UNIVERSITY OF CAMBRIDGE

NATURAL SCIENCES TRIPOS Part IA

Wednesday 9 June 2004 9 to 12

MATHEMATICS (2)

Before you begin read these instructions carefully:

You may submit answers to no more than **six** *questions. All questions carry the same number of marks.*

The approximate number of marks allocated to a part of a question is indicated in the right hand margin.

Write on **one** *side of the paper only and begin each answer on a separate sheet.*

Questions marked with an asterisk (*) require a knowledge of B course material.

At the end of the examination:

For **each question** *you have attempted, attach a* **blue** *cover sheet to your answer and write the question number and letter (for example,* **3B***) in the 'section' box on the cover sheet.*

List all the questions you attempted on the **yellow** *master cover sheet.*

Every cover sheet must bear your candidate number and your desk number.

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1A **(a)** Let $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ be vectors representing the sides of a triangle. Express $\mathbf{c}$ in terms of $\mathbf{a}$ and $\mathbf{b}$, and draw a diagram showing these vectors. [3]

Hence derive the cosine rule

$$c^2 = a^2 + b^2 - 2ab\cos C$$ [3]

and the sine rule

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c},$$ [4]

where $a = |\mathbf{a}|$, $b = |\mathbf{b}|$ and $c = |\mathbf{c}|$, and A, B and C are the internal angles of the triangle.

(b) Consider an arbitrary tetrahedron with four faces f_1, f_2, f_3 and f_4. Let the vectors $\mathbf{V}_1$, $\mathbf{V}_2$, $\mathbf{V}_3$ and $\mathbf{V}_4$ have magnitudes equal to the areas of f_1, f_2, f_3 and f_4, respectively, and directions outwardly normal to the faces. Show that

$$\sum_{i=1}^{4} \mathbf{V}_i = \mathbf{0}.$$ [10]

2A A periodic function $f(x)$ is defined by

$$f(x) = \begin{cases} 0 & \text{for } -5 < x < 0, \\ 6 & \text{for } 0 < x < 5, \end{cases}$$

and $f(x) = f(x - 10)$, so that the period is 10. Draw a graph of the function $f(x)$ for $-5 < x < 15$. [3]

Find the Fourier series, with period 10, for $f(x)$. [13]

What values does the Fourier series converge to at $x = -5$, $x = 0$ and $x = 5$? [4]

3B **(a)** Show that the product of any number of orthogonal matrices is also orthogonal. [2]

(b) Let $\mathbf{E}_1$ and $\mathbf{E}_2$ be symmetric 2×2 matrices with the properties

$$\mathbf{E}_1^2 = \mathbf{E}_2^2 = \mathbf{I}, \quad \mathbf{E}_1\mathbf{E}_2 + \mathbf{E}_2\mathbf{E}_1 = \mathbf{0}, \qquad (*)$$

where $\mathbf{I}$ is the 2×2 identity matrix, and let

$$\mathbf{A} = \cos\left(\frac{\theta}{2}\right)\mathbf{I} + \sin\left(\frac{\theta}{2}\right)\mathbf{E}_1\mathbf{E}_2,$$

where θ is a scalar.

(i) Show that $\mathbf{E}_1$ and $\mathbf{E}_2$ are orthogonal. Show also that $\mathbf{A}$ is orthogonal but not symmetric unless $\sin(\theta/2) = 0$. [6]

(ii) Show that

$$\mathbf{A}\mathbf{E}_1\mathbf{A}^T = \cos\theta\mathbf{E}_1 - \sin\theta\mathbf{E}_2$$

and write down the corresponding expression for $\mathbf{A}\mathbf{E}_2\mathbf{A}^T$. [7]

(iii) Find explicitly all $\mathbf{E}_2$ that satisfy $(*)$ in the case

$$\mathbf{E}_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

[5]

4B **(a)** Find the area under the curve given parametrically by

$$x = \cos\theta, \quad y = 1 + \cos 2\theta, \quad 0 < \theta < \pi\,.$$

[6]

(b) A curve is given parametrically by

$$x = \cos\theta, \quad y = \cos(4\theta), \quad 0 < \theta < \pi\,.$$

(i) Evaluate dy/dx as a function of θ. Find the gradient dy/dx when $\theta = 0$. What point in the (x, y) plane does this correspond to? [6]

(ii) It may be shown that

$$\frac{d^2y}{dx^2} = 16(6\cos^2\theta - 1).$$

Sketch the curve in the (x, y) plane, indicating maxima, minima and points for which $\dfrac{d^2y}{dx^2} = 0$, and the corresponding values of θ. [8]

5C If a football match ends in a draw, there may be a 'penalty shoot-out'. In one version of a penalty shoot-out, each team initially takes 5 shots at the goal. If one team scores more times than the other, then that team wins.

(i) Two teams take part in a penalty shoot-out. The probability that any player scores from a single shot is p, where $0 < p < 1$. Prove carefully that the probability, α, that neither side has won at the end of the initial 10-shot period is given by

$$\alpha = \sum_{n=0}^{5} \binom{5}{n}^2 (1-p)^{2n} p^{10-2n} \ .$$

(Note that $\binom{5}{n}$ is another notation for ${}^5C_n$.) [8]

In the case $p = \frac{1}{2}$, evaluate α and find the probability that the first team to shoot wins at the end of the initial period. [3]

(ii) If neither team has won at the end of the initial period, the teams take another shot each. If one team scores and the other does not, then the team that scores wins; if neither team wins, then the teams take another shot each and the process continues until one of the teams wins. Find the probability, for $0 < p < 1$, that a total of $10 + 2k + 2$ shots ($k \geq 0$) are required to win the game. Verify that the probability that, eventually, one team or the other will win is 1. [9]

6C* Use the method of separation of variables to find the most general solution $\phi(x, y)$ of the equation

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = -25\phi$$

given that ϕ satisfies the boundary conditions

$$\phi(x, 0) = \phi(x, \pi) = \phi(0, y) = \phi(\pi, y) = 0 \ .$$ [15]

What is the most general solution of the equation

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = +25\phi$$

that satisfies the boundary conditions $\phi(x, 0) = \phi(x, \pi) = \phi(0, y) = \phi(\pi, y) = 0$? You should justify your answer. [5]

7D **(a)** A surface S_1 is defined by the equation $x^3 + xy + yz = 1$ and $\frac{1}{2} \leqslant x \leqslant 1$, $\frac{1}{2} \leqslant y \leqslant 1$.

(i) Find the vector field $\mathbf{u}$ that is normal to S_1 and which satisfies $\mathbf{u} \cdot \mathbf{k} = 1$, where $\mathbf{k} = (0, 0, 1)$. [5]

(ii) Let $d\mathbf{S}$ be the surface area element of S_1 that satisfies $d\mathbf{S} \cdot \mathbf{k} > 0$. Explain why

$$d\mathbf{S} = \frac{\mathbf{n} dx dy}{\cos\alpha} \, ,$$

where $\mathbf{n}$ is the unit normal to S_1 and $\cos\alpha = \mathbf{n} \cdot \mathbf{k} > 0$. Hence or otherwise show that

$$d\mathbf{S} = \left(\frac{3x^2 + y}{y}, \frac{x+z}{y}, 1 \right) dx dy \, .$$

[5]

(iii) A vector field $\mathbf{F}$ is defined by $\mathbf{F} = (0, y^2, x)$. Find $\int_{S_1} \mathbf{F} \cdot d\mathbf{S}$. [6]

(b) A surface S_2 is defined by $x^3 + xy = 1$ and $-1 \leqslant z \leqslant 1$, $x > 0$. A vector field $\mathbf{G}$ is defined by $\mathbf{G} = (0, 0, x)$. Explain why $\int_{S_2} \mathbf{G} \cdot d\mathbf{S} = 0$, where $d\mathbf{S}$ is now the element of area for S_2. [4]

8D **(a)** Find, by any method, the first three non-zero terms in the Taylor expansion about $x = 0$ of the following functions.

(i) $\dfrac{\log(1+x)}{1-x}$; [4]

(ii) $\dfrac{1}{1+\sin x}$; [4]

(iii) $\log(\cos x)$. [7]

(b) Let

$$f(x) = \sum_{i=0}^{\infty} a_i x^i \quad \text{and} \quad g(x) = \sum_{i=0}^{\infty} b_i x^i$$

and let $\sum_{i=0}^{\infty} c_i x^i$ be the Taylor expansion about $x = 0$ of the function $f(x)g(x)$.

(i) Find c_0, c_1 and c_2 in terms of a_0, a_1, a_2, b_0, b_1 and b_2.

(ii) Give a general expression for c_i as a finite sum of products of the coefficients a_j and b_j. [5]

9E **(a)** The function $F(x,y)$ is defined by $F(x,y) = xy^2 \exp(x^2 y)$. Verify that

$$\frac{\partial^2 F}{\partial x \partial y} = \frac{\partial^2 F}{\partial y \partial x} \, .$$

[7]

(b) The function f is given by $f(a,b) = a^p b^q$.

(i) If b remains constant and a changes by $k\%$, where k is small, derive a formula for the approximate percentage change in f, to second order in k. [5]

(ii) If a and b change by $k\%$ and $m\%$, respectively, where k and m are small, derive a formula expressing the percentage change in f, to second order in k and m. [8]

10E* **(a)** Let a be an integer greater than 1. Show by means of a sketch that

$$\int_a^{a+1} \frac{1}{x^3} dx < \frac{1}{a^3} < \int_a^{a+1} \frac{1}{(x-1)^3} dx \, .$$

Hence show that

$$\int_a^{\infty} \frac{1}{x^3} dx < \sum_{n=a}^{\infty} \frac{1}{n^3} < \int_a^{\infty} \frac{1}{(x-1)^3} dx \, .$$

[6]

(b) Let p be an integer greater than 1. The real number k_p is defined, for each p, by

$$\int_5^{\infty} \frac{1}{(x-k_p)^p} dx = \sum_{n=5}^{\infty} \frac{1}{n^p} .$$

You may assume that $k_p < 5$.

(i) Using the result of part (a), or otherwise, show that $0 < k_3 < 1$. [4]

(ii) Evaluate k_2 and k_4 to three places of decimals, given that

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6} \quad \text{and} \quad \sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{90} \, .$$

[5]

(iii) Assuming that $k_3 \approx (k_2 + k_4)/2$, estimate $\sum_{n=1}^{\infty} \frac{1}{n^3}$. [5]

11F Consider the second order linear differential equations

$$\frac{d^2y}{dx^2} + \frac{4}{x}\frac{dy}{dx} + \frac{2}{x^2}y = 0, \qquad (1)$$

$$\frac{d^2y}{dx^2} + \frac{1}{x}\frac{dy}{dx} + \frac{1}{x^2}y = 0. \qquad (2)$$

(i) Make the substitution $x = \mathrm{e}^z$ in each of equations (1) and (2) to obtain in each case a second order linear equation for $y(z)$ in the form

$$\frac{d^2y}{dz^2} + \alpha\frac{dy}{dz} + \beta y = 0,$$

where α and β are constants. [10]

(ii) Hence or otherwise find the solutions to equations (1) and (2) that satisfy the conditions $y = 0$, $dy/dx = 1$ at $x = 1$. [10]

12F* **(a)** A rectangular box has width a, depth b and height c. The box has no top. The front of the box, of area ac, is made of glass costing £2 per unit area, while the remaining walls and base are made from wood costing £1 per unit area.

Using the method of Lagrange multipliers, find the shape of box that has the least cost for a given volume V, and determine this cost in terms of V. [10]

(b) Show that $(x, y) = (3/2, 3/2)$ and $(x, y) = (-1/2, 5/2)$ are stationary points of the function

$$\frac{1}{3}(x^3 + y^3) + 2xy - \frac{21}{4}(x + y),$$

and find the other two stationary points. Determine the nature of each stationary point. [10]