UNIVERSITY OF CAMBRIDGE

NATURAL SCIENCES TRIPOS Part IA

Monday 13 June 2005 9 to 12

MATHEMATICS (1)

Before you begin read these instructions carefully:

You may submit answers to no more than **six** *questions. All questions carry the same number of marks.*

The approximate number of marks allocated to a part of a question is indicated in the right hand margin.

Write on **one** *side of the paper only and begin each answer on a separate sheet.*

Questions marked with an asterisk (*) require a knowledge of B course material.

At the end of the examination:

Each question has a number and a letter (for example, ***3B****).*

Answers must be tied up in **separate** *bundles, marked* ***A, B, C, D, E*** *or* ***F*** *according to the letter affixed to each question.* ***Do not join the bundles together****. For each bundle, a blue cover sheet* **must** *be completed and attached to each bundle, with the appropriate letter written in the section box.*

A **separate** *yellow master cover sheet listing all the questions attempted* **must** *also be completed.*

Every cover sheet must bear your examination number and desk number.

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1A Find, by any method, the first non-zero term in the Taylor series about $x = 0$ of

(a)

$$\frac{1}{\sin^2 x} - \frac{1}{x^2},$$

[6]

(b)

$$\frac{\sin x - \tan^{-1} x}{x^2 \ln(1+x)},$$

[7]

(c)

$$\frac{e^{2x} - 2e^x + 1}{\cos(3x) - 2\cos(2x) + \cos x}.$$

[7]

[*You may assume, if you wish, the standard Taylor series expansions for exponential, trigonometric and logarithmic functions.*]

2A* Using suffix notation, demonstrate whether the following statements are true or false:

(a) Any square matrix can be written as the sum of a symmetric and an antisymmetric matrix. [2]

(b) If A and B are any square symmetric matrices, then the product AB is symmetric. [3]

(c) If B is a square symmetric matrix, then the matrix $A^T BA$ is symmetric for any square matrix A. [5]

(d) If A is an antisymmetric square matrix and B is a symmetric square matrix, then $A_{ij}B_{ij} = 0$. [4]

(e) If A is an orthogonal matrix, then any square matrix B has the same eigenvalues as $A^T BA$. [6]

3B

(a) It is known that one person out of a group of N people committed a certain crime, and that (in the absence of further evidence) each person in the group is equally likely to be guilty. A suspect is chosen at random from the group and a DNA test is performed. The probability of a positive DNA match being obtained when a suspect is not guilty is p and the probability of no match being obtained when the suspect is guilty is 0. One of the suspects is tested and a positive DNA match is obtained. What is now the probability that the suspect is guilty?

[8]

(b) A party is attended by N people (including the host). Assuming that birthdays are distributed evenly over the year, and ignoring the effects of leap years, answer the following:

(i) Two people meet at random at the party. What is the probability that they share the same birthday?

[2]

(ii) What is the probability that at least one other person at the party has the same birthday as the host? How large would N have to be to make this probability greater than 50%?

[4]

(c) In a certain country, the probability of any given candidate passing the driving test on a first attempt is 3/5, on a second attempt is 4/5 and on a third attempt is 7/8. Assuming that a candidate does not retake the test after passing, what is the mean number of attempts for candidates who passed in 3 or fewer attempts?

[6]

4B A continuous random variable T has a probability distribution function $f(t)$ given by

$$f(t) = \begin{cases} \alpha e^{-\alpha t} & t > 0, \\ 0 & t \leqslant 0, \end{cases}$$

where $\alpha > 0$.

(a)

(i) Find $P(0 \leqslant T \leqslant t_0)$ in terms of α and t_0. [2]

(ii) Verify that f is correctly normalised, and find the mean and variance of T. [5]

(iii) What is the probability that T exceeds its mean? [3]

(b) Suppose that T represents the length of time that a given piece of equipment operates before failing.

(i) If the equipment has not failed for a time s, show that the probability that it will not fail during a further time τ is independent of s.

[5]

(ii) It costs £(C/α^2) to produce the given piece of equipment, and the manufacturer receives £Q for every unit of time that the equipment operates before failing. What value of α should be chosen to maximise the expected net profit? [5]

5C The vertices of a tetrahedron $O,\ P,\ Q,\ R$ have coordinates $(0,0,0),\ (2,1,1),\ (1,2,2)$ and $(0,0,3)$ respectively.

Find by vector methods

(a) the angle between the faces OPR and OQR [5]

(b) the angle between the vector OP and the normal to the face PQR [5]

(c) the area of the face PQR [5]

(d) the shortest distance from the origin to the plane containing P, Q and R. [5]

6C*

(a) State clearly the divergence theorem. [4]

(b) Calculate directly the surface integral $\int \mathbf{F} \cdot d\mathbf{S}$ for the vector field $\mathbf{F} = (x, 2y, 3z)$ over the surface of a sphere of radius a centred at the origin. [10]

(c) Calculate div $\mathbf{F}$ and find the integral of div $\mathbf{F}$ over the volume V of the sphere. [6]

7D*

(a) Express $\ln(N!)$ as a sum. With the use of a suitable diagram, show that

$$(N+1)\ln(N+1) - N + 1 - 2\ln 2 \geqslant \ln(N!) \geqslant N\ln N - N + 1$$

for all integers $N \geqslant 1$. [4]

(b) It is suggested that an equation of the form

$$\ln(N!) \approx (N+a)\ln N - N + b$$

might give a good estimate of $\ln(N!)$. Given that $\ln(10!) = 15.104413$ and $\ln(50!) = 148.477767$ (both to 6 decimal places), find suitable values of a and b (to 4 significant figures). [6]

Use this formula to estimate

(i) $\ln(500!)$ [2]

(ii) $\ln(1000!)$ [2]

(iii) $\ln(N_A!)$ where $N_A = 6.022137 \times 10^{23}$. [2]

(c) Estimate

$$\ln(1 \times 3 \times 5 \times 7 \times \ldots \times 997 \times 999)\,.$$

[4]

8D

(a) Evaluate

$$\int_1^2 \int_1^2 \int_1^2 \frac{dx\ dy\ dz}{x^2 y^3 z^4}\,.$$

[3]

(b) Evaluate using polar coordinates

$$\int_0^\infty dx \int_0^\infty dy \frac{yx^2}{x^2+y^2} e^{-(x^2+y^2)}.$$

[6]

(c)

(i) Verify that

$$\int \tanh(xy)\text{sech}(xy)dy = -\frac{1}{x}\text{sech}(xy) + c\,,$$

where c is independent of y. [2]

(ii) By using the above result to express the quantity

$$\int_0^\infty \frac{1}{x}(\text{sech}x - \text{sech}(\alpha x))dx$$

(where $\alpha > 1$) as a double integral, or otherwise, determine its value. [9]

9E Obtain the general solutions $y(x)$ of the following differential equations

(a)

$$\frac{dy}{dx} = \frac{y^2+1}{\cos^2 x}\,.$$

[6]

(b)

$$x\frac{dy}{dx} + (x+\alpha)y = e^{-x},$$

where α is a constant. Include in your answer the result for the special case $\alpha = 0$.
[8]

(c)

$$\frac{dy}{dx} + 4xy = 2x(y^2+1).$$

[6]

10E Find the solution $y(x)$ of the differential equation

$$\frac{d^2y}{dx^2} - (a+b)\frac{dy}{dx} + aby = e^{cx},$$

where a, b and c are constants, and subject to the conditions $y(0) = y'(0) = 0$, in the two cases

(a) $a \neq b \neq c \neq a$, [10]

(b) $a \neq b = c$. [10]

11F Find and determine the nature of each of the stationary points of the function

$$f(x) = (x + x^3)e^{-x^2}.$$

where x is a real variable. [6]

Sketch a graph of the function $f(x)$. [4]

Evaluate the definite integrals

(a)

$$\int_{-\infty}^{\infty} f(x)dx,$$

[3]

(b)

$$\int_{0}^{\infty} f(x)dx,$$

[4]

(c)

$$\int_{0}^{\infty} f(x)\frac{df(x)}{dx}dx.$$

[3]

12F Let $f(x)$ be the periodic function, of period 2, defined by

$$f(x) = 1 - |x|, \quad -1 \leqslant x \leqslant 1.$$

(a) Sketch $f(x)$. [3]

(b) Find the Fourier cosine series of $f(x)$. [10]

(c) Hence show that

$$\sum_{m=0}^{\infty} \frac{1}{(2m+1)^2} = \frac{\pi^2}{8}.$$

[7]

END OF PAPER