UNIVERSITY OF CAMBRIDGE

NATURAL SCIENCES TRIPOS Part IB & II (General)

Friday 30 May 2003 9 to 12

MATHEMATICS (2)

Before you begin read these instructions carefully:

You may submit answers to no more than **six** *questions. All questions carry the same number of marks.*

The approximate number of marks allocated to a part of a question will be indicated in the right hand margin.

Write on **one** *side of the paper only and begin each answer on a separate sheet.*

At the end of the examination:

Each question has a number and a letter (for example, **6C***).*

Answers must be tied up in **separate** *bundles, marked* **A, B or C** *according to the letter affixed to each question.*

Do not join the bundles together.

For each bundle, a blue cover sheet must be completed and attached to the bundle.

A **separate** *yellow master cover sheet listing all the questions attempted* **must** *also be completed.*

Every cover sheet must bear your examination number and desk number.

1C In spherical polar coordinates (r, θ, ϕ), Poisson's equation for an axisymmetric electric potential $\Phi(r, \theta)$ can be expressed as

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial \Phi}{\partial r}\right) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial \Phi}{\partial \theta}\right) = -\frac{\rho}{\epsilon_0}\,,$$

where $\rho(r, \theta)$ is the electric charge density and ϵ_0 is a constant.

In the case when $\rho = 0$, use the method of separation of variables to obtain the general solution for the potential Φ in terms of Legendre polynomials $P_n(\cos\theta)$. You should state clearly the differential equation in x satisfied by $P_n(x)$. [11]

Suppose now that the charge density distribution is given by

$$\rho = \begin{cases} 4\epsilon_0 \cos\theta\,, & \text{in } 0 \leq r < a; \\ 0\,, & \text{in } r \geq a. \end{cases}$$

On the assumptions that the potential and its derivative should be continuous everywhere, and that $\Phi \to 0$ as $r \to \infty$, find the potential everywhere. [9]

[*You may quote the result that* $P_1(x) = x$.]

2C For functions Φ and Ψ defined in a volume V bounded by a surface S with outward unit normal $\mathbf{n}$, show that

$$\iiint_V \left(\Phi\nabla^2\Psi - \Psi\nabla^2\Phi\right) \mathrm{d}V = \iint_S \left(\Phi\nabla\Psi - \Psi\nabla\Phi\right)\cdot\mathbf{n}\,\mathrm{d}S\,.$$ [2]

Suppose that within the volume V, Φ satisfies Poisson's equation

$$\nabla^2\Phi = \sigma(\mathbf{x})\,,$$

where $\sigma(\mathbf{x})$ is a known function. State the equation satisfied by a Green's function $G(\mathbf{x};\mathbf{x}_0)$ for this equation, where $\mathbf{x}_0$ is a point within the volume V. Show that

$$\iint_S \nabla G\cdot\mathbf{n}\,\mathrm{d}S = 1\,.$$ [3]

On the surface S, Φ satisfies the Neumann condition

$$\mathbf{n}\cdot\nabla\Phi = f(\mathbf{x})\,.$$

Choose, with justification, a suitable boundary condition for $G(\mathbf{x};\mathbf{x}_0)$ on the surface S, and hence derive an integral expression that specifies $\Phi(\mathbf{x}_0)$ up to an undetermined constant. [5]

In the case when V is the region $z \geq 0$, use the method of images to calculate $G(\mathbf{x};\mathbf{x}_0)$. If $\sigma = 0$ throughout V, $\Phi \to 0$ as $|\mathbf{x}| \to \infty$, and on the surface $z = 0$

$$f = \delta(x)\delta(y)\,,$$

find Φ. [10]

3A The complex variable z is given by $z = x + iy$ where x and y are real. The function $f(z)$ is

$$f(z) = \left(1 - z^2\right)^{\frac{1}{2}}\,,$$

and is defined so that it is real and positive on the real axis in the range $-1 < x < 1$. Explain how $f(z)$ can be made single valued in the complex plane by the introduction of cuts running along the real axis $x < -1$ and $x > 1$. Evaluate $f(z)$ on the upper and lower sides of both cuts. [5]

Explain how to use a complex contour to evaluate the integral

$$I = \int_1^\infty \frac{\mathrm{d}x}{x(x^2-1)^{\frac{1}{2}}}\,.$$ [5]

Evaluate I. [10]

4A Let $f = u+iv$ be a complex differentiable function of a complex variable $z = x+iy$ where x, y, u and v are real. Derive the Cauchy-Riemann equations for u and v. [5]

Show that the Cauchy-Riemann equations imply that each of u and v satisfy Laplace's equation in two dimensions. [3]

Find the most general complex differentiable function f if

$$u = x^2 - y^2 \,,$$ [5]

and also if

$$u = x/(x^2 + y^2) \,.$$ [7]

5C Define the Laplace transform $\bar{f}(p)$ of a function $f(t)$ defined for $t \geq 0$. Derive expressions for the Laplace transform of $f'(t)$ and $tf(t)$. [5]

Evaluate $\bar{f}(p)$ when $f(t) = t^n$ for $n = 0, 1, 2 \ldots$. [2]

Find one solution to the equation

$$tg'' + (1-t)g' + 2g = 0 \,,$$

by first deriving an equation for $\bar{g}'(p)$ (assuming that it is defined), by then solving that equation for $\bar{g}(p)$, and finally by inverting the transform. [13]

[*You may quote the results that*

$$\int_0^\infty t^m \mathrm{e}^{-t}\,\mathrm{d}t = m! \qquad \text{and} \qquad \frac{p-3}{p^2-p} = \frac{3}{p} - \frac{2}{p-1} \,. \quad]$$

6C (a) Given a non-zero vector v_i , define

$$P_{ij} = \delta_{ij} - v_i v_j / v_k v_k \,,$$

where δ_{ij} is the kronecker delta.

Verify that P_{ij} is a tensor, and show for any vector u_i which is orthogonal to v_i that P_{ij} satisfies (i) $P_{ij} v_j = 0$ and (ii) $P_{ij} u_j = u_i$. [6]

(b) For any tensor T_{ik} in R^3, prove directly from the transformation property of a tensor that the quantities

$$\alpha = T_{ii}, \quad \beta = T_{ik} T_{ki}, \quad \gamma = T_{ik}\ T_{km}\ T_{mi}$$

are invariant under rotation of the coordinate axes. [6]

Suppose now that T_{ik} is a symmetric tensor. Express the invariants α, β and γ in terms of the tensor's eigenvalues. Deduce that the cubic equation for the eigenvalues λ is

$$\lambda^3 + c_2 \alpha \lambda^2 + c_1 (\alpha^2 - \beta) \lambda + c_0 (\alpha^3 - 3\alpha\beta + 2\gamma) = 0 \,,$$

where c_0, c_1 and c_2 are constants to be determined. [8]

[*The summation convention is assumed throughout this question.*]

7B Consider four point particles, each of mass m, at the corners of a square, each pair of particles being connected by a light unstretched spring of constant k (in total, therefore, the system has six springs). Take the origin of coordinates at the centre of the square with the four equilibrium points corresponding to $(-1, 1)$, $(1, 1)$, $(-1, -1)$, $(1, -1)$ for the first, second, third and fourth particle respectively. Each of these particles is displaced by (x_i, y_i), $i = 1, 2, 3, 4$ with respect to the equilibrium points. Find the potential energy matrix. [6]

(i) Show that the modes $(x_1, y_1, x_2, y_2, x_3, y_3, x_4, y_4) = (1, 0, 1, 0, 1, 0, 1, 0)$, $(0, 1, 0, 1, 0, 1, 0, 1)$, and $(1, 1, 1, -1, -1, 1, -1, -1)$ correspond to zero frequency oscillations and interpret the result. [4]

(ii) Determine the normal mode and frequency of oscillation corresponding to the 'breathing mode' in which all the particles move along radial lines away from the centre. [5]

(iii) Because of the reflection symmetry in the $x-$axis, the system is invariant under the double interchange of y_1 with $-y_3$ and y_2 with $-y_4$. Therefore seek an eigenvector of the form $(x_1, y_1, x_2, y_2, x_3, y_3, x_4, y_4) = (0, \alpha, 0, \beta, 0, -\alpha, 0, -\beta)$, and determine α, β and the frequency of oscillation. Deduce the normal mode and the frequency of oscillation for the motion with the reflection symmetry in the $y-$axis. [5]

[*Note: there is no need to solve the characteristic equation to find the eigenvalues from the eigenvalue equation.*]

8A List the axioms satisfied by the elements of a group. Define the terms *subgroup* and *coset*. Show that the order of a subgroup is a divisor of the order the group. [5]

By writing out the appropriate multiplication table verify that the set generated by multiplications of the matrices

$$A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad B = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix},$$

satisfies the group axioms. [5]

List all the subgroups of the group. [5]

Give a complete set of cosets based on one of the subgroups. [5]

9A Explain what is meant by the statement that there is a *homomorphism* from the group G to the group F. [4]

Define the *kernel*, K, of the map from G to F and show that K is a *normal subgroup* of G. [10]

Show that there is a homomorphism from the group S_3, the permutation group on three objects, to the group S_2, the permutation group on two objects. Describe the homomorphism in detail and identify its kernel. [6]

10A The dihedral group D_4 is the symmetry group of a square. List the *five* conjugacy classes of D_4, explaining the meaning of each element. [4]

Prove that the elements in each of the five classes are conjugate to each other. [6]

Calculate the character table of D_4. [10]

[*You may quote results based on the orthogonality theorem without proof.*]