UNIVERSITY OF CAMBRIDGE

NATURAL SCIENCES TRIPOS Part IB & II (General)

Friday 3 June 2005 9 to 12

MATHEMATICS (2)

Before you begin read these instructions carefully:

You may submit answers to no more than ***six*** *questions. All questions carry the same number of marks.*

The approximate number of marks allocated to a part of a question is indicated in the right hand margin.

Write on **one** *side of the paper only and begin each answer on a separate sheet.*

At the end of the examination:

Each question has a number and a letter (for example, ***6A****).*

Answers must be tied up in **separate** *bundles, marked* ***A, B or C*** *according to the letter affixed to each question.*

Do not join the bundles together.

For each bundle, a blue cover sheet must be completed and attached to the bundle.

A **separate** *yellow master cover sheet listing all the questions attempted* **must** *also be completed.*

Every cover sheet must bear your examination number and desk number.

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1C State the divergence theorem for a vector field $\mathbf{F}(\mathbf{r})$ in a simply connected region R with surface S. [2]

Let $u(\mathbf{r})$ and $v(\mathbf{r})$ be scalar fields that tend to zero as $|\mathbf{r}| \to \infty$. Use the divergence theorem to show that

$$\int \left[u(\nabla^2 - m^2)v - v(\nabla^2 - m^2)u\right] dV = 0 \quad , \qquad \textbf{(A)}$$

where the integration is over all space. [4]

A scalar field $\phi(\mathbf{r})$, that tends to zero as $|\mathbf{r}| \to \infty$, satisfies the equation

$$(\nabla^2 - m^2)\phi(\mathbf{r}) = \rho(\mathbf{r}) \quad , \qquad \textbf{(B)}$$

where $\rho(\mathbf{r})$ tends to zero rapidly as $|\mathbf{r}| \to \infty$. Use equation **(A)** to show that

$$\phi(\mathbf{r}) = \int G(\mathbf{r}, \mathbf{r}')\rho(\mathbf{r}')dV' \quad .$$

[2]

where the Green's function $G(\mathbf{r}, \mathbf{r}')$ satisfies

$$(\nabla^2 - m^2)G(\mathbf{r}, \mathbf{r}') = \delta(\mathbf{r} - \mathbf{r}')$$

and $G(\mathbf{r}, \mathbf{r}') = G(\mathbf{r}', \mathbf{r})$.

Assuming that $G(\mathbf{r}, \mathbf{r}')$ depends only on $|\mathbf{r} - \mathbf{r}'|$ compute the Green's function. [4]

Solve equation **(B)** for the case

$$\rho(\mathbf{r}) = \frac{e^{-\beta r}}{r} \quad ,$$

where $r = |\mathbf{r}|$. Hence show that

$$\frac{1}{4\pi} \int \frac{e^{-m|\mathbf{r}-\mathbf{r}'|}}{|\mathbf{r} - \mathbf{r}'|} \frac{e^{-\beta r'}}{r'} dV' = -\frac{1}{r} \frac{e^{-\beta r} - e^{-mr}}{\beta^2 - m^2} \quad .$$

[8]

You may use the result that

$$\nabla^2\psi(r) = \frac{1}{r}\frac{d^2}{dr^2}(r\psi(r)) \quad .$$

2C In plane polar coordinates, (r, θ), Laplace's equation for a potential $\phi(r, \theta)$ is

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \phi}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 \phi}{\partial \theta^2} = 0 \quad .$$

Using the method of separation of variables show that the general solution of Laplace's equation in a domain $a \le r \le b$, $0 \le \theta < 2\pi$, has the form

$$\begin{aligned}\phi(r,\theta) &= A + B\log r \\ &\quad + \sum_{n=1}^{\infty}[r^n(A_n\cos n\theta + B_n\sin n\theta) + r^{-n}(C_n\cos n\theta + D_n\sin n\theta)] \quad .\end{aligned}$$

where A_n, B_n, C_n and D_n are constants. [10]

Obtain the solution for $\phi(r, \theta)$ given the boundary conditions $\phi(a, \theta) = U$ and $\phi(b, \theta) = V\cos\theta$. [10]

3A Let $f = u + iv$ be a complex differentiable function of a complex variable $z = x + iy$, where x, y, u and v are real. Derive the Cauchy-Riemann equations for u and v. [5]

Find the most general analytic function $f(z)$ whose real part is

$$u = e^{-x}(x\sin y - y\cos y) \ . \qquad [5]$$

Use the Cauchy-Riemann equations to show that $f(z) = \cosh z$ is analytic. [5]

Find the terms a_{-1}, a_0 in the Laurent expansion of $\dfrac{\cosh z}{z^2 - 1}$ about $z = 1$. [5]

4A What is meant by a pole of a complex function $f(z)$ and what is meant by its residue? [2]

Write down the Cauchy theorem for a function $f(z)$ which is analytic in a simply connected domain R, and state the residue theorem for a function $g(z)$ which is analytic in R except at a finite number of poles. [2]

By integrating the function $f(z) = \dfrac{e^{iaz} - e^{ibz}}{z^2}$, where $a, b > 0$, along a closed contour containing no poles, evaluate

$$\int_{-\infty}^{\infty} \frac{\cos ax - \cos bx}{x^2} dx.$$

[7]

Find the poles of the function $g(z) = \dfrac{e^{az}}{\cosh \pi z}$, and calculate their residues. [2]

By integrating $g(z)$ along a closed rectangular contour containing one pole, evaluate

$$\int_{-\infty}^{\infty} \frac{e^{ax}}{\cosh \pi x} dx$$

where $-\pi < a < \pi$. [7]

5C Given a function $u(t)$ on the range $t \geq 0$, define its Laplace transform $\bar{u}(p)$. Calculate the Laplace transform of $\dfrac{du}{dt}$ in terms of $\bar{u}(p)$ and $u(0)$. Obtain the corresponding results for the second and third derivatives of $u(t)$. [2]

Let $v(t) = tu(t)$. Show that

$$\bar{v}(p) = -\frac{d}{dp}\bar{u}(p) \quad .$$

[4]

Hence compute the Laplace transforms of the functions

$$v_n(t) = t^n e^{-t} \quad ,$$

for $n = 0, 1, 2, 3$. [6]

Consider the differential equation

$$\frac{d^3y}{dt^3} + 3\frac{d^2y}{dt^2} + 3\frac{dy}{dt} + y = e^{-t} \quad ,$$

in the range $t \geq 0$. Use the method of the Laplace transform to find the solution when $y(t)$ satisfies the initial conditions, $y(0) = 0$, $\dot{y}(0) = 0$ and $\ddot{y}(0) = 0$. [8]

6A The differential operator ∂_i is defined by $\partial_i = \dfrac{\partial}{\partial x_i}$. Show that ∂_i is a tensor of rank one. [2]

Two vectors $\mathbf{u}$ and $\mathbf{w}$ are related by $\mathbf{w} = \nabla \times \mathbf{u}$. By considering individual components and using the summation convention, derive the following identities:

(i) $\mathbf{w} \times \mathbf{u} = \mathbf{u} \cdot \nabla \mathbf{u} - \nabla \left(\frac{|\mathbf{u}|^2}{2} \right)$ [4]

(ii) $\nabla \times (\mathbf{w} \times \mathbf{u}) = (\mathbf{u} \cdot \nabla)\mathbf{w} + (\nabla \cdot \mathbf{u})\mathbf{w} - (\nabla \cdot \mathbf{w})\mathbf{u} - (\mathbf{w} \cdot \nabla)\mathbf{u}$ [7]

If $\mathbf{u}$ is the velocity of an inviscid, incompressible flow, then the equations of motion can be written as:

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla)\mathbf{u} = -\nabla H$$

$$\boldsymbol{\nabla} \cdot \mathbf{u} = 0$$

for some function H, which depends on the density, pressure, and gravitational potential. Using the identities derived above, show that

$$\frac{\partial \mathbf{w}}{\partial t} + (\mathbf{u} \cdot \nabla)\mathbf{w} = (\mathbf{w} \cdot \boldsymbol{\nabla})\mathbf{u}$$

where $\mathbf{w}$ is defined as above (in this case $\mathbf{w}$ is known as the 'vorticity'). [7]

7C A light string of length $4a$ is fixed at its ends and is under tension T. A particle of mass $4m$ is attached at the centre of the string. A particle of mass $3m$ is attached a distance a from the left end of the string another particle also of mass $3m$ is attached a distance a from the right end of the string.

The particles can undergo small oscillations transverse to the string in a plane containing the string. Let the displacement of the central particle be z and the displacements of the other two particles be x and y respectively. Given that the potential energy of the system is

$$V = \frac{1}{2} m\Omega^2 \left[x^2 + (x-z)^2 + (y-z)^2 + y^2 \right] \quad ,$$

where

$$\Omega^2 = \frac{T}{ma} \quad .$$

Obtain the equations of motion for the particles. [4]

Find the normal mode frequencies. [6]

Calculate the ratios for the displacements x, y and z in each of the normal modes. [6]

Give sketches of the three normal modes. [4]

8B Given a finite group G of order $|G|$ and a subgroup H of order $|H|$ define a left coset of H in G. State Lagrange's theorem relating the order of a group and those of its subgroups. [2]

Construct the multiplication table of the following set of matrices, and verify that they form a group under matrix multiplication:

$$\begin{aligned} I &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \\ A &= \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \\ B &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \\ C &= \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} \end{aligned}$$

[9]

If H is a group in which every element other than the identity, e, has order 2, show that H is Abelian. [9]

9B Define a homomorphism and an isomorphism between two groups G_1 and G_2. [2]

Verify that the set $A(G)$ of isomorphisms from a group G to itself forms a group, under the group operation defined by the composition of maps. [9]

Define what is meant by a finite cyclic group H. By showing that there is an isomorphism onto the set of complex numbers $\{e^{2\pi ik/n}\}$, for $k = 1, ..., n$, explain why a finite cyclic group G of order n has no proper subgroup if n is prime, and find two proper subgroups otherwise. [9]

10B Define a representation of a group G and explain what is meant by a reducible representation and an irreducible representation. [3]

If $\mathbf{D}$ is an n-dimensional representation and S an invertible $n \times n$ matrix, show that the map $\tilde{\mathbf{D}}$ defined by $\tilde{\mathbf{D}}(g) = S\mathbf{D}(g)S^{-1}$ is a representation. Define the characters of a representation, and the conjugacy classes of a group. Show that (i) characters of $\mathbf{D}$ and $\tilde{\mathbf{D}}$ are the same, and that (ii) each element of a given conjugacy class has the same order. [7]

List the elements of the group of symmetries of the square. Find the conjugacy classes, and deduce the number of irreducible representations, quoting any theorems you use. [10]

END OF PAPER