Before you begin read these instructions carefully:

You may submit answers to no more than six questions. All questions carry the same number of marks.

The approximate number of marks allocated to a part of a question will be indicated in the right hand margin.

Write on one side of the paper only and begin each answer on a separate sheet.

At the end of the examination:

Each question has a number and a letter (for example, 6C).

Answers must be tied up in separate bundles, marked A, B or C according to the letter affixed to each question.

Do not join the bundles together.

For each bundle, a blue cover sheet must be completed and attached to the bundle.

A separate yellow master cover sheet listing all the questions attempted must also be completed.

Every cover sheet must bear your examination number and desk number.
In spherical polar coordinates \((r, \theta, \phi)\), Poisson’s equation for an axisymmetric electric potential \(\Phi(r, \theta)\) can be expressed as
\[
\frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \frac{\partial \Phi}{\partial r} \right) + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial \Phi}{\partial \theta} \right) = -\frac{\rho}{\epsilon_0},
\]
where \(\rho(r, \theta)\) is the electric charge density and \(\epsilon_0\) is a constant.

In the case when \(\rho = 0\), use the method of separation of variables to obtain the general solution for the potential \(\Phi\) in terms of Legendre polynomials \(P_n(\cos \theta)\). You should state clearly the differential equation in \(x\) satisfied by \(P_n(x)\). \[11\]

Suppose now that the charge density distribution is given by
\[
\rho = \begin{cases} 
4\epsilon_0 \cos \theta, & \text{in } 0 \leq r < a; \\
0, & \text{in } r \geq a.
\end{cases}
\]

On the assumptions that the potential and its derivative should be continuous everywhere, and that \(\Phi \to 0\) as \(r \to \infty\), find the potential everywhere. \[9\]

\[You \ may \ quote \ the \ result \ that \ P_1(x) = x.\]
For functions $\Phi$ and $\Psi$ defined in a volume $V$ bounded by a surface $S$ with outward unit normal $\mathbf{n}$, show that

$$\iiint_V \left( \Phi \nabla^2 \Psi - \Psi \nabla^2 \Phi \right) \, dV = \iint_S (\Phi \nabla \Psi - \Psi \nabla \Phi) \cdot \mathbf{n} \, dS.$$ \[2\]

Suppose that within the volume $V$, $\Phi$ satisfies Poisson’s equation

$$\nabla^2 \Phi = \sigma(\mathbf{x}),$$

where $\sigma(\mathbf{x})$ is a known function. State the equation satisfied by a Green’s function $G(\mathbf{x}; \mathbf{x}_0)$ for this equation, where $\mathbf{x}_0$ is a point within the volume $V$. Show that

$$\iint_S \nabla G \cdot \mathbf{n} \, dS = 1.$$ \[3\]

On the surface $S$, $\Phi$ satisfies the Neumann condition

$$\mathbf{n} \cdot \nabla \Phi = f(\mathbf{x}).$$

Choose, with justification, a suitable boundary condition for $G(\mathbf{x}; \mathbf{x}_0)$ on the surface $S$, and hence derive an integral expression that specifies $\Phi(\mathbf{x}_0)$ up to an undetermined constant. \[5\]

In the case when $V$ is the region $z \geq 0$, use the method of images to calculate $G(\mathbf{x}; \mathbf{x}_0)$. If $\sigma = 0$ throughout $V$, $\Phi \to 0$ as $|\mathbf{x}| \to \infty$, and on the surface $z = 0$

$$f = \delta(x)\delta(y),$$

find $\Phi$. \[10\]

The complex variable $z$ is given by $z = x + iy$ where $x$ and $y$ are real. The function $f(z)$ is

$$f(z) = (1 - z^2)^{\frac{1}{2}},$$

and is defined so that it is real and positive on the real axis in the range $-1 < x < 1$. Explain how $f(z)$ can be made single valued in the complex plane by the introduction of cuts running along the real axis $x < -1$ and $x > 1$. Evaluate $f(z)$ on the upper and lower sides of both cuts. \[5\]

Explain how to use a complex contour to evaluate the integral

$$I = \int_1^\infty \frac{dx}{x(x^2-1)^{\frac{1}{2}}}.$$ \[5\]

Evaluate $I$. \[10\]
4A Let \( f = u + iv \) be a complex differentiable function of a complex variable \( z = x + iy \) where \( x, y, u \) and \( v \) are real. Derive the Cauchy-Riemann equations for \( u \) and \( v \). [5]

Show that the Cauchy-Riemann equations imply that each of \( u \) and \( v \) satisfy Laplace’s equation in two dimensions. [3]

Find the most general complex differentiable function \( f \) if
\[
u = x^2 - y^2,
\] [5]

and also if
\[
u = x/(x^2 + y^2).
\] [7]

5C Define the Laplace transform \( \bar{f}(p) \) of a function \( f(t) \) defined for \( t \geq 0 \). Derive expressions for the Laplace transform of \( f'(t) \) and \( tf(t) \). [5]

Evaluate \( \bar{f}(p) \) when \( f(t) = t^n \) for \( n = 0, 1, 2, \ldots \). [2]

Find one solution to the equation
\[
tg'' + (1 - t)g' + 2g = 0,
\]
by first deriving an equation for \( \bar{g}'(p) \) (assuming that it is defined), by then solving that equation for \( \bar{g}(p) \), and finally by inverting the transform. [13]

[You may quote the results that
\[
\int_0^\infty t^m e^{-t} \, dt = m! \quad \text{and} \quad \frac{p - 3}{p^2 - p} = \frac{3}{p} - \frac{2}{p - 1}.
\] ]
(a) Given a non-zero vector $v_i$, define

$$P_{ij} = \delta_{ij} - v_i v_j / v_k v_k,$$

where $\delta_{ij}$ is the kronecker delta.

Verify that $P_{ij}$ is a tensor, and show for any vector $u_i$ which is orthogonal to $v_i$ that $P_{ij}$ satisfies (i) $P_{ij} v_j = 0$ and (ii) $P_{ij} u_j = u_i$.

(b) For any tensor $T_{ik}$ in $R^3$, prove directly from the transformation property of a tensor that the quantities

$$\alpha = T_{ii}, \quad \beta = T_{ik} T_{ki}, \quad \gamma = T_{ik} T_{km} T_{mi}$$

are invariant under rotation of the coordinate axes.

Suppose now that $T_{ik}$ is a symmetric tensor. Express the invariants $\alpha$, $\beta$ and $\gamma$ in terms of the tensor’s eigenvalues. Deduce that the cubic equation for the eigenvalues $\lambda$ is

$$\lambda^3 + c_2 \alpha \lambda^2 + c_1 (\alpha^2 - \beta) \lambda + c_0 (\alpha^3 - 3 \alpha \beta + 2 \gamma) = 0,$$

where $c_0$, $c_1$ and $c_2$ are constants to be determined.

[The summation convention is assumed throughout this question.]

7B Consider four point particles, each of mass $m$, at the corners of a square, each pair of particles being connected by a light unstretched spring of constant $k$ (in total, therefore, the system has six springs). Take the origin of coordinates at the centre of the square with the four equilibrium points corresponding to $(-1, 1), (1, 1), (-1, -1), (1, -1)$ for the first, second, third and fourth particle respectively. Each of these particles is displaced by $(x_i, y_i), i = 1, 2, 3, 4$ with respect to the equilibrium points. Find the potential energy matrix.

(i) Show that the modes $(x_1, y_1, x_2, y_2, x_3, y_3, x_4, y_4) = (1, 0, 1, 0, 1, 0, 1, 0), (0, 1, 0, 1, 0, 1, 0, 1), (1, 1, -1, -1, -1, 1, -1, -1)$ correspond to zero frequency oscillations and interpret the result.

(ii) Determine the normal mode and frequency of oscillation corresponding to the ‘breathing mode’ in which all the particles move along radial lines away from the centre.

(iii) Because of the reflection symmetry in the $x$–axis, the system is invariant under the double interchange of $y_1$ with $-y_3$ and $y_2$ with $-y_4$. Therefore seek an eigenvector of the form $(x_1, y_1, x_2, y_2, x_3, y_3, x_4, y_4) = (0, \alpha, 0, \beta, 0, -\alpha, 0, -\beta)$, and determine $\alpha, \beta$ and the frequency of oscillation. Deduce the normal mode and the frequency of oscillation for the motion with the reflection symmetry in the $y$–axis.

[Note: there is no need to solve the characteristic equation to find the eigenvalues from the eigenvalue equation.]
8A List the axioms satisfied by the elements of a group. Define the terms subgroup and coset. Show that the order of a subgroup is a divisor of the order the group. 

By writing out the appropriate multiplication table verify that the set generated by multiplications of the matrices

\[ A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \]

satisfies the group axioms.

List all the subgroups of the group.

Give a complete set of cosets based on one of the subgroups.

9A Explain what is meant by the statement that there is a homomorphism from the group \( G \) to the group \( F \).

Define the kernel, \( K \), of the map from \( G \) to \( F \) and show that \( K \) is a normal subgroup of \( G \).

Show that there is a homomorphism from the group \( S_3 \), the permutation group on three objects, to the group \( S_2 \), the permutation group on two objects. Describe the homomorphism in detail and identify its kernel.

10A The dihedral group \( D_4 \) is the symmetry group of a square. List the five conjugacy classes of \( D_4 \), explaining the meaning of each element.

Prove that the elements in each of the five classes are conjugate to each other.

Calculate the character table of \( D_4 \).

[You may quote results based on the orthogonality theorem without proof.]