UNIVERSITY OF CAMBRIDGE

NATURAL SCIENCES TRIPOS Part IA

Wednesday 14th June 2006 9 to 12

MATHEMATICS (2)

Before you begin read these instructions carefully:

You may submit answers to no more than ***six*** *questions. All questions carry the same number of marks.*

The approximate number of marks allocated to a part of a question is indicated in the right-hand margin.

Write on **one** *side of the paper only and begin each answer on a separate sheet.*

Questions marked with an asterisk (*) require a knowledge of B course material.

At the end of the examination:

Each question has a number and a letter (for example, ***3B****).*

Tie up the answers in **separate** *bundles, marked* ***A, B, C, D, E*** *or* ***F*** *according to the letter affixed to each question.* ***Do not join the bundles together****.*

For each bundle, complete and attach a blue cover sheet, with the appropriate letter written in the section box.

Complete a **separate** *yellow master cover sheet listing all the questions attempted.*

Every cover sheet must bear your examination number and desk number.

STATIONERY REQUIREMENTS
6 blue cover sheets and treasury tags
Yellow master cover sheet

Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1A

(a) If

$$y = \sin^{-1}\left(\frac{x}{\sqrt{1+x^2}}\right),$$

find $\dfrac{\mathrm{d}y}{\mathrm{d}x}$ as a function of x. [4]

(b) Find the first non-zero term in the Taylor series about $x = 0$ of

$$\frac{x\sin(\sin x) - \sin^2 x}{x^4}.$$ [8]

(c) Evaluate (without using a calculator)

$$\int_0^{\pi/6} \frac{\mathrm{d}x}{(\sin x)^{\frac{1}{2}}(\cos x)^{\frac{7}{2}}}.$$ [8]

[*Hint*: substitute $\tan x = u$.]

2A*

(a) A light ray moving in the direction $\mathbf{s}$ is reflected off a mirror A whose unit normal is $\mathbf{a}$. The reflected ray has direction

$$\mathbf{s}' = \mathbf{A}\,\mathbf{s}\,,$$

where the matrix $\mathbf{A}$ has components

$$A_{ij} = \delta_{ij} - 2a_i a_j\,,$$

and a_i are the components of the vector $\mathbf{a}$.

Show that $\mathbf{A}^2 = \mathbf{I}$ and $\mathbf{A} = \mathbf{A}^{\mathrm{T}}$, where $\mathbf{A}^{\mathrm{T}}$ denotes the transposed matrix and $\mathbf{I}$ is the unit matrix. [6]

(b) The light ray is subsequently reflected off a second mirror B whose unit normal is $\mathbf{b}$, so that the reflected ray now has direction

$$\mathbf{s}'' = \mathbf{B}\,\mathbf{s}'\,,$$

where

$$B_{ij} = \delta_{ij} - 2b_i b_j\,.$$

Calculate $\mathbf{BA} - \mathbf{AB}$, and show that the direction of the ray would be the same if the light ray had first been reflected off B and then A if and only if $\mathbf{a}$ is either parallel or perpendicular to $\mathbf{b}$. [6]

(c) Show, by considering the identity $\mathbf{x} = \mathbf{a}(\mathbf{x}\cdot\mathbf{a}) + \mathbf{b}(\mathbf{x}\cdot\mathbf{b}) + \mathbf{c}(\mathbf{x}\cdot\mathbf{c})$, that if $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ are three orthonormal vectors then

$$a_i a_j + b_i b_j + c_i c_j = \delta_{ij}\,.$$

Hence, or otherwise, show that if a laser beam is sent from the earth to the moon and is reflected off a 'corner reflector' with three orthogonal reflectors, then it will return in exactly the direction opposite to that with which it started out. [8]

3B

(a) Which of the following vector fields, given in Cartesian coordinates, is conservative?

(i)
$$\mathbf{F}_1 = y\,\mathbf{i} + [z\cos(yz) + x]\,\mathbf{j} + y\cos(yz)\,\mathbf{k}\,,$$

(ii)
$$\mathbf{F}_2 = \exp(xy)\,\mathbf{i} + \exp(x+y)\,\mathbf{j}\,,$$

(iii)
$$\mathbf{F}_3 = (2xyz + \sin x)\,\mathbf{i} + x^2 z\,\mathbf{j} + x^2 y\,\mathbf{k}\,.$$

In each case, if $\mathbf{F}$ is a conservative vector field, find a scalar potential ϕ such that $\mathbf{F} = \boldsymbol{\nabla}\phi$. [10]

(b) Calculate directly the line integral

$$\int_C \mathbf{F}_3 \cdot \mathrm{d}\mathbf{r}\,,$$

where the integration path C is

(i) a straight line from $(0,0,0)$ to (π,π,π), [5]

(ii) the curve defined by a series of straight lines from $(0,0,0)$ to $(0,0,\pi)$ then to $(0,\pi,\pi)$ and finally to (π,π,π). [5]

4B

The element of vector area for a surface S, given in Cartesian coordinates by the equation $f(x, y, z) = 0$, can be expressed as

$$\mathrm{d}\mathbf{S} = \frac{\mathbf{n}}{\cos\alpha}\, \mathrm{d}x\, \mathrm{d}y\,,$$

where $\alpha < \pi/2$ is the angle between the unit vector $\mathbf{k}$ in the z-direction and the unit normal $\mathbf{n}$ of S.

(a) Show how to construct from $f(x, y, z)$ a vector field $\mathbf{F}(x, y, z)$ such that

$$\mathrm{d}\mathbf{S} = \mathbf{F}\, \mathrm{d}x\, \mathrm{d}y$$

and $\mathbf{F} \cdot \mathbf{k} = 1$. [6]

(b) Evaluate the element of vector area $\mathrm{d}\mathbf{S}$ for the surface S given by

$$x^2(1 + y) + y^2 z = 1$$

and bounded by $0 < x < 1$, $0 < y < 2$. [6]

(c) The flux of a vector field $\mathbf{G}$ through the surface S, as specified in part (b), is defined by

$$I = \int_S \mathbf{G} \cdot \mathrm{d}\mathbf{S}\,.$$

Calculate the magnitude $|I_1 - I_2|$ of the difference between the fluxes of $\mathbf{G}_1$ and $\mathbf{G}_2$ through S, where $\mathbf{G}_1$ and $\mathbf{G}_2$ are:

$$\mathbf{G}_1 = y^2\, \mathbf{i} + x\, \mathbf{k}\,,$$

$$\mathbf{G}_2 = y^2\, \mathbf{i} + y^3\, \mathbf{j} + x\, \mathbf{k}\,.$$ [8]

5C

An 'apple' is represented by a solid of revolution defined in spherical polar coordinates by

$$r = a(1 - \cos\theta)\,.$$

(a) Assuming the usual correspondence between spherical polar coordinates and Cartesian coordinates (x, y, z), sketch the cross-section of the apple in the $x = 0$ plane, indicating the points where $\theta = 0$, $\pi/2$ and π. [6]

(b) Assuming $\theta_0 > \pi/2$, what is the volume of an infinitesimal slice of the apple bounded by planes intersecting the surface at constant θ values θ_0 and $\theta_0 + \mathrm{d}\theta$? [4]

(c) By using the substitution $\cos\theta = u$, evaluate the total volume of the part of the apple having $\theta > \pi/2$. [10]

6C*

(a) Find the derivative of

$$\int_{x^2}^{x^3} \frac{\cos(xt)}{t}\,\mathrm{d}t$$

with respect to the parameter x. [7]

(b) By means of a sketch, show that if n is an integer greater than 1, then

$$\int_n^{n+1} \frac{1}{x}\,\mathrm{d}x < \frac{1}{n} < \int_n^{n+1} \frac{1}{x-1}\,\mathrm{d}x.$$

[3]

Using this result, show that the quantity

$$\lim_{N\to\infty} \left\{ \left(\sum_{n=1}^{N} \frac{1}{n} \right) - \ln N \right\}$$

is finite, and obtain upper and lower bounds on it. (You should show explicitly that these bounds are positive.) [10]

7D

Solve the following differential equations, by determining which are exact and using an integrating factor for those that are not.

(a)
$$(2xy^2 + 4)\,\mathrm{d}x + 2(x^2y - 3)\,\mathrm{d}y = 0\,,$$ [6]

(b)
$$(y^2 - x)\,\mathrm{d}x + 2y\,\mathrm{d}y = 0\,,$$ [8]

(c)
$$(\cos x - x\sin x + y^2)\,\mathrm{d}x + 2xy\,\mathrm{d}y = 0\,.$$ [6]

8D

(a) A species of bird always lays a nest of four eggs. Each egg may be white (with probability p) or brown (with probability $1-p$).

(i) Using the notation $\mathrm{W}_n\mathrm{B}_{4-n}$, list all possible nest contents together with their probabilities of occurrence. [6]

(ii) Taking $p = 3/4$, find the most common nest content. [2]

(b) The discrete variable X assumes values $x_i = i$ $(i = 1, \ldots, 6)$ with probabilities $p_i = 1/6$. Calculate:

(i) the expectation value of X , [2]

(ii) the expectation value of X^2 , [2]

(iii) the variance of X . [2]

(c) The continuous variable X in the interval $[1, 6]$ has the probability distribution function

$$f(x) = \begin{cases} \alpha\,, & 1 \leqslant x \leqslant 3\,, \\ 0\,, & 3 < x < 4\,, \\ \alpha\,, & 4 \leqslant x \leqslant 6\,. \end{cases}$$

Calculate:

(i) the value of α , [2]

(ii) the variance of X . [4]

9E

(a) Use the method of separation of variables to show that the general solution of the differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} + (y-a)(y-b) = 0\,, \qquad (*)$$

where a and b are constants and $b \neq a$, is

$$y = \frac{a\,\mathrm{e}^{a(x+c)} - b\,\mathrm{e}^{b(x+c)}}{\mathrm{e}^{a(x+c)} - \mathrm{e}^{b(x+c)}}\,,$$

where c is an arbitrary constant. [6]

(b) The function $z(x)$ is related to $y(x)$, as found in part (a), by

$$\frac{\mathrm{d}z}{\mathrm{d}x} = yz\,. \qquad (**)$$

Find the general solution $z(x)$. [4]

(c) Find $y(x)$ and $z(x)$ in the special case when $b = a$. [6]

(d) By eliminating y between equations $(*)$ and $(**)$, find the second-order linear differential equation with constant coefficients satisfied by $z(x)$. [4]

10E*

(a) Define what is meant by the statement that the series

$$\sum_{n=1}^{\infty} u_n \qquad (*)$$

is convergent. [4]

(b) If $u_n = w_{n+1} - w_n$, state a necessary and sufficient condition on w_n for the series $(*)$ to converge. [2]

(c) State the comparison test for the convergence of a series of positive terms. [3]

(d) By considering the derivatives of each side of the inequality, and the values of each side at $x = 0$, or otherwise, show that

$$1 - (1+x)^{-p} > px(1+x)^{-(p+1)} \,,$$

provided that $p > 0$ and $x > 0$. Hence, by letting $x = 1/n$, deduce that

$$n^{-p} - (n+1)^{-p} > p(n+1)^{-(p+1)} \,,$$

provided that $p > 0$ and $n > 0$. [5]

(e) Combining the results from parts (b), (c) and (d), show that if $k > 1$ then the series $\sum_{n=1}^{\infty} n^{-k}$ is convergent. [6]

11F

Let $f(x,y)$ be a function of two variables. x and y can be rewritten in terms of two new variables $u = u(x,y)$ and $v = v(x,y)$.

(a) Use the chain rule to find $\left(\frac{\partial f}{\partial x}\right)_y$ in terms of $\left(\frac{\partial f}{\partial u}\right)_v$, $\left(\frac{\partial f}{\partial v}\right)_u$, $\left(\frac{\partial u}{\partial x}\right)_y$ and $\left(\frac{\partial v}{\partial x}\right)_y$. [3]

(b) Find expressions for $\frac{\partial^2 f}{\partial x^2}$ and $\frac{\partial^2 f}{\partial y^2}$ in terms of derivatives of f with respect to u and v. [6]

(c) Suppose that

$$x = u\cos v\,,$$
$$y = u\sin v\,.$$

Evaluate $\frac{\partial^2 f}{\partial x^2}$ and $\frac{\partial^2 f}{\partial y^2}$ in terms of derivatives of f with respect to u and v. [6]

(d) A solution to

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0$$

is $x^2 - y^2$. Use your results to show that $u^2\cos(2v)$ is a solution to

$$\frac{\partial^2 f}{\partial u^2} + \frac{1}{u}\frac{\partial f}{\partial u} + \frac{1}{u^2}\frac{\partial^2 f}{\partial v^2} = 0\,.$$

[5]

12F

A function of two variables $f(x,y)$ is given by

$$f(x,y) = \frac{x+y}{x^2+y^2+1}$$

and represents the height of the point (x,y) above the (x,y)-plane.

(a) Find the extrema of this function. [6]

(b) Determine, by examining the second derivatives of f, whether each extremum is a maximum, a minimum or a saddle point. [8]

(c) Hence sketch a contour plot of f. [6]

END OF PAPER