UNIVERSITY OF CAMBRIDGE

NATURAL SCIENCES TRIPOS Part IB & II (General)

Monday 26 May 2003 1.30 to 4.30

MATHEMATICS (1)

Before you begin read these instructions carefully:

You may submit answers to no more than **six** *questions. All questions carry the same number of marks.*

The approximate number of marks allocated to a part of a question will be indicated in the right hand margin.

Write on **one** *side of the paper only and begin each answer on a separate sheet.*

At the end of the examination:

Each question has a number and a letter (for example, **6B***).*

Answers must be tied up in **separate** *bundles, marked* **A, B or C** *according to the letter affixed to each question.*

Do not join the bundles together.

For each bundle, a blue cover sheet must be completed and attached to the bundle.

A **separate** *yellow master cover sheet listing all the questions attempted* **must** *also be completed.*

Every cover sheet must bear your examination number and desk number.

1B Elliptic cylindrical coordinates (u, v, z) are defined in terms of the Cartesian coordinates (x, y, z) by

$$x = a \cosh u \cos v,$$

$$y = a \sinh u \sin v,$$

$$z = z,$$

where $u \geq 0$, $0 \leq v < 2\pi$, $-\infty < z < \infty$. Sketch the surfaces of constant u and the surfaces of constant v. Show that the coordinate surfaces intersect at right angles and thus these coordinates are orthogonal coordinates. [8]

Find the metric coefficients (h_u, h_v, h_z) defined by

$$|\mathrm{d}r|^2 = \mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2 = h_u^2 \mathrm{d}u^2 + h_v^2 \mathrm{d}v^2 + h_z^2 \mathrm{d}z^2\,,$$

and show that $h_u = h_v$. [4]

Obtain the divergence, curl and Laplace's equation in the coordinates (u, v, z). [8]

2B A bar of length L made of material with diffusivity ν, is initially at a temperature of 0°C. One end of the bar ($x = 0$) is held at 0°C and the other is supplied with heat at a constant rate per unit area H, so that

$$k \frac{\partial u}{\partial x}(L, t) = H\,,$$

where $u(x, t)$ is the temperature distribution and k is the thermal conductivity.

(i) State the equation with the appropriate boundary and initial conditions for the temperature distribution within the bar after a time t. [3]

(ii) Transform the problem into the equivalent homogeneous one by writing

$$u(x, t) = v(x, t) + w(x)\,,$$

where $w(x)$ satisfies the equation for the temperature distribution and its boundary conditions (but not the initial condition). [4]

(iii) Solve the resulting problem for $v(x, t)$ using the method of separation of variables. [13]

3B Define the convolution $h(t) = f * g$ of two functions $f(t)$ and $g(t)$ and prove the relation between the Fourier transforms of h, f and g. [5]

Find the Fourier transform of the unit rectangular distribution:

$$f(t) = \begin{cases} 1, & |t| < 1; \\ 0, & \text{otherwise.} \end{cases}$$ [3]

Determine the convolution of f with itself and deduce the transform of the convolution. [6]

Formulate Parseval's theorem and apply it to f and $f * f$ to obtain the values of

$$\int_{-\infty}^{\infty} \frac{\sin^2 k}{k^2}\, \mathrm{d}k$$

and

$$\int_{-\infty}^{\infty} \frac{\sin^4 k}{k^4}\, \mathrm{d}k\,.$$ [6]

4B The $n \times n$ matrix $\mathbf{A}$, with elements a_{ij}, is real and symmetric. Describe how a real orthogonal matrix $\mathbf{T}$ may be constructed such that $\mathbf{T}^{-1}\mathbf{A}\mathbf{T}$ is diagonal. [5]

Hence describe how to construct the orthogonal matrix $\mathbf{S}$ which, under a change of variables

$$\mathbf{y} = \mathbf{S}\mathbf{x},$$

transforms the quadratic form

$$\sum_{i,j=1}^{n} a_{ij} x_i x_j$$

into

$$\sum_{i=1}^{n} \lambda_i y_i^2\,,$$

where you should identify the λ_i. [5]

Show that the expression

$$x_1^2 + x_2^2 - 3x_3^2 + 2x_1x_2 + 6x_1x_3 - 6x_2x_3$$

can be transformed in this way into

$$2y_1^2 + 3y_2^2 - 6y_3^2,$$

giving the $y_i (i = 1, 2, 3)$ explicitly in terms of $x_i (i = 1, 2, 3)$. [10]

5B Prove that the eigenvalues of an Hermitian matrix are real. Prove also that the eigenvectors corresponding to distinct eigenvalues of an Hermitian matrix are orthogonal. [7]

Prove that the eigenvalues of a unitary matrix $\mathbf{U}$ have unit modulus. [3]

Suppose that a unitary matrix $\mathbf{U}$ is written as $\mathbf{A} + i\mathbf{B}$, where $\mathbf{A}$ and $\mathbf{B}$ are symmetric real matrices with non-degenerate eigenvalues.

(a) Show that $\mathbf{A}$ and $\mathbf{B}$ commute and $\mathbf{A}^2 + \mathbf{B}^2 = 1$. [4]

(b) Show that the eigenvectors of $\mathbf{A}$ are also eigenvectors of $\mathbf{B}$, and give the eigenvalues of $\mathbf{A}$ and $\mathbf{B}$ in terms of the eigenvalues of $\mathbf{U}$. [6]

6B Define the terms *ordinary point* and *regular singular point* for a second order linear differential equation, and explain briefly the reason for distinguishing between them. [4]

Find power series solutions about $z = 0$ of

$$y'' - 2zy' + \lambda y = 0.$$ [8]

For what values of λ does the equation possess a polynomial solution? Find the solutions corresponding to two eigenvalues λ of your choice. [8]

7A Consider the eigenvalue problem

$$y'' + \frac{2}{x}y' + \lambda y = 0\,,$$

where the function $y(x)$ satisfies the boundary conditions that $y(0)$ is finite and $y(\pi) = 0$. By expressing the equation in self-adjoint form, or otherwise, show that eigenfunctions $u(x)$ and $v(x)$ corresponding to distinct eigenvalues satisfy the orthogonality relation

$$\int_0^\pi x^2 u(x)v(x)\,\mathrm{d}x = 0\,.$$

[5]

By using the change of dependent variable $z = xy$, or otherwise, find the set of eigenfunctions and corresponding eigenvalues. [6]

Show that the solution $y(x)$ satisfying the equation

$$y'' + \frac{2}{x}y' - \beta y = 1\,, \quad \text{where} \quad \beta > 0\,,$$

and the boundary conditions that $y(0)$ is finite and $y(\pi) = 0$, has the form

$$y(x) = 2\sum_{n=1}^{\infty} \frac{(-1)^n}{n(n^2+\beta)}\frac{\sin nx}{x}\,.$$

[9]

8A Find the values of p for which $y = x^p$ is a solution of

$$\left(\frac{\mathrm{d}^2}{\mathrm{d}x^2} + \frac{1}{x}\frac{\mathrm{d}}{\mathrm{d}x} - \frac{1}{x^2}\right) y = 0\,.$$

[2]

The function $y(x)$ is defined on the range $x \geq 0$ and satisfies the differential equation

$$\left(\frac{\mathrm{d}^2}{\mathrm{d}x^2} + \frac{1}{x}\frac{\mathrm{d}}{\mathrm{d}x} - \frac{1}{x^2}\right) y = \mathrm{e}^{-x}\,, \qquad (*)$$

and the boundary conditions $y(0) = 0$, and $y(x) \to 0$ as $x \to \infty$. Find the Green's function for this problem. [7]

Use it to obtain the solution to equation $(*)$. [11]

9C Derive the Euler-Lagrange equation that makes the functional

$$I_1 = \int_a^b F_1(z, r(z), r'(z))\, \mathrm{d}z$$

stationary for given values of $r(a)$ and $r(b)$. [5]

Generalise this result to derive the Euler-Lagrange equations that make the functional

$$I_2 = \int_a^b F_2(z, r(z), r'(z), \theta(z), \theta'(z))\, \mathrm{d}z$$

stationary for given values of $r(a)$, $r(b)$, $\theta(a)$ and $\theta(b)$. [3]

With respect to cylindrical polar coordinates (r, θ, z), an optical medium has a variable refractive index $\mu(r, \theta)$ independent of z. Show that if light moves along a path specified by $r = r(z)$, $\theta = \theta(z)$ then

$$(r'^2 + r^2\theta'^2 + 1)\, \mu^{-2}$$

is constant along the path. [6]

Show that if

$$\mu \equiv \mu(r)\,,$$

then light can travel along paths with constant radius. Find the solution for $\theta(z)$ in this case and describe the path. [6]

10C For functions $y(x)$, $p(x)$, $q(x)$ and $w(x)$ defined on $a \leq x \leq b$, let

$$F[y] = \int_a^b \left(py'^2 + qy^2\right)\,\mathrm{d}x\,, \qquad G[y] = \int_a^b wy^2\,\mathrm{d}x\,,$$

and

$$\Lambda[y] = \frac{F[y]}{G[y]}\,.$$

On the assumption that $p > 0$ and $w > 0$ for $a < x < b$, and subject to suitable boundary conditions which should be stated, show that the stationary values of $\Lambda[y]$ are the eigenvalues, λ, of the Sturm-Liouville differential equation

$$-\frac{\mathrm{d}}{\mathrm{d}x}\left(p(x)\frac{\mathrm{d}y}{\mathrm{d}x}\right) + q(x)y - \lambda w(x)y = 0\,.$$

Show also that the functions for which $\Lambda[y]$ is stationary are the corresponding eigenfunctions. [7]

Consider now the choice $p = 1$, $q = 0$ and $w = x$ (for $0 \leq x \leq \pi$), so that $y(x)$ satisfies the differential equation

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + \lambda xy = 0\,.$$

By use of a suitable polynomial trial function find an estimate of the lowest eigenvalue, λ_0, for the choice of boundary conditions $y(0) = y(\pi) = 0$. Also find an estimate of the lowest eigenvalue using a suitable trigometric trial function. [10]

Indicate, with reasoning, whether these estimates are greater or less than λ_0 and why you might expect these estimates to be close to each other. [3]