MATHEMATICS (2)

Before you begin read these instructions carefully:

You may submit answers to no more than six questions. All questions carry the same number of marks.

The approximate number of marks allocated to a part of a question is indicated in the right hand margin.

Write on one side of the paper only and begin each answer on a separate sheet.

Questions marked with an asterisk (*) require a knowledge of B course material.

At the end of the examination:

Each question has a number and a letter (for example 3B).

Answers must be tied up in separate bundles, marked A, B, C, D, E or F according to the letter affixed to each question. Do not join the bundles together.

For each bundle, a blue cover sheet must be completed and attached to the bundle, with the appropriate letter written in the section box.

A separate yellow master cover sheet listing all the questions attempted must also be completed.

Every cover sheet must bear your examination number and desk number.
1A

(a) Find the equation of the plane that is perpendicular to the vector \(-\mathbf{i} - 2\mathbf{j} + 2\mathbf{k}\) and passes through the point A \((2, -1, 1)\). Determine the shortest distance of this plane from the origin. \([8]\)

(b) A second plane passes through the points A, B \((1, 1, -1)\) and C \((3, -1, 2)\). Find a unit vector normal to this plane, and a vector along the line of intersection of the two planes. \([12]\)

2A

Show that there are no constant or cosine terms in the Fourier series for an odd function over the range \(-\pi \leq x \leq \pi\). \([8]\)

Find the Fourier series for the function \(f(x) = \sin ax\), where \(-\pi \leq x \leq \pi\) and \(a\) is not an integer. \([12]\)

[You may find the following trigonometric identities useful:
\[
\begin{align*}
\cos(\theta \pm \phi) &= \cos \theta \cos \phi \mp \sin \theta \sin \phi, \\
\sin(\theta \pm \phi) &= \sin \theta \cos \phi \pm \cos \theta \sin \phi.
\end{align*}
\]
]

3B

(a) Express the real matrix
\[
M = \begin{pmatrix} 0 & b \\ a & 0 \end{pmatrix}
\]
as the sum of a real symmetric matrix, \(S\), and a real antisymmetric matrix \(A\).

Determine the values of \(a\) and \(b\) for which both \(S\) and \(A\) are orthogonal matrices. \([12]\)

(b) Determine the eigenvalues and normalised eigenvectors of the matrix
\[
\begin{pmatrix} 5 & -2 \\ -2 & 2 \end{pmatrix}.
\]

Verify that the eigenvectors are orthogonal. \([8]\)
4B*

(a) State carefully the divergence theorem and Stokes’ theorem.

(b) In Cartesian coordinates and components, the vector field \( \mathbf{F} \) is given by

\[
\mathbf{F} = (x^2yz, xy^2z, xyz^2).
\]

Evaluate \( \int_S \mathbf{F} \cdot d\mathbf{S} \), where \( S \) is the surface of the cube

\[
0 \leq x \leq 1, \quad 0 \leq y \leq 1, \quad 0 \leq z \leq 1.
\]

(c) In Cartesian coordinates and components, the vector field \( \mathbf{G} \) is given by

\[
\mathbf{G} = (4y, 3x, 2z).
\]

Evaluate \( \int_S (\nabla \times \mathbf{G}) \cdot d\mathbf{S} \), where \( S \) is the open hemispherical surface

\[
x^2 + y^2 + z^2 = r^2, \quad z \geq 0.
\]

5C

(a) It is known that \( n \) people out of a population of \( N \) suffer from a certain disease, and that the other \( N - n \) people do not. The test for the disease has a probability \( a \) of producing a correct positive result when used on a sufferer and a probability \( b \) of producing a false positive result when used on a non-sufferer. The test is positive when done on me. What is the probability that I am a sufferer?

(b) A random variable \( X \) has density function \( f(t) \) given by

\[
f(t) = Ae^{-kt}, \quad \text{for} \quad t \geq 0,
\]

where \( A \) and \( k \) are constants. Find, in terms of \( k \):

(i) the value of \( A \);

(ii) the probability that \( X \geq 3 \) given that \( X \geq 1 \);

(iii) the expectation value of \( X \).
6C Solve the equations

\[\begin{align*}
3x + 2y + 7z &= a, \\
x + 4y + 6z &= b, \\
y + z &= c,
\end{align*}\]

finding \(x, y\) and \(z\) in terms of \(a, b\) and \(c\). \[12\]

Write the equations in matrix form \(Ax = d\), where

\[
\mathbf{x} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \quad \text{and} \quad \mathbf{d} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}.
\]

Hence find

\[
\begin{pmatrix} 3 & 2 & 7 \\ 1 & 4 & 6 \\ 0 & 1 & 1 \end{pmatrix}^{-1}.
\]

\[6\]

7D Let \(E\) be the ellipsoid

\[\frac{x^2}{a^2} + \frac{y^2}{b^2} + z^2 = 1,\]

where \(a > \sqrt{2}\) and \(b > \sqrt{2}\). Find the normal to the surface of \(E\). \[5\]

Let \(S\) be the part of the surface of \(E\) defined by

\[0 \leq x \leq 1, \quad 0 \leq y \leq 1, \quad \text{and} \quad z > 0,\]

and let \(\mathbf{F}\) be the vector field defined by \(\mathbf{F} = (-y, x, 0)\). Explain why \(\int_S \mathbf{F} \cdot d\mathbf{S} = 0\) in the case \(a = b\). \[5\]

Given that the surface area element of \(S\) is given by

\[
d\mathbf{S} = \left(\frac{x}{a^2 z}, \frac{y}{b^2 z}, 1\right) \, dx \, dy,
\]

find \(\int_S \mathbf{F} \cdot d\mathbf{S}\) in the case \(a \neq b\). \[10\]

Paper 2
(a) The $n \times n$ matrix $P$ has components $P_{ij} = \delta_{ij} - a_i a_j - b_i b_j$ where $n > 1$ and $a_i$ and $b_i$ are the components of two orthogonal unit vectors $a$ and $b$, respectively.

(i) Evaluate $\text{tr}(P)$.

(ii) Using the summation convention, or otherwise, show that $P^2 = P$.

(iii) Evaluate $Pa$ and hence find $\det(P)$. [12]

(b) Let $S$ be a symmetric $n \times n$ matrix and let $A$ be an antisymmetric $n \times n$ matrix.

(i) Show that $S^2$ is symmetric.

(ii) Show that $\text{tr}(SA) = 0$. [8]
9E

(a) The internal energy $U$ of a gas can be regarded as a function of the entropy $S$ and the volume $V$. It is given that

$$dU = TdS - pdV,$$

where $T$ and $p$ denote temperature and pressure, respectively. The enthalpy $H$ is defined by

$$H = U + pV.$$

Show that

$$\left( \frac{\partial T}{\partial V} \right)_S = - \left( \frac{\partial p}{\partial S} \right)_V$$

and also that

$$\left( \frac{\partial T}{\partial p} \right)_S = \left( \frac{\partial V}{\partial S} \right)_p.$$

(b) The heat capacities of a gas at constant pressure, $C_p$, and constant volume, $C_V$, are defined by

$$C_p = \left( \frac{\partial H}{\partial T} \right)_p, \quad C_V = \left( \frac{\partial U}{\partial T} \right)_V.$$

Using the definition of $H$ above, show that

$$C_p - C_V = \left( \frac{\partial U}{\partial T} \right)_p + p \left( \frac{\partial V}{\partial T} \right)_p - \left( \frac{\partial U}{\partial T} \right)_V.$$

By considering $U$ as a function of $T$ and $V$, show that

$$\left( \frac{\partial U}{\partial T} \right)_p = \left( \frac{\partial U}{\partial V} \right)_T \left( \frac{\partial V}{\partial T} \right)_p + \left( \frac{\partial U}{\partial T} \right)_V$$

and hence that

$$C_p - C_V = \left[ \left( \frac{\partial U}{\partial V} \right)_T + p \right] \left( \frac{\partial V}{\partial T} \right)_p.$$

Evaluate $C_p - C_V$ for an ideal gas, for which

$$\left( \frac{\partial U}{\partial V} \right)_T = 0 \quad \text{and} \quad pV = NkT,$$

where $N$ is the number of gas molecules present in volume $V$ and $k$ is Boltzmann’s constant.
10E

(a) The distance \( r(x, y) \) of the point \((x, y)\) from the origin of two-dimensional Cartesian space is given by

\[
r(x, y) = \sqrt{x^2 + y^2}.
\]

Obtain formulae for \( \frac{\partial r}{\partial x} \), \( \frac{\partial r}{\partial y} \) and also for \( \frac{\partial^2 r}{\partial x^2} \), \( \frac{\partial^2 r}{\partial y^2} \) and \( \frac{\partial^2 r}{\partial x \partial y} \).

Without further calculation, write down a formula for \( \frac{\partial^2 r}{\partial y \partial x} \).

(b) Let \( g(x, y) \) be a function defined on the \( xy \)-plane. The coordinates \( u \) and \( v \) are defined by

\[
\begin{align*}
  u &= x \cos \theta + y \sin \theta, \\
  v &= -x \sin \theta + y \cos \theta,
\end{align*}
\]

where \( \theta \) is a constant. Show that

\[
\frac{\partial^2 g}{\partial x^2} + \frac{\partial^2 g}{\partial y^2} = \frac{\partial^2 g}{\partial u^2} + \frac{\partial^2 g}{\partial v^2}.
\]

11F Consider the differential equation

\[
\frac{d^2 y}{dx^2} - 2 \frac{dy}{dx} + y = 2x \sin x.
\]

Find a particular solution, of the form

\[
y(x) = (a + bx) \sin x + (c + dx) \cos x,
\]

where the constants \( a, b, c, \) and \( d \) are to be determined.

Hence find the solution \( y(x) \) that satisfies the initial conditions \( y = 0 \) and \( dy/dx = 0 \) at \( x = 0 \).
Consider the stationary points of the function
\[ u(x, y) = 2(x - y)^2 + (x + y)^2, \]
subject to the constraint \( v(x, y) = 0 \), where
\[ v(x, y) = x^2 - y - \frac{1}{4}. \]

By considering the function \((u - \lambda v)\), where \(\lambda\) is a Lagrange multiplier, show that any stationary points \((x, y)\) satisfy:
\[ x = \frac{\lambda}{(6\lambda - 16)}, \quad y = \frac{\lambda(3 - \lambda)}{(6\lambda - 16)}. \]  

Deduce that \(\lambda\) obeys the cubic equation
\[ 3\lambda^3 - 21\lambda^2 + 48\lambda - 32 = 0. \]

You may assume that this cubic equation has only one real root (so that there is a unique stationary point). Sketch the contours of \(u\) in relation to the curve \(C\) given by \(v = 0\), and explain why the function \(u\) takes its minimum value on \(C\) at this stationary point.