UNIVERSITY OF CAMBRIDGE

NATURAL SCIENCES TRIPOS Part IB & II (General)

Friday 2 June 2006 9 to 12

MATHEMATICS (2)

Before you begin read these instructions carefully:

You may submit answers to no more than ***six*** *questions. All questions carry the same number of marks.*

The approximate number of marks allocated to a part of a question is indicated in the right hand margin.

Write on **one** *side of the paper only and begin each answer on a separate sheet.*

At the end of the examination:

Each question has a number and a letter (for example, ***6A****).*

Answers must be tied up in **separate** *bundles, marked* ***A, B or C*** *according to the letter affixed to each question.*

Do not join the bundles together.

For each bundle, a blue cover sheet must be completed and attached to the bundle.

A **separate** *yellow master cover sheet listing all the questions attempted* **must** *also be completed.*

Every cover sheet must bear your examination number and desk number.

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1C

The function $\Phi(\mathbf{r})$ vanishes as $|\mathbf{r}| \to \infty$ (where $\mathbf{r} = (x, y, z)$) and satisfies the three-dimensional Poisson equation

$$\nabla^2 \Phi(\mathbf{r}) = \sigma(\mathbf{r}) \,.$$

Write down the solution of this equation in terms of a Green function $G(\mathbf{r}; \mathbf{r_0})$. [3]

What equation does $G(\mathbf{r}; \mathbf{r_0})$ satisfy and what boundary conditions need to be imposed? Show that

$$G(\mathbf{r}; \mathbf{r_0}) = \frac{-1}{4\pi} \frac{1}{|\mathbf{r} - \mathbf{r_0}|} \,.$$

[7]

When $\sigma(\mathbf{r}) = 0$, a solution of the Poisson equation in the half-space $x > 0$ bounded by the surface $x = 0$ can be written as

$$\Phi(\mathbf{r_0}) = \int_{-\infty}^{\infty} dy \int_{-\infty}^{\infty} dz \, f(y, z) \, \frac{\partial \tilde{G}(\mathbf{r}; \mathbf{r_0})}{\partial x} \,,$$

where the Green function now satisfies $\tilde{G}(\mathbf{r}; \mathbf{r_0}) = 0$ when $x = 0$ or $x_0 = 0$ and vanishes as $|\mathbf{r}| \to \infty$. Using the method of images, or otherwise, determine the Green function appropriate for this situation. [5]

Show that

$$\Phi(\mathbf{r_0}) = \frac{x_0}{2\pi} \int_{-\infty}^{\infty} dy \int_{-\infty}^{\infty} dz \frac{f(y, z)}{(x_0^2 + (y - y_0)^2 + (z - z_0)^2)^{3/2}} \,.$$

[5]

2C

The two-dimensional electrostatic potential, $\Phi(r,\theta)$, satisfies the Laplace equation in polar coordinates

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \Phi}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2 \Phi}{\partial \theta^2}=0\ ,$$

where Φ is periodic in θ.

Use the method of separation of variables to write the general solution for the potential as

$$\Phi(r,\theta)=A_0+C_0\ln r+\sum_{n=1}^{\infty}\left[\left(A_n r^n+C_n r^{-n}\right)\cos n\theta+\left(B_n r^n+D_n r^{-n}\right)\sin n\theta\right]\ ,$$

where A_n, B_n, C_n and D_n are constants. [10]

A conducting wire forms a circle of radius a. Along the semi-circle $0<\theta<\pi$ it is kept at potential V_1, while it is kept at a potential V_2 along the semi-circle $\pi<\theta<2\pi$. Give a series expansion for the potential inside the circle and show that the sum of the series is

$$\Phi(r,\theta)=\frac{V_1+V_2}{2}+\frac{V_1-V_2}{\pi}\tan^{-1}\left(\frac{2ar\sin\theta}{a^2-r^2}\right)\ .$$

[10]

[*You may use the fact that* $\tan^{-1}y=\dfrac{1}{2i}\ln\left(\dfrac{1+iy}{1-iy}\right)$ *and that* $\ln(1+x)=-\sum_{n=1}^{\infty}\dfrac{(-x)^n}{n}$.]

3A

(i) Locate all the singularities of the following functions of the complex variable z, including any at infinity, and state their nature (i.e. simple pole, branch point, etc):

a) $\dfrac{z^2-3z}{z^2+2z+2}$, b) $\dfrac{z^8+z^4+2}{(z-1)^3(3z+2)^2}$, c) $\dfrac{\ln(z+3i)}{z^2}$.

[6]

(ii) State what it means if a function is said to be *analytic* in some region. Write down the Cauchy-Riemann equations. Use the Cauchy-Riemann equations to prove that $\frac{d}{dz}e^z=e^z$ and $\frac{d}{dz}\sin z=\cos z$, where z is a complex number. [6]

(iii) Find $v(x,y)$ such that $f(z)=(u+iv)$ is analytic and express $f(z)$ in terms of z where:

a) $u(x,y)=2x(1-y)$, b) $u(x,y)=3x^2y+2x^2-y^3-2y^2$. [8]

4A

Write down a formula for the residue of a pole of order N. [1]

Evaluate

$$\int_0^\infty \frac{1}{(x^2+1)^2(x^2+4)}dx$$

by integrating a suitable function around a chosen contour in the complex plane. [9]

Show that $\dfrac{1}{\sin z}$ may be written as $\dfrac{1}{z}\left(1+\dfrac{z^2}{3!}+O(z^4)\right)$ for small z and use this to write down the first two terms in the power series expansion of $\cot z$ for small z. Use your answer to find the residue of $f(z) = \dfrac{\pi\cot\pi z}{z^2}$ for the pole at $z=0$. Find the residues of $f(z)$ for poles at $z \neq 0$. [5]

By integrating $f(z)$ around a square contour S_N with vertices at the four points $(\pm 1 \pm i)(N+\frac{1}{2})$, prove that

$$\sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{\pi^2}{6}\ .$$

[You may use the fact that $|\cot\pi z| \leq C$ for all points on S_N for all N, where C is a constant.] [5]

5C

Define the Laplace transform $\bar{f}(p)$ of a function $f(t)$. Write down the inversion formula that expresses $f(t)$ in terms of $\bar{f}(p)$, clearly specifying the path of integration. [4]

The Laplace transform of a function $x(t)$ is given by

$$\bar{x}(p) = \bar{f}(p)\,\bar{g}(p)\,.$$

Derive the formula

$$x(t) = \int_0^t g(t-\tau)\,f(\tau)\,d\tau\,.$$

[6]

Consider the equation

$$y''(t) - y(t) = g(t)\,,$$

with boundary conditions $y(0)=1$ and $y'(0)=1$, where $g(t)$ is an arbitrary function. Show that

$$y(t) = \int_0^t \sinh(t-\tau)\,g(\tau)\,d\tau\ + e^t\,.$$

[10]

6A

(i) Write down the transformation law for a tensor of rank n. What is the definition of an *isotropic tensor*? Prove that the most general isotropic tensor of rank two is $\lambda\delta_{ij}$, where λ is a scalar, by considering rotations by 90° of a general tensor of rank two T_{ij}. [6]

(ii) Prove that any tensor of rank two can be expressed as the sum of a symmetric tensor S_{ij} and an anti-symmetric tensor A_{ij}. (You should include a proof that **S** and **A** are tensors.) The vector **a** is defined by $a_i = \frac{1}{2}\epsilon_{ijk}A_{jk}$; show that $A_{ij} = \epsilon_{kij}a_k$ and write down the matrix components of **A** in terms of **a**. [7]

(iii) Let $\mathbf{v}(\mathbf{x})$ be a vector field. Given that $\partial_i = \dfrac{\partial}{\partial x^i}$ is a tensor of rank one, prove that $\nabla \times \mathbf{v}$ is a vector field.

Use suffix notation to show:

$$\nabla \times (\nabla \times \mathbf{v}) = \nabla(\nabla \cdot \mathbf{v}) - \nabla^2 \mathbf{v} \qquad (*)$$

In the absence of charges and currents, Maxwell's equations for an electric field **E** and a magnetic field **B** are given by:

$$\nabla \cdot \mathbf{E} = 0\ , \qquad \nabla \cdot c\mathbf{B} = 0\ ,$$

$$\nabla \times c\mathbf{B} - \frac{1}{c}\frac{\partial \mathbf{E}}{\partial t} = 0\ , \qquad \nabla \times \mathbf{E} + \frac{1}{c}\frac{\partial c\mathbf{B}}{\partial t} = 0\ ,$$

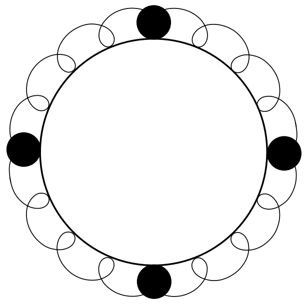
where c is the speed of light. Using the identity $(*)$, show that **E** and **B** both satisfy the wave equation

$$-\nabla^2 \mathbf{v} + \frac{1}{c^2}\frac{\partial^2 \mathbf{v}}{\partial t^2} = 0\ .$$

[7]

7C

Define the *normal modes*, *normal frequencies* and *normal coordinates* of a system of interacting particles subject to small oscillations. [6]



Consider four point particles of mass m that are constrained to move on the surface of a frictionless cylinder of radius R. Adjacent pairs of beads are connected by springs of spring constant k and equilibrium length $\frac{\pi R}{2}$. The beads are free to move only in the angular direction around the cylinder, as in the diagram, so the length of any spring is proportional to the angular separation of the particles it connects (and the force of gravity can be neglected).

Write down the Lagrangian for the system. Find the normal modes and normal frequencies. Describe the motions corresponding to each normal mode. [14]

8B

Write down the axioms for a set G to be a group. State Lagrange's theorem. [5]

If G is a group, and $g \in G$, define the *order* of g. Show that the order of g divides $|G|$, the order of the group.

Show that if G is a group and $|G| = p$ where p is prime, then G is isomorphic to C_p, the order p cyclic group.

If G is a group such that every element except for the identity has order 2, show that G is abelian. [9]

Now suppose that G is a group with $|G| = 6$. Prove the following statements:

(i) At least one element of the group must be of order 3 or 6.

(ii) It is not possible for there to be two elements of the group, x and y, of order 3 with $x \neq y$ and $x \neq y^2$.

[4]

Hence show that either G is isomorphic to C_6, or that $G = \{I, x, x^2, h_1, h_2, h_3\}$ where x is of order 3 and h_1, h_2, h_3 are of order 2. [2]

9B

Let G be a group. Suppose that $H \subset G$. What does it mean for H to be a *normal subgroup* of G?

Suppose that G_1, G_2 are groups, and Φ is a mapping $\Phi : G_1 \to G_2$. Give the definition for the map Φ to be a *homomorphism*. [2]

If $\Phi : G_1 \to G_2$ is a homomorphism, show that the kernel of Φ is a normal subgroup of G_1. [8]

Consider the following function of n real variables $x_1, \ldots, x_n$:

$$f(x_1, \ldots, x_n) = \prod_{i<j} (x_i - x_j) \ .$$

Let Σ_n be the permutation group. Suppose that under the action of $\sigma \in \Sigma_n$,

$$x_i \to x_{\sigma(i)}$$

for $i = 1, \ldots, n$. If $\sigma \in \Sigma_n$ is a transposition, determine its action on the polynomial f. [6]

Suppose that $\pi \in \Sigma_n$ is a permutation that can be written as a product of an even number of transpositions. Deduce that all decompositions of π into a product of transpositions must contain an even number of transpositions. [4]

10B

Let D be a representation of G; i.e. a homomorphism $D : G \to GL(n, \mathbb{C})$, where $GL(n, \mathbb{C})$ is the group of $n \times n$ invertible complex matrices. What does it mean for a vector subspace $W \subset \mathbb{C}^n$ to be an *invariant subspace* with respect to D? What does it mean for D to be *irreducible*? [4]

Let $D_1 : G \to GL(n, \mathbb{C})$ be a representation, and define

$$\mathbf{D}_2(g) = \left[\mathbf{D}_1(g^{-1})\right]^\dagger$$

where $\dagger$ denotes the hermitian conjugate. Show that D_2 is a representation. [6]

Suppose that W is an invariant subspace of $\mathbb{C}^n$ with respect to D_2. Show that $W_\perp$ is an invariant subspace of $\mathbb{C}^n$ with respect to D_1, where $W_\perp$ is the vector space of vectors orthogonal to W. Hence show that if D_1 is irreducible then D_2 must also be irreducible. [10]

END OF PAPER