UNIVERSITY OF CAMBRIDGE

NATURAL SCIENCES TRIPOS Part IB & II (General)

Tuesday 31 May 2005 9 to 12

MATHEMATICS (1)

Before you begin read these instructions carefully:

You may submit answers to no more than ***six*** *questions. All questions carry the same number of marks.*

The approximate number of marks allocated to a part of a question is indicated in the right hand margin.

Write on **one** *side of the paper only and begin each answer on a separate sheet.*

At the end of the examination:

Each question has a number and a letter (for example, ***6A****).*

Answers must be tied up in **separate** *bundles, marked* ***A, B or C*** *according to the letter affixed to each question.*

Do not join the bundles together.

For each bundle, a blue cover sheet must be completed and attached to the bundle.

A ***separate*** *yellow master cover sheet listing all the questions attempted* ***must*** *also be completed.*

Every cover sheet must bear your examination number and desk number.

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1B For $\phi(\boldsymbol{r}) = \exp(ikr)/(4\pi r)$ evaluate $\nabla\phi$ in Cartesian coordinates, where $\boldsymbol{r} = (x, y, z)$, $r = |\boldsymbol{r}|$, and k is a positive real number. [6]

State Stokes' theorem for a vector field $\boldsymbol{F}$. [2]

Using Cartesian coordinates show that

$$(\boldsymbol{F} \cdot \nabla)\boldsymbol{F} \;=\; \frac{1}{2}\nabla f^2 \;-\; \boldsymbol{F} \times (\nabla \times \boldsymbol{F})$$

where $f = |\boldsymbol{F}|$. [7]

For orthorgonal curvilinear coordinates q_i, for $i = 1, 2, 3$, show that

$$\nabla\psi = \sum_{i=1}^{3} \frac{1}{h_i}\frac{\partial\psi}{\partial q_i}\boldsymbol{e}_i$$

explaining the definition of the quantities h_i and $\boldsymbol{e}_i$, where ψ is a scalar function. [5]

2A The heat flow in a laterally insulated bar placed along the x-axis from $x = 0$ to $x = \pi$ is governed by the heat equation

$$\frac{\partial T}{\partial t} = c^2 \frac{\partial^2 T}{\partial x^2}, \qquad c^2 = \frac{\kappa}{\sigma\rho}$$

where $T(x, t)$ is the temperature, κ is the thermal conductivity, σ is the specific heat and ρ is the density of the material. The ends of the bar are kept at temperature $T = 0$.

Using the technique of separation of variables, find the functions $F_n(x)$ and $G_n(t)$ which allow $T(x, t)$ to be written as

$$T(x,t) = \sum_{n=1}^{\infty} a_n F_n(x) G_n(t) \ .$$

[10]

If the initial temperature is given by

$$f(x) = \begin{cases} x & \text{if } 0 < x < \pi/2 \\ \pi - x & \text{if } \pi/2 < x < \pi \end{cases}$$

find a_n. [8]

Verify that the infinite series converges. [2]

3B Define the Fourier transform of a function $f(x)$ and write down its inverse transform. [2]

For a function $g(x)$, derive the Fourier transform of $d^n g/dx^n$ in terms of the Fourier transform of g for any positive integer n, under the assumption that the derivative exists and that $d^n g(x)/dx^n \to 0$ as $|x| \to \infty$ for each n. [7]

For a function $f(x,t)$ with given initial condition $f(x,0)$ and which satisfies

$$\frac{\partial f}{\partial t} = \alpha \frac{\partial^2 f}{\partial x^2}$$

and $|f(x,t)| \to 0$, $|\partial f/\partial x| \to 0$ as $|x| \to \infty$ for all t, use Fourier transforms to derive the solution for $f(x,t)$ in terms of $f(x,0)$. Evaluate this explicitly in the case $f(x,0) = \exp(-x^2)$. [11]

4B Suppose A is a Hermitian $n \times n$ matrix such that $A^2 = A$ and $\det A \neq 1$. By examining the eigenvalues of A, or otherwise, show that $\det A = 0$ and $\mathrm{Tr}\ A = m$ where m is an integer less than n. [6]

Define a scalar product in a complex vector space and state what is meant by an orthonormal basis. [4]

If $\boldsymbol{x}$ and $\boldsymbol{y}$ are two elements of an orthonormal basis find the distance $||\boldsymbol{x} - \boldsymbol{y}||$ between them. [4]

Given an $n \times n$ Hermitian matrix B and a vector $\boldsymbol{x}$ in an n-dimensional vector space over the complex numbers, show that $\boldsymbol{x}^\dagger B \boldsymbol{x}$ is real. [6]

5B Define what is meant by a unitary matrix and an orthogonal matrix. [2]

Find the eigenvalues and eigenvectors of the matrix A where

$$A = \begin{pmatrix} a & 0 & b \\ 0 & c & 0 \\ b & 0 & d \end{pmatrix}$$

where a, b, c, and d are real constants, in the case $a + d = 0$. [8]

Schwarz's inequality for any pair of vectors $\boldsymbol{a}, \boldsymbol{b}$ in an n-dimensional vector space is $| < \boldsymbol{a}, \boldsymbol{b} > | \leq ||\boldsymbol{a}||\ ||\boldsymbol{b}||$. Prove this and hence or otherwise show that $||\boldsymbol{a}+\boldsymbol{b}|| \leq ||\boldsymbol{a}||+||\boldsymbol{b}||$. [10]

6A Define a regular singular point of a second-order linear differential equation. [1]

Consider Bessel's differential equation

$$x^2 y'' + xy' + (x^2 - \nu^2)y = 0$$

where ν is real and non-negative. Show that a series solution is

$$y_1(x) = x^\nu \sum_{m=0}^{\infty} a_{2m} x^{2m}$$

where a_{2m} is defined in terms of a recurrence relation which you should find. [10]

In the case where ν is not an integer, find a second solution $y_2(x)$. [2]

In the case where ν is an integer, set $a_0 = (2^\nu \nu!)^{-1}$, and hence find the solution of the recurrence relation for the term a_{2m} in $y_1(x)$. Discuss the convergence of the series. [7]

7A The second-order Sturm–Liouville linear differential operator $\mathcal{L}$ is given by

$$\mathcal{L} = -\frac{d}{dx}\left(p(x)\frac{d}{dx}\right) - q(x)\ ,$$

where $p(x)$ and $q(x)$ are real functions defined for $a \leq x \leq b$, with $p(x) > 0$ for $a < x < b$. Consider the eigenvalue equation

$$\mathcal{L}y = \lambda w(x) y$$

where $w(x)$ is a real function with $w(x) > 0$ for $a < x < b$ and $y(a) = y(b) = 0$. Show that any two eigenfunctions corresponding to different eigenvalues λ are orthogonal with respect to an inner product with weight function $w(x)$. [8]

By considering the substitution $x = e^v$, or otherwise, find the eigenvalues and orthonormal eigenfunctions of the equation

$$x^2 \frac{d^2 y}{dx^2} + \lambda y = 0$$

when $y(1) = y(2) = 0$. [12]

8A Find the Green's function $G(x, \xi)$ that satisfies

$$\frac{d^2G}{dx^2} + 4\frac{dG}{dx} + (4 + a^2)G = \delta(x - \xi)$$

where a is real, subject to $\frac{dG}{dx}(0, \xi) = 0$ and $G(0, \xi) = 0$. [10]

Hence find the solution $y(x)$ of the equation

$$y'' + 4y' + 8y = \sin 2x$$

with $y(0) = 0$ and $y'(0) = 0$. [10]

You may use the following identity:

$$\sin A \sin B = \frac{1}{2}\left[\cos(A - B) - \cos(A + B)\right]$$

9C The functions $p(x)$, $q(x)$ and $w(x)$ are defined on the interval $0 \le x \le 1$. They are all positive in the interval except that $p(0) = 0$.

Consider the twice differentiable functions $y(x)$ that are subject to the boundary conditions $y(1) = 0$, $y(0)$ is finite, and the constraint

$$\int_0^1 w(x)\,(y(x))^2\,dx = 1 \quad .$$

Show that such a function $y(x)$ that renders the integral

$$I = \int_0^1 \left(p(x)y'^2 + q(x)y^2\right) dx \quad ,$$

stationary, is a solution of the Sturm-Liouville eigenvalue problem

$$(p(x)y')' - q(x)y + \lambda w(x)y = 0 \quad ,$$

where λ is the eigenvalue. [6]

Explain briefly the Rayleigh-Ritz method for estimating the lowest eigenvalue. [4]

Consider the eigenvalue problem on the interval $0 \le x \le 1$,

$$(xy')' + \lambda xy = 0 \quad .$$

By using the trial wave function

$$y = A(1 - x^2) \quad ,$$

where A is a constant that you should determine, obtain an estimate of the lowest eigenvalue. [8]

The exact value (to four decimal places) is 5.7815. Why is your estimate bigger than this? [2]

10C The Euler-Lagrange equation

$$\frac{d}{dx}\frac{\partial f}{\partial y'} - \frac{\partial f}{\partial y} = 0 \quad ,$$

is satisfied by the function $y(x)$ that renders the integral

$$I = \int_a^b f(x, y, y')dx \quad ,$$

stationary, subject to appropriate boundary conditions. Show that if $f(x, y, y')$ does not depend explicitly on x, then $y(x)$ also satisfies the first integral

$$f - y'\frac{\partial f}{\partial y'} = k \quad ,$$

where k is a constant. [8]

A flexible fence of length $2l$, is attached at its ends to two points on a straight wall a distance $2a$ apart. The values of l and a satisfy the inequalities $a < l < \pi a/2$. Find the shape of the flexible fence that maximises the area between the fence and the wall? [12]

END OF PAPER