UNIVERSITY OF CAMBRIDGE

NATURAL SCIENCES TRIPOS Part IB & II (General)

Tuesday 30 May 2006 9 to 12

MATHEMATICS (1)

Before you begin read these instructions carefully:

You may submit answers to no more than ***six*** *questions. All questions carry the same number of marks.*

The approximate number of marks allocated to a part of a question is indicated in the right hand margin.

Write on **one** *side of the paper only and begin each answer on a separate sheet.*

At the end of the examination:

Each question has a number and a letter (for example, ***6A****).*

Answers must be tied up in ***separate*** *bundles, marked* ***A, B or C*** *according to the letter affixed to each question.*

Do not join the bundles together.

For each bundle, a blue cover sheet must be completed and attached to the bundle.

A ***separate*** *yellow master cover sheet listing all the questions attempted* ***must*** *also be completed.*

Every cover sheet must bear your examination number and desk number.

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1B

Let x, y, z be Cartesian co-ordinates, and let Φ be a scalar field.

Define the gradient $\nabla\Phi$ in Cartesian co-ordinates.

Let q_1, q_2, q_3 be orthogonal curvilinear co-ordinates. Show that the gradient of Φ in orthogonal curvilinear co-ordinates is

$$\nabla\Phi = \frac{\partial\Phi}{\partial q_1}\frac{\mathbf{e}_1}{h_1} + \frac{\partial\Phi}{\partial q_2}\frac{\mathbf{e}_2}{h_2} + \frac{\partial\Phi}{\partial q_3}\frac{\mathbf{e}_3}{h_3}$$

and define the quantities $\mathbf{e}_i$ and h_i which appear in this expression. [8]

Oblate spheroidal co-ordinates (R, θ, ϕ) are defined by

$$\begin{aligned} x &= \cosh R \cos\theta \cos\phi \\ y &= \cosh R \cos\theta \sin\phi \\ z &= \sinh R \sin\theta \ . \end{aligned}$$

Show that the co-ordinate surfaces associated with R, θ, ϕ intersect at right angles. [8]

Show that for these co-ordinates,

$$h_R = h_\theta = \sqrt{\sinh^2 R + \sin^2\theta}, \qquad h_\phi = \cosh R \cos\theta \ .$$

[4]

2A

Vibrations of a violin string are governed by the one-dimensional wave equation

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} \ ,$$

where $u(x,t)$ is the deflection of the string and c is a constant. The string is fixed at the ends $x = 0$ and $x = L$. The initial deflection is $f(x)$ and the initial velocity is $g(x)$. Use the method of separation of variables to find the deflection at later times in terms of an infinite series

$$u(x,t) = \sum_{n=1}^{\infty} \left[a_n F_n(t) + b_n G_n(t) \right] H_n(x) \ ,$$

where $F_n(t)$, $G_n(t)$, $H_n(x)$, a_n and b_n should all be specified. [12]

An initial displacement of the string is given by

$$f(x) = \begin{cases} \dfrac{2kx}{L} & \text{if } 0 < x < \frac{L}{2} \\[2ex] \dfrac{2k(L-x)}{L} & \text{if } \frac{L}{2} < x < L \end{cases}$$

Sketch this function. [2]

Find the solution corresponding to the string being released from rest with the above initial displacement. [6]

3B

The Fourier transform of a function f is given by

$$\tilde{f}(k) = \int_{-\infty}^{\infty} f(x) e^{-ikx} dx \ .$$

Give the expression for the inverse Fourier transform.

Let $\alpha > 0$ and define for non-negative integer n:

$$f_n(x) = \begin{cases} x^n e^{-\alpha x} & x > 0 \\ 0 & x \leqslant 0 \end{cases}$$

Show that

$$\tilde{f}_0(k) = \frac{1}{\alpha + ik}$$

and

$$\tilde{f}_n(k) = \frac{n}{\alpha + ik} \tilde{f}_{n-1}(k) \ , \qquad n = 1, 2, \ldots$$

Hence compute $\tilde{f}_n(k)$. [8]

Prove the following identity (Parseval's theorem) for functions f and g:

$$\int_{-\infty}^{\infty} [f(x)]^* g(x) dx = \frac{1}{2\pi} \int_{-\infty}^{\infty} [\tilde{f}(k)]^* \tilde{g}(k) dk \ ,$$

where * denotes complex conjugation. [8]

Hence show that

$$\int_{-\infty}^{\infty} \frac{1}{(\alpha^2 + k^2)^{n+1}} dk = \frac{2\pi (2n)!}{(n!)^2 (2\alpha)^{2n+1}} \ .$$

[4]

[In this question all functions may be assumed to be sufficiently integrable so that their Fourier transforms and inverse Fourier transforms are well-defined.]

4B

(i) Let $\mathbf{a} \in \mathbb{R}^n$ be a fixed n-component real vector, $\mathbf{a} \neq 0$. Let A^+ and A^- be real $n \times n$ matrices with components

$$(A^{\pm})_{ij} = \delta_{ij} \pm a_i a_j \ .$$

Obtain the eigenvalues of $A^{\pm}$ and describe the corresponding eigenvectors. [4]

Show that A^+ is always invertible and obtain necessary and sufficient conditions for A^- to be invertible. [4]

(ii) Let $\mathbf{b} \in \mathbb{R}^3$ be a fixed vector, $\mathbf{b} \neq 0$. Let B^+ and B^- be real 3×3 matrices with components

$$(B^{\pm})_{ij} = \delta_{ij} \pm b_i b_j + \epsilon_{ijk} b_k \ .$$

By choosing a suitable basis for $\mathbb{R}^3$, or otherwise, determine the real eigenvalues of $B^{\pm}$ and the corresponding eigenvectors. [6]

Show that B^+ is invertible and obtain necessary and sufficient conditions for B^- to be invertible. [6]

5B

What does it mean for a $n \times n$ square matrix to be *diagonalizable*? [2]

Suppose that A is a $n \times n$ square matrix such that $A^p = 0$ for some positive integer p. Show that A has 0 as an eigenvalue. Show also that A cannot be diagonalizable unless $A = 0$. [6]

Let B and C be the matrices

$$B = \begin{pmatrix} 4+2\alpha & -2 & -2-4\alpha \\ 3\alpha & -3 & 9-6\alpha \\ 2+\alpha & -1 & -1-2\alpha \end{pmatrix}$$

for $\alpha \in \mathbb{R}$, and

$$C = \begin{pmatrix} 0 & 2 & 6 \\ 3 & 3 & 3 \\ 3 & 1 & -3 \end{pmatrix}$$

By considering the characteristic polynomials of B and C, determine whether B and C are diagonalizable. [12]

6A

Define a *singular point* and a *regular singular point* of a homogeneous second-order linear ordinary differential equation. Identify two singular points of the following equation and determine their nature:

$$2x(1-2x)y'' + (12x^2 - 4x + 1)y' - 2(4x^2+1)y = 0 \qquad (*)$$

[3]

(i) Find two linearly independent solutions to $(*)$ in the form of power series about $x = 0$. The coefficients of the power series should be given in terms of recurrence relations, which you are not required to solve. [10]

(ii) Write down the coefficient of the x^n term in the Taylor series expansions about $x = 0$ of e^x and e^{2x}. Demonstrate that these series converge for all x. [3]

(iii) Use your answers to (i) and (ii) to verify that the general solution to $(*)$ can be written as

$$y = A\sqrt{x}e^x + Be^{2x} \ .$$

[4]

7C Show how to express the eigenvalue equation

$$\frac{d^2y}{dx^2} + u(x)\frac{dy}{dx} + v(x)y + \lambda w(x)y = 0\,,$$

(where $u(x), v(x), w(x)$ are real functions and $w(x) > 0$ for $a \leqslant x \leqslant b$) in Sturm–Liouville form, [6]

$$\frac{d}{dx}\left[p(x)\frac{dy}{dx}\right] + q(x)y + \lambda r(x)y = 0\,.$$

Suppose that y satisfies the boundary conditions

$$k_1 y(a) + k_2 y'(a) = 0\,, \qquad l_1 y(b) + l_2 y'(b) = 0\,,$$

where k_1, k_2, l_1, l_2 are constants. Show that two eigenfunctions y_m, y_n, with distinct eigenvalues $\lambda_m \neq \lambda_n$ satisfy the orthogonality condition

$$\begin{aligned}\int_a^b r(x) y_m y_n dx = \frac{1}{\lambda_m - \lambda_n}\big[&p(b)(y_n'(b)y_m(b) - y_m'(b)y_n(b)) \\ &- p(a)(y_n'(a)y_m(a) - y_m'(a)y_n(a))\big] = 0\,.\end{aligned}$$

[4]

Find the eigenfunctions and the values of the eigenvalue λ that satisfy the equation

$$y'' + 2\alpha y' + (\alpha^2 + \lambda)y = 0\,,$$

where $y(0) = y(\pi) = 0$. [6]

Put the equation in Sturm–Liouville form and hence determine the orthogonality condition for the eigenfunctions. [4]

8A

By first finding the appropriate Green's function, solve the differential equation

$$y''(x) + k^2 y(x) = f(x) \ ,$$

where k is a non-zero real number for the cases:

$$y(0) = y'(0) = 0 \ , \qquad f(x) = 2k \cos kx \ ,$$

[10]

and

$$y(0) = y\left(\frac{\pi}{6k}\right) = 0 \ , \qquad f(x) = \frac{k^2}{2} \ .$$

[10]

[*You may use the identity* $\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$.]

9C

The function $y(x)$ makes the value of

$$F = \int_a^b \left(p(x)\,[y'(x)]^2 - q(x)\,[y(x)]^2\right)\,dx$$

stationary subject to the condition $G = 1$, where

$$G = \int_a^b r(x)\,[y(x)]^2\,dx\ ,$$

and with the boundary conditions $y(a) = y(b) = 0$ (where $p(x), q(x), r(x)$ are real functions and $r(x) > 0$ for $a \leqslant x \leqslant b$). Use Euler's equation to show that $y(x)$ satisfies the Sturm–Liouville equation

$$\frac{d}{dx}\left[p(x)\frac{dy}{dx}\right] + q(x)y + \lambda r(x)y = 0\,.$$

Show that the stationary values of $\Lambda = F/G$ are eigenvalues corresponding to the values of λ. [8]

Consider the equation

$$\frac{d}{dx}\left(x\frac{dy}{dx}\right) - xy + \lambda y = 0\,,$$

where $y(0)$ is finite and $y \to 0$ in the limit $x \to \infty$, and λ denotes an eigenvalue. Use the Rayleigh–Ritz method to obtain an estimate of the smallest eigenvalue by using a trial function $y^{(1)}_{trial} = e^{-\beta x}$. [5]

Now use instead the trial function $y^{(2)}_{trial} = A - x$ for $0 \le x \le A$, $y = 0$ for $A \le x \le \infty$ to give a new estimate of the smallest eigenvalue. [5]

Which of these two estimates is the better and why? [2]

[*You may wish to use the fact that* $\int_0^\infty x\,e^{-kx}\,dx = 1/k^2$.]

10C State the Euler equation obtained by minimizing $\int_{x_1}^{x_2} f(x, y(x), y'(x))\, dx$, with $y(x_1)$ and $y(x_2)$ fixed at the boundaries. [3]

If f is not an explicit function of x, show that

$$y' \frac{\partial f}{\partial y'} - f$$

is constant. [3]

A bead slides down a frictionless wire, starting at rest at $x = 0$, $y = 0$ and reaching a point B at $x = x_B$, $y = y_B$ after a time t. By considering the bead's total energy, show that its velocity at any point during its motion is given by $v = \sqrt{2gy}$. [4]

Hence, show that the time $T[y]$ taken to reach B depends on the shape of the wire $y(x)$ according to

$$T[y] = \frac{1}{\sqrt{2g}} \int_0^{x_B} \sqrt{\frac{1 + y'^2}{y}}\, dx\,.$$

Consider the variational problem that determines the shape of the wire that minimizes the time (the *brachistochrone*) and show that the quantity $y(1 + y'^2)$ is constant for such a shape. Hence, determine the parametric equations of the brachistochrone,

$$x = c(\theta - \tfrac{1}{2}\sin 2\theta)\,, \qquad y = c\,\sin^2\theta\,,$$

where c is a constant and θ parameterizes the curve. [5]

Show that if $x_B = l$ and $y_B = 0$ the minimum time is equal to $\sqrt{2\pi l/g}$. [5]

END OF PAPER