UNIVERSITY OF CAMBRIDGE

NATURAL SCIENCES TRIPOS Part IA

Monday 7 June 2004 9 to 12

MATHEMATICS (1)

Before you begin read these instructions carefully:

You may submit answers to no more than **six** *questions. All questions carry the same number of marks.*

The approximate number of marks allocated to a part of a question is indicated in the right hand margin.

Write on **one** *side of the paper only and begin each answer on a separate sheet.*

Questions marked with an asterisk (*) require a knowledge of B course material.

At the end of the examination:

For **each question** *you have attempted, attach a* **blue** *cover sheet to your answer and write the question number and letter (for example,* **3B***) in the 'section' box on the cover sheet.*

List all the questions you attempted on the **yellow** *master cover sheet.*

Every cover sheet must bear your candidate number and your desk number.

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1A The components of the vectors **a**, **b** and **c** are (a_1, a_2, a_3), (b_1, b_2, b_3) and (c_1, c_2, c_3), respectively. Write down, in terms of the components, the scalar triple product $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})$. Show that

$$\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c}\,.$$ [4]

(i) Using vector methods, find the equation, in the form $\alpha x + \beta y + \gamma z = \delta$, of the plane passing through the points $(2, -1, 2)$, $(1, 2, 3)$ and $(4, 1, 0)$. [12]

(ii) Let **i**, **j** and **k** be mutually orthogonal unit vectors. Find the volume of the parallelepiped three of whose edges are $\mathbf{i} + 3\mathbf{j}$, $-\mathbf{i} + 3\mathbf{j} - 2\mathbf{k}$ and $\mathbf{j} + \mathbf{k}$. [4]

2A* State the comparison test and the ratio test for assessing the convergence of the infinite series $\sum_{n=0}^{\infty} v_n$, where $v_n > 0$. [4]

Determine whether the following series converge or diverge. You may use without proof standard results relating to the series $\sum_{n=1}^{\infty} n^s$.

(i) $\displaystyle\sum_{n=1}^{\infty} \frac{\ln n}{2n^3 + 1}$ [4]

(ii) $\displaystyle\sum_{n=0}^{\infty} \frac{n}{n + 4}$ [4]

(iii) $\displaystyle\sum_{n=1}^{\infty} \sin^2\left(\frac{1}{n}\right)$ [4]

(iv) $\displaystyle\sum_{n=0}^{\infty} n^4 \exp(-n^2)$ [4]

3B* State Stokes's theorem, giving a careful definition of all the quantities involved. [3]

(i) The vector field **A** is given in Cartesian coordinates and components by

$$\mathbf{A} = (2x - y, -yz^2, -y^2 z).$$

Show that

$$\nabla \times \mathbf{A} = (0, 0, 1).$$ [3]

(ii) A tetrahedron is defined by the vertices, in Cartesian coordinates, $(0, 0, 0)$, $(1, 0, 0)$, $(0, 1, 0)$ and $(0, 0, 1)$. An open surface S is defined from this tetrahedron as the three faces which include the vertex $(0, 0, 1)$. Let C be the boundary of this surface (which lies in the plane $z = 0$).

Explicitly verify Stokes's theorem for **A**, C and S. [14]

4B **(a)** By expressing the integrand in partial fractions, evaluate

$$\int_a^b \frac{x-1}{x(x+1)}\,dx,$$

where a and b are positive, giving your answer as the natural logarithm of a rational function of a and b. [7]

(b) Evaluate

$$\int \tan^2 x\,dx.$$

[7]

(c) Sketch the function $\ln x$ for $1 < x < e$ and use your sketch to put the following integrals in ascending order of numerical value:

(i) $I_1 = \int_1^e \ln x\,dx$; (ii) $I_2 = \int_1^e \ln(x^2)\,dx$; (iii) $I_3 = \int_1^e (\ln x)^2\,dx$. [6]

5C **(a)** The random variable X can take any non-negative value, according to

$$\mathrm{P}(0 \le X \le x) = k\left(1 - e^{-x}\right)\ ,$$

where k is a constant. Find the value of k, the probability density function for X, and the mean and variance of X. [7]

(b) The random variable Y can take the value -1 or any non-negative value, according to

$$\mathrm{P}(Y = -1) = m\,; \qquad \mathrm{P}(0 \le Y \le y) = \frac{1 - e^{-y}}{2}\ ,$$

where m is a constant. Find:

(i) the value of m; [3]

(ii) the mean of Y; [3]

(iii) the variance of Y; [3]

(iv) the probability that $Y \ge 1$ given that $Y \ge 0$. [4]

6C Let

$$\mathbf{A} = \begin{pmatrix} 1 & 3 & 0 \\ 2 & k & 0 \\ 4 & 0 & 1 \end{pmatrix}.$$

(i) In the case $k \neq 6$, find $\mathbf{A}^{-1}$ and hence solve the equations

$$\begin{aligned} x + 3y &= 2 \\ 2x + 4y &= 3 \\ 4x + z &= 1\ . \end{aligned}$$

[7]

(ii) In the case $k = 1$, find $\mathbf{B}$ given that

$$\mathbf{BA} = \begin{pmatrix} 2 & 6 & 0 \\ 2 & 1 & 0 \\ 8 & 0 & 2 \end{pmatrix}.$$

[7]

(iii) In the case $k = 6$, find all the solutions to the equation $\mathbf{Ax} = \mathbf{0}$. [6]

7D* **(a)** The $n \times n$ matrix $\mathbf{S}$ is symmetric and the $n \times n$ matrix $\mathbf{A}$ is antisymmetric. Show that $\mathrm{tr}(\mathbf{AS}) = 0$. [5]

(b) The 3×3 matrices $\mathbf{M}$ and $\mathbf{N}$ have components

$$M_{ij} = \delta_{ij} + 2\epsilon_{ijk} a_k \quad \text{and} \quad N_{ij} = a_i b_j ,$$

where a_i and b_j are the components of two non-zero 3-dimensional vectors $\mathbf{a}$ and $\mathbf{b}$.

(i) Evaluate $\mathrm{tr}(\mathbf{M})$. [3]

(ii) Find the value of $\det(\mathbf{M})$. [5]

(iii) Using suffix notation, or otherwise, show that $\mathbf{MN} = \mathbf{N}$. [4]

(iv) Deduce, using (ii) and (iii) above, the value for $\det(\mathbf{N})$. [3]

8D The force fields $\mathbf{F}$ and $\mathbf{G}$ are given by

$$\mathbf{F} = \begin{pmatrix} 3x^2yz^2 \\ 2x^3yz \\ x^3z^2 \end{pmatrix} , \qquad \mathbf{G} = \begin{pmatrix} 3x^2y^2z \\ 2x^3yz \\ x^3y^2 \end{pmatrix} ,$$

respectively.

(i) Compute the line integrals $\int \mathbf{F} \cdot d\mathbf{x}$ and $\int \mathbf{G} \cdot d\mathbf{x}$ along the straight line from $(0,0,0)$ to $(1,1,1)$. [4]

(ii) Compute the line integrals $\int \mathbf{F} \cdot d\mathbf{x}$ and $\int \mathbf{G} \cdot d\mathbf{x}$ along the path $\mathbf{x}(t) = (t, t^2, t^2)$ from $(0,0,0)$ to $(1,1,1)$. [6]

(iii) Show that $\mathbf{G}$ is conservative and find a function Φ such that $\nabla\Phi = \mathbf{G}$. Is $\mathbf{F}$ conservative? [10]

99E **(a)** Give expressions for the following complex numbers in the form $a + ib$, where a and b are real:

(i) e^{2i}; (ii) $\ln(-1)$; (iii) $\ln(i)$; (iv) 10^i; (v) i^i . [10]

(b) Define $\cosh x$ and $\sinh x$ in terms of e^x and e^{-x}. [2]

Show that:

(i) $\cosh^2 x - \sinh^2 x = 1$; (ii) $\tanh(-x) = -\tanh x$; (iii) $\dfrac{d}{dx}\cosh x = \sinh x$. [3]

(c) Evaluate, simplifying your answers where possible:

(i) $\displaystyle\int_{-1000}^{1000} \tanh^5 x \, dx$; [1]

(ii) $\displaystyle\int_{0}^{\ln 2} \tanh x \, dx$. [4]

10E **(a)** By considering $(\cos\theta + i\sin\theta)^5$, express $\cos(5\theta)$ in terms of $\cos\theta$. [5]

(b) Describe fully the curve in the Argand diagram whose equation is

$$|z + 1 + i| = 8\ .$$ [3]

Describe fully the three loci determined, as z moves round this curve, by the three complex numbers u, v and w defined as follows:

(i) $u = 2x + iy$ (where $z = x + iy$);

(ii) $v = z + 4 + 3i$;

(iii) $w = iv$. [6]

(c) Find the equations of the loci in the x–y plane described by:

(i) $(x, y) = (a\cos\theta, b\sin\theta)$;

(ii) $(x, y) = (a\cosh\theta, b\sinh\theta)$;

where a and b are fixed real numbers and θ varies. Describe, and sketch on the same axes, each locus. [6]

11F **(a)** Find the general solution of the equation

$$(1 - x^2)\frac{dy}{dx} = xy^2.$$ [6]

(b) By means of the substitution $y = z^{-\frac{1}{2}}$, or otherwise, find the solution of the equation

$$\frac{dy}{dx} = y - y^3$$

with the initial condition $y(0) = \frac{1}{2}$. [7]

(c) By means of an integrating factor, or otherwise, find the general solution of the equation

$$\frac{dy}{dx}\sin x - y\cos x = \cos^3 x.$$ [7]

12F A solid right circular cone C of height h and base radius a is bounded by the surfaces $r = a(1 - z/h)$ and $z = 0$, where r and z are cylindrical polar coordinates, so that r is the distance from the axis of the cone. If the density ρ is given by

$$\rho(r) = \rho_0 \left(1 + \frac{r}{a}\right),$$

find the total mass $M = \int_C \rho \, dV$ of the cone. [10]

Find also the distance d of the centre of mass of the cone from its base, using the formula

$$Md = \int_C z\rho \, dV \ .$$

[10]