UNIVERSITY OF
CAMBRIDGE

NATURAL SCIENCES TRIPOS Part IA

Monday 12th June 2006 9 to 12

MATHEMATICS (1)

Before you begin read these instructions carefully:

You may submit answers to no more than ***six*** *questions. All questions carry the same number of marks.*

The approximate number of marks allocated to a part of a question is indicated in the right-hand margin.

Write on ***one*** *side of the paper only and begin each answer on a separate sheet.*

Questions marked with an asterisk (*) require a knowledge of B course material.

At the end of the examination:

Each question has a number and a letter (for example, ***3B****).*

Tie up the answers in ***separate*** *bundles, marked* ***A, B, C, D, E*** *or* ***F*** *according to the letter affixed to each question.* **Do not join the bundles together**.

For each bundle, complete and attach a blue cover sheet, with the appropriate letter written in the section box.

Complete a ***separate*** *yellow master cover sheet listing all the questions attempted.*

Every cover sheet must bear your examination number and desk number.

STATIONERY REQUIREMENTS
6 blue cover sheets and treasury tags
Yellow master cover sheet

Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1A

(a) A matrix $\mathbf{A}$ satisfies

$$\mathbf{A} = 5\mathbf{A}^{\mathrm{T}},$$

where $^{\mathrm{T}}$ denotes the transpose. Find $\mathbf{A}$. [4]

(b) If $\mathbf{B} = -\mathbf{B}^{\mathrm{T}}$ and $\det(\mathbf{I}+\mathbf{B}) \neq 0$, where $\mathbf{I}$ is the unit matrix, show that

$$\mathbf{C} = (\mathbf{I}-\mathbf{B})(\mathbf{I}+\mathbf{B})^{-1}$$

is an orthogonal matrix.

Is there any non-zero choice of $\mathbf{B}$ such that

(i) $\mathbf{C}$ is symmetric?

(ii) $\mathbf{C}$ is antisymmetric?

In each case, if so, give an example; if not, explain why. [7]

(c) If $\mathbf{D}$ is a square matrix and $\det(\mathbf{I}-\mathbf{D}) \neq 0$, show that

$$(\mathbf{I}-\mathbf{D})^{-1} = \sum_{n=0}^{\infty} \mathbf{D}^n,$$

where $\mathbf{D}^0 = \mathbf{I}$. (Assume that the right-hand side of the equation exists.) [4]

Use your result to find $(\mathbf{I}-\mathbf{D})^{-1}$ if

$$\mathbf{D} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}.$$

[5]

2A

(a)

(i) Find the eigenvalues and eigenvectors of the matrix

$$\mathbf{M} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} .$$

Verify that the eigenvectors are orthogonal. [5]

(ii) Construct a matrix $\mathbf{R}$ whose rows are the normalised eigenvectors of the matrix $\mathbf{M}$. Is $\mathbf{R}$ an orthogonal matrix? Evaluate

$$\mathbf{RMR}^{\mathrm{T}} .$$

Is $\mathbf{M}$ an invertible matrix? Give a reason for your answer. [8]

(b) For what real values of λ does the equation

$$\begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 5 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \lambda \begin{pmatrix} x \\ y \\ z \end{pmatrix} ,$$

where $\theta \neq n\pi$ for any integer n, have non-vanishing real solutions $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$? How many are there, and what are they? [7]

3B

(a) Give expressions for the following complex numbers in the form $a+\mathrm{i}b$, where a and b are real:

(i)
$$\frac{2+\mathrm{i}}{(1-\mathrm{i})^2}\,,$$
[3]

(ii)
$$\left(\frac{\mathrm{i}+1}{\mathrm{i}-1}\right)^3\,.$$
[3]

(b) Find all solutions of the following equations and give the results in the form $z=a+\mathrm{i}b$, where a and b are real:

(i)
$$z^3=-8\mathrm{i}\,,$$
[3]

(ii)
$$z^2=2(1+\sqrt{3}\,\mathrm{i})\,,$$
[3]

(iii)
$$z=\mathrm{i}^{(1-\mathrm{i})}\,.$$
[3]

(c) Show that the solutions of the equation

$$2\sin z+\cos z=\mathrm{i}\sin z$$

are given by

$$z=\left(n\pi-\frac{\pi}{8}\right)-\frac{1}{4}\mathrm{i}\ln 2\,,$$

where n is an arbitrary integer.

[5]

4B*

(a) State Stokes' theorem, explaining carefully the meaning of any symbols you use and indicating the orientation of any integrations. [4]

(b) A vector field $\mathbf{F}$ is given in Cartesian coordinates by

$$\mathbf{F} = (2xz^3, yz^2, x^2z^2)\,.$$

Evaluate $\boldsymbol{\nabla} \times \mathbf{F}$. [4]

(c) Can $\mathbf{F}$ be obtained from a scalar potential ϕ such that $\mathbf{F} = \boldsymbol{\nabla}\phi$? [2]

(d) Verify Stokes' theorem for $\mathbf{F}$ on the triangle ABC defined by $A = (1,1,0)$, $B = (0,1,0)$ and $C = (0,1,1)$. [10]

5C

A, B and C are three points with position vectors $\mathbf{a} = a\,\mathbf{i}$, $\mathbf{b} = b\,\mathbf{j}$ and $\mathbf{c} = c\,\mathbf{k}$ respectively, where a, b and c are positive scalars. P is the plane passing through A, B and C.

(a) Find a normal to the plane P. [3]

(b) What is the vector equation of P? [3]

(c) Define D as the point on P closest to the origin. What is the distance of D from the origin? [4]

(d) A position vector $\mathbf{r}$ is given by

$$\mathbf{r} = \lambda\,\mathbf{a} + \mu\,\mathbf{b} + \nu\,\mathbf{c}$$

where λ, μ and ν are scalars. What condition must λ, μ and ν satisfy if $\mathbf{r}$ is to lie in P? [3]

(e) Find the values of λ, μ and ν corresponding to the point D. [4]

(f) If $\mathbf{d}$ is the position vector of D, find the angle between the vectors $\mathbf{a} - \mathbf{d}$ and $\mathbf{b} - \mathbf{c}$. [3]

6C

(a) Let $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ be three vectors in 3-dimensional space.

(i) Show that

$$(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c} = \mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) .$$ [5]

(ii) Evaluate

$$(\mathbf{a} \times \mathbf{b}) \times \mathbf{c} - \mathbf{a} \times (\mathbf{b} \times \mathbf{c})$$

in terms of $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ and their scalar products. [9]

(b) In a spherical coordinate system with coordinates (r, θ, ϕ), the position vectors $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ correspond to points with coordinates $(r_0, \theta_0, 0)$, $(r_0, \theta_0, 2\pi/3)$ and $(r_0, \theta_0, 4\pi/3)$ respectively.

By converting these coordinates into Cartesian form, or otherwise, find the volume of a parallelepiped having these vectors as edges. [6]

7D

(a) A test is developed to find whether a patient carries a gene found in 0.1% of the population. The probability that a carrier tests negative is 1% while the probability that a non-carrier tests positive is 5%.

(i) What is the probability that a randomly selected patient tests positive? (Note that such a patient is either a carrier or a non-carrier.) [4]

Write equations describing the probability of the cases:

(ii) that a patient with a positive test carries the gene, [2]

(iii) that a patient with a negative test carries the gene, [2]

and evaluate each numerically (to 3 significant figures).

(b) Players A and B roll a six-sided die in turn. If a player rolls:

1 or 2: that player wins and the game ends;

3: the other player wins and the game ends;

4, 5 or 6: the turn passes to the other player.

A starts the game by throwing the die. What is the probability:

(i) that B gets a first throw and wins on it? [2]

(ii) that A wins before A's second throw? [3]

(iii) that A wins, if the game is played until there is a winner? [7]

8D*

(a) Use the method of Lagrange multipliers to find the locations and values of the maxima and minima of the function

$$x^2y + 2y$$

subject to the constraint

$$x^2 + y^2 = 1\,.$$ [8]

(b) The planes

$$2x + y - z = 1$$

and

$$x - y + z = 2$$

intersect in the line L. Without finding L explicitly, use the method of Lagrange multipliers to find the point on L closest to the origin. [12]

9E

(a) Give Taylor's formula for the series expansion of a function $f(x)$ about the point $x = a$. [3]

(b) Find, by any method, the first four non-vanishing terms in the Taylor series of:

(i)
$$\ln x \qquad \text{about } x = 1\,,$$ [5]

(ii)
$$\arctan x \qquad \text{about } x = 1\,,$$ [5]

(iii)
$$\frac{x}{e^x - 1} \qquad \text{about } x = 0\,.$$ [7]

10E

Solve the following differential equations for $y(x)$, subject to $y(0) = y'(0) = 0$:

(a)
$$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} + 4y = \cos x\,,$$
[6]

(b)
$$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} + 4y = \cos^2 x\,,$$
[6]

(c)
$$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} - 2\frac{\mathrm{d}y}{\mathrm{d}x} + y = (1+x)\,\mathrm{e}^x\,.$$
[8]

11F

The periodic function $f(x)$ of period 2π is defined by

$$\begin{aligned} f(x) &= \sin 2x \qquad \text{for} \quad 0 \leqslant x \leqslant \pi\,, \\ f(x) &= -\sin 2x \qquad \text{for} \quad -\pi \leqslant x \leqslant 0\,. \end{aligned}$$

(a) Sketch $f(x)$. [4]

(b) Find the Fourier expansion of $f(x)$. [10]

(c) Hence show that
$$\sum_{m=1}^{\infty} \frac{1}{(2m+1)^2 - 4} = \frac{1}{3}\,.$$
[6]

12F*

(a) Describe the method of separation of variables for Laplace's equation in two dimensions,
$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0\,.$$
[5]

(b) A function $f(x, y)$ satisfies Laplace's equation inside a square region $0 \leqslant x \leqslant a$, $0 \leqslant y \leqslant a$. On the sides $x = 0$, $x = a$ and $y = 0$, f vanishes. On the side $y = a$, f takes the constant value F.

Find $f(x, y)$ everywhere inside the square region. [15]

END OF PAPER