considered as one single answer.)*

Questions marked with an asterisk (*) require a knowledge of B course material.

At the end of the examination:

Tie up all of your section A answer in a single bundle, with a completed blue cover sheet.

Each section B question has a number and a letter (for example, ***3Z****).*

Section B answers must be tied up in ***separate*** *bundles, marked* ***R, S, T, X, Y*** *or* ***Z*** *according to the letter affixed to each question.* ***Do not join the bundles together****. For each bundle, a blue cover sheet* ***must*** *be completed and attached to each bundle, with the appropriate letter written in the section box.*

A ***separate*** *yellow master cover sheet listing all the questions attempted* ***must*** *also be completed. (Your section A answer may be recorded just as A: there is no need to list each individual short question.)*

Every cover sheet must bear your examination number and desk number.

STATIONERY REQUIREMENTS
6 blue cover sheets and treasury tags
Yellow master cover sheet
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

SECTION A

1. Given that $x = 1$ is a solution of

$$2x^3 + x^2 - 5x + 2 = 0$$

find all the solutions of this equation. [2]

2. Differentiate

   a) $\cos 5x$ [1]

   b) $x^2 e^x$. [1]

3. Integrate

   a) $\dfrac{1}{\sqrt{3+2x}}$ [1]

   b) $x \ln x$. [1]

4. Determine the radius and the position of the centre of the circle

$$x^2 + y^2 + 6y + 5 = 0.$$

[2]

5. Evaluate

   a) $\displaystyle\sum_{n=0}^{100}(2n+1)$ [1]

   b) $\displaystyle\sum_{n=0}^{\infty}\left(\frac{1}{3}\right)^n$. [1]

6. A curve is given parametrically by

$$x = (1+t)^2$$
$$y = 12t - t^3.$$

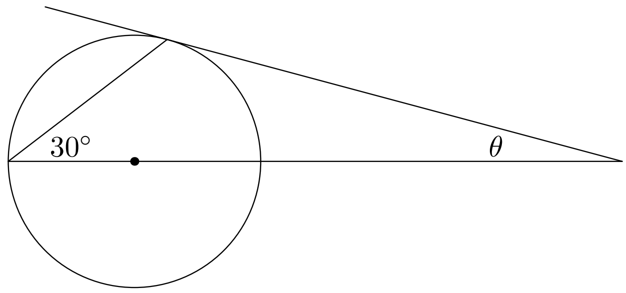
Calculate its slope $\dfrac{\mathrm{d}y}{\mathrm{d}x}$ at the point $(1, 0)$. [2]

7. Find all solutions of the equation

$$\sin 2\theta - 2\cos^2\theta = 0$$

in the range $0 \leqslant \theta \leqslant \pi$. [2]

8. Determine the angle θ. The picture is not drawn accurately. [1]



9. Write $\sqrt{5}\left(\frac{\sqrt{5}+1}{2} + \frac{2}{\sqrt{5}+1}\right)$ in its simplest form. [1]

10. Sketch the graphs of

a) $\cos^2 x$ for $0 \leqslant x \leqslant \pi$ [1]

b) $\frac{1}{1-e^x}$ for $-\infty < x < \infty$. [1]

11.

a) For what value of k is the line

$$y = kx$$

tangent to the curve

$$y = e^{2x}?$$

[1]

b) What is the point of tangency? [1]

SECTION B

1Y

(a) An operator K, capable of acting on an arbitrary vector $\mathbf{x}$, is defined as

$$K\mathbf{x} = \mathbf{a}(\mathbf{b} \cdot \mathbf{x}).$$

Show that

$$K^2\mathbf{x} = (\mathbf{a} \cdot \mathbf{b})K\mathbf{x},$$

where $K^2\mathbf{x} \equiv K(K\mathbf{x})$. [4]

(b) An operator H, capable of operating on an arbitrary vector $\mathbf{x}$, is defined as

$$H\mathbf{x} = \alpha\, \mathbf{n} \wedge \mathbf{x},$$

in terms of a unit vector $\mathbf{n}$ and a constant α.

Show that

$$H^3\mathbf{x} = -\alpha^2 H\mathbf{x}.$$

[6]

(c) If I is the identity operator ($I\mathbf{x} = \mathbf{x}$), and if G is defined by

$$G = I + H + H^2/2! + H^3/3! + \cdots,$$

show that

$$G\mathbf{x} = \mathbf{x} + \mathbf{n} \wedge \mathbf{x} \sin\alpha + \mathbf{n} \wedge (\mathbf{n} \wedge \mathbf{x})(1 - \cos\alpha).$$

[10]

2T

(a) Find the real and imaginary parts of the following expressions:

$$\sqrt{i}; \qquad \ln(-e); \qquad 2^i; \qquad (i^i)^i.$$

[8]

(b) Write down expressions for $\sin x$ and $\cos x$ in terms of e^{ix} and e^{-ix}. [2]

By considering the complex expression for $\sin^3 x$, express $\sin 3x$ in terms of $\sin x$. [4]

Find all the solutions to $\sin 3x = 2\sin x$. [2]

(c) Find the roots of

$$z^2 + 2bz + 1 = 0,$$

and draw a diagram showing their location on the Argand diagram as b takes real values in the range -1 to 1. [4]

3Z

(a) Find, by any method, the first two non-vanishing terms in the Taylor expansion of the following functions about $x = 0$:

(i) $\arctan x$, [4]

(ii) $\dfrac{1}{a^x} - 1$ for real $a > 0$, [4]

(iii) $\ln\left(\cos^2 x\right)$. [6]

(b) Let

$$f(x) = \sum_{m=0}^{\infty} f_m x^m, \text{ with } f_0 \neq 0,$$

and

$$g(x) = \sum_{n=1}^{\infty} g_n x^n \text{ so that } g(0) = 0.$$

Give the first *four* terms of the Taylor expansion of the composite function $f[g(x)]$ in terms of the coefficients f_i and g_i. [6]

4X

(a) The probability of the number n of persons passing a certain checkpoint during a day is

$$P(n;\lambda) = \frac{\lambda^n e^{-\lambda}}{n!},$$

which defines a Poisson distribution with parameter λ.

Show that

$$\sum_{n=0}^{\infty} P(n;\lambda) = 1.$$

[2]

The probability that any given person is male is p. Show that the probability that k males and l females pass the checkpoint during a day is

$$P(k \text{ males, } l \text{ females}) = \binom{k+l}{k} \frac{p^k(1-p)^l \lambda^{(k+l)} e^{-\lambda}}{(k+l)!} \; .$$

[4]

Hence show that the probability that k males pass (independent of the number of females passing) during the day conforms to a Poisson distribution with parameter λp, i.e. the probability is $P(k;\lambda p)$. [4]

[*Note:* $\binom{n}{k}$ *is alternative notation for* ${}^nC_k$*.*]

(b) A proportion 0.1 of members of a large population have a certain viral disease, and a further proportion 0.2 are carriers of the virus. A test for the presence of the virus shows positive with probability 0.95 if the person tested has the disease, 0.9 if the person is a carrier and 0.05 if the person in fact is free of the virus.

Calculate the probability that any given person tests positive. [3]

Calculate the probability that a person who tests negative in fact has the virus (i.e. either has the disease or is a carrier). [7]

5Z

(a) Solve the following differential equations by determining which are exact and using an integrating factor for any that are not:

(i) $(1+e^y)\cos x\,\mathrm{d}x + e^y \sin x\,\mathrm{d}y = 0\ ,$ [5]

(ii) $y\left(x^2+\ln y\right)\,\mathrm{d}x + x\,\mathrm{d}y = 0\ .$ [7]

The solutions may be left in the form of implicit equations for y.

(b) Solve the differential equation

$$(1+x)\frac{\mathrm{d}y}{\mathrm{d}x} - 3y = (1+x)^5$$

with the initial condition $y = 3/2$ at $x = 0$. [8]

6T

The interaction energy between two dipoles with orientations θ_1 and θ_2 is

$$U(\theta_1,\theta_2) = -3\cos(\theta_1+\theta_2) - \cos(\theta_1-\theta_2).$$

Find all of the values of (θ_1,θ_2) in the region $0 \leqslant \theta_1 \leqslant \pi$, $0 \leqslant \theta_2 \leqslant \pi$ which correspond to stationary values of U. [8]

By evaluating the second partial derivatives of U, determine whether each of these points is a minimum, a maximum, or a saddle point. [6]

Sketch a contour plot of $U(\theta_1,\theta_2)$ in the region $0 \leqslant \theta_1 \leqslant \pi$, $0 \leqslant \theta_2 \leqslant \pi$. [6]

7R

(i) Evaluate

$$\int_{y=0}^{1}\int_{x=2y}^{2} e^{x^2}\,\mathrm{d}x\,\mathrm{d}y$$

by first changing the order of integration. [8]

(ii) A lens is given in cylindrical polar coordinates (r,θ,z) by $r^2+z^2 < 2$, $z > 1$, $0 < \theta < 2\pi$ and has density $\rho = kz$.

Calculate the mass of the lens. [6]

Calculate the area of the curved surface of the lens using integration. [6]

8S

(a) Below are statements about square matrices $\mathbf{A}$ and $\mathbf{B}$ and vectors $\mathbf{x}$ and $\mathbf{y}$.

(i) If both $\mathbf{x}$ and $\mathbf{y}$ are eigenvectors of $\mathbf{A}$ then $\mathbf{x}+\mathbf{y}$ is an eigenvector of $\mathbf{A}$. [3]

(ii) If $\mathbf{x}$ is an eigenvector of both $\mathbf{A}$ and $\mathbf{B}$ then $\mathbf{x}$ is an eigenvector of $\mathbf{A}+\mathbf{B}$. [3]

(iii) If $\mathbf{x}$ is an eigenvector of both $\mathbf{A}$ and $\mathbf{B}$ then $\mathbf{x}$ is an eigenvector of $\mathbf{AB}$. [3]

Indicate which of these statements is true and which is false. If a statement is true, give the eigenvalue of the eigenvector in question assuming $\mathbf{Ax} = a\mathbf{x}$, $\mathbf{Bx} = b\mathbf{x}$ and $\mathbf{Ay} = c\mathbf{y}$. If a statement is false, find a counter-example disproving it.

(b) $\mathbf{A}$ is a square matrix with the property $\mathbf{A}^2 = \mathbf{A}$.

(i) Determine the eigenvalues of $\mathbf{A}$, indicating how you found them. [2]

(ii) Find all the eigenvalues of $\mathbf{B} = \mathbf{I} - \mathbf{A}$. [1]

(c) The matrix

$$\mathbf{H} = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 0 \end{pmatrix}.$$

(i) Find the eigenvalues of $\mathbf{H}$. [4]

(ii) Determine the eigenvector for the lowest eigenvalue of $\mathbf{H}$. [4]

9X*

(a) Let $f(x)$ be a positive but decreasing function of x for all $x > 0$, so that $f(x_1) > f(x_2) > 0$ for all x_1, x_2 such that $x_2 > x_1 > 0$, and define

$$S_n = \int_n^{n+1} f(x)\,\mathrm{d}x.$$

Demonstrate, with the aid of a diagram or otherwise, that

$$S_n < f(n) < S_{n-1}.$$

[4]

Deduce that the sum $\sum_{n=2}^{\infty} f(n)$ converges or diverges with the integral

$$\int_2^{\infty} f(x)\,\mathrm{d}x,$$

[3]

and that, in the case of convergence,

$$\sum_{n=2}^{\infty} f(n) = \sum_{n=2}^{N} f(n) + R_N,$$

where

$$\int_{N+1}^{\infty} f(x)\,\mathrm{d}x < R_N < \int_N^{\infty} f(x)\,\mathrm{d}x.$$

[3]

[*Note: the lower limit* $n = 2$ *is arbitrary, and chosen just for consistency with what follows.*]

Determine whether the series

$$\sum_{n=2}^{\infty} \frac{1}{n\ln(n)}, \quad \sum_{n=2}^{\infty} \frac{1}{n(\ln(n))^2}$$

converge or diverge, and obtain bounds on the remainder term R_N in the case(s) of convergence. [6]

(b) State the ratio test for determining whether a series $\sum_{n=1}^{\infty} a_n$ converges absolutely or diverges. Investigate the convergence or otherwise of the series

$$\sum_{n=1}^{\infty} \frac{n}{2^n - 1}, \quad \sum_{n=1}^{\infty} \frac{(-1)^{n+1} 2^n}{n^5}.$$

[4]

10Y*

(a) Let

$$f(x) = \int_0^x e^{-t} t^{-1} (t/x)^a \, \mathrm{d}t,$$

where $a > 0$.

Find $\mathrm{d}f/\mathrm{d}x$ and $\mathrm{d}^2 f/\mathrm{d}x^2$. [4]

Show that f satisfies the differential equation

$$x\frac{\mathrm{d}^2 f}{\mathrm{d}x^2} + (1 + a + x)\frac{\mathrm{d}f}{\mathrm{d}x} + af = 0.$$

[4]

(b) Consider

$$g(x) = \int_0^x \frac{h(t)}{\sqrt{x-t}} \mathrm{d}t,$$

where $h(0) \neq 0$.

(i) Explain why we cannot obtain $\mathrm{d}g/\mathrm{d}x$ by the direct application of the method used in (a). [2]

(ii) Integrate by parts to show that

$$g(x) = 2h(0)\sqrt{x} + 2\int_0^x h'(t)\sqrt{x-t}\, \mathrm{d}t.$$

[4]

(iii) Use this to find an expression for $\mathrm{d}g/\mathrm{d}x$. [6]

END OF PAPER