UNIVERSITY OF CAMBRIDGE

NATURAL SCIENCES TRIPOS Part IB & II (General)

Monday 27 May 2002 1.30 to 4.30

MATHEMATICS (1)

Before you begin read these instructions carefully:

You may submit answers to no more than ***six*** *questions. All questions carry the same number of marks.*

The approximate number of marks allocated to a part of a question will be indicated in the right hand margin.

Write on **one** *side of the paper only and begin each answer on a separate sheet.*

At the end of the examination:

Each question has a number and a letter (for example, ***6C****).*

Answers must be tied up in ***separate*** *bundles, marked* ***A, B or C*** *according to the letter affixed to each question.*

Do not join the bundles together.

For each bundle, a blue cover sheet must be completed and attached to the bundle.

A ***separate*** *yellow master cover sheet listing all the questions attempted* ***must*** *also be completed.*

Every cover sheet must bear your examination number and desk number.

1A In a curvilinear system a point P has coordinates (u, v, w) such that

$$u^2 = \frac{1}{2}(s + x), \quad v^2 = \frac{1}{2}(s - x) \quad \text{and} \quad w = z,$$

where (x, y, z) are the rectangular Cartesian coordinates of P and $s = \sqrt{x^2 + y^2}$ is its distance from the z-axis. Describe the surfaces $u = \text{const}$, $v = \text{const}$ and $w = \text{const}$ and sketch the loci of intersections of $u^2 = 0,\ 1,\ 2$ and $v^2 = 0,\ 1,\ 2$ with the x, y plane. [5]

Express x, y and z explicitly in terms of u, v and w in such a way that, when u is defined so that $u \geq 0$, y and v have the same sign and the point P is uniquely determined by u, v and w. [4]

Show that u, v and w are orthogonal curvilinear coordinates and find the coefficients h_u, h_v and h_w such that dl, the distance between points (u, v, w) and $(u + du, v + dv, w + dw)$, is given by

$$dl^2 = h_u^2 du^2 + h_v^2 dv^2 + h_w^2 dw^2$$

in the limit $dl \to 0$. [7]

If $\phi = \phi(u, v)$ only express $\nabla^2\phi$ in terms of derivatives with respect to u and v. [4]

[*You may use the following formulae*

$$\nabla\phi = \frac{1}{h_u}\frac{\partial\phi}{\partial u}\boldsymbol{e}_u + \frac{1}{h_v}\frac{\partial\phi}{\partial v}\boldsymbol{e}_v + \frac{1}{h_w}\frac{\partial\phi}{\partial w}\boldsymbol{e}_w$$

and

$$\nabla.\boldsymbol{A} = \frac{1}{h_u h_v h_w}\left\{\frac{\partial}{\partial u}(h_v h_w A_u) + \frac{\partial}{\partial v}(h_w h_u A_v) + \frac{\partial}{\partial w}(h_u h_v A_w)\right\}.]$$

2A The temperature in an insulated silver rod placed along the x-axis from $x = 0$ to $x = l$ obeys the diffusion equation

$$\frac{\partial T}{\partial t} = \nu \frac{\partial^2 T}{\partial x^2}.$$

At $t = 0$ the rod is uniformly of temperature $T = 0$. For $t > 0$ the end at $x = 0$ is held at $T = 0$ while the end at $x = l$ is held at $T = T_0$.

What is the steady state temperature distribution along the rod at large t? [5]

By looking for separable solutions of the form $T(x,t) = \xi(x)\theta(t)$ find the temperature distribution for any $t > 0$. [10]

[*Hint: You can use the steady state solution as the boundary condition when* $t \to \infty$ *to determine the spatial eigenfunctions.*]

The heat capacity of the bar is γ per unit length so that the total heat in the bar is

$$H = \int_0^l \gamma T \, dx.$$

Find the rate at which heat is absorbed by the bar at time $t > 0$. If your answer is an infinite series ensure that it converges. [5]

3A Define the Fourier transform $\tilde{f}(k)$ of a function $f(x)$ and write down its inverse. [3]

Show that the Fourier transform of $\frac{d^2 f}{dx^2}$ is

$$-k^2 \tilde{f}(k).$$

[4]

By taking the Fourier transform with respect to x find the function $\phi(x,y)$, $-\infty < x < \infty$, $0 \le y < \infty$ that satisfies Laplace's equation

$$\nabla^2 \phi = \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0,$$

with $\phi(x,0) = \delta(x)$ and $\phi(x,y) \to 0$ as $y \to \infty$. [9]

Verify your solution satisfies $\nabla^2 \phi = 0$. [4]

4B Suppose $\mathbf{A}$ is an $n \times n$ matrix:

Prove that $\mathbf{A}^n = \mathbf{0}$ if and only if all eigenvalues of $\mathbf{A}$ vanish.

[You may use the fact that a matrix satisfies its own characteristic equation] [5]

By considering the quadratic form $Q = \mathbf{x}^T \mathbf{A} \mathbf{x}$, or otherwise, prove that if the matrix $\mathbf{A}$ is symmetric then $\mathbf{A}^n = \mathbf{0}$ implies $\mathbf{A} = \mathbf{0}$. [5]

For the following matrix

$$\mathbf{T} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & \alpha \\ 1 & 1 & \beta \end{pmatrix}$$

find the values of α and β for which all the eigenvalues of $\mathbf{T}$ vanish and verify that in that case $\mathbf{T}^3 = \mathbf{0}$. [5]

Consider the transformation $\mathbf{x}' = \mathbf{T}\mathbf{x}$ for three dimensional vectors $\mathbf{x}$ and $\mathbf{x}'$. Show that all $\mathbf{x}'$ are confined to a single plane for any given $\mathbf{x}$.

What is the effect of $\mathbf{T}^2$ and $\mathbf{T}^3$ operating on $\mathbf{x}$? [5]

5B (a) Let $\mathbf{A}$ and $\mathbf{B}$ be $n \times n$ Hermitian matrices ($\mathbf{A} = \mathbf{A}^\dagger$, $\mathbf{B} = \mathbf{B}^\dagger$) with distinct eigenvalues. Show that:

(i) $\mathbf{H} = i\,(\mathbf{AB} - \mathbf{BA})$ is Hermitian, [4]

(ii) the eigenvectors of $\mathbf{A}$ and $\mathbf{B}$ are identical if and only if $\mathbf{AB} = \mathbf{BA}$, [6]

(iii) the matrix $\mathbf{N} = \mathbf{A} + i\mathbf{B}$ can be diagonalised if and only if $\mathbf{N}\mathbf{N}^\dagger = \mathbf{N}^\dagger\mathbf{N}$. [5]

(b) If $\mathbf{C}$ is a unitary matrix and $\mathbf{A}$ is Hermitian show that $\left(\mathbf{C}^{-1}\mathbf{A}\mathbf{C}\right)^n$ has real eigenvalues if n is a positive integer. [5]

[*You may quote the properties of the eigenvalue and eigenvectors of Hermitian matrices without proof.*]

6C The differential equation

$$(1-x^2)y'' - xy' + m^2 y = 0\,, \qquad (*)$$

has two linearly independent solutions about the origin of the form $y = x^\sigma \sum_{n=0}^{\infty} a_n x^n$ (with $a_0 \neq 0$).

(a) Is the origin $x = 0$ an ordinary or singular point of this differential equation? Determine the two appropriate values of σ and find recurrence relations between the a_n's for the two cases. [8]

(b) Show that if m is an integer then there always exists a polynomial solution, denoted by $T_m(x)$. How many non-zero terms do these polynomials contain? Find the first four polynomials $T_m(x)$ (i.e. $m = 0, 1, 2, 3$) given the normalization $T_m(1) = 1$. [7]

(c) Use the ratio test to discuss the convergence of the non-polynomial solutions on the interval $-1 \leq x \leq 1$. Comment on the relationship of the radius of convergence R to the location of the singular points of the differential equation $(*)$. [5]

7C (a) Consider the general eigenvalue equation

$$y'' - b(x)y' + c(x)y = -\lambda d(x) y\,,$$

subject to homogeneous boundary conditions $y(0) = y(1) = 0$. Find suitable functions $p(x)$, $q(x)$ and $w(x)$ which enable this equation to be re-expressed in Sturm-Liouville form

$$-\left(p(x)y'\right)' + q(x)y = \lambda w(x) y\,.$$ [4]

(b) Find eigenfunctions and eigenvalues for the equation

$$y'' + \lambda y = 0\,, \qquad (\dagger)$$

subject to the boundary conditions $y(0) = 0$ and $y'(\pi/2) = 0$. Determine an appropriate normalization for these eigenfunctions. [5]

Use these to obtain an eigenfunction expansion as a solution of the inhomogeneous equation

$$y'' + \kappa y = x\,,$$

subject to the same boundary conditions as in $(\dagger)$ and where κ is a constant (not an eigenvalue). [7]

Hence, by choosing appropriate values for κ and x (or otherwise), show that

$$\frac{\pi^4}{96} = 1 + \frac{1}{3^4} + \frac{1}{5^4} + \frac{1}{7^4} + \ldots$$

[4]

8C Consider the inhomogeneous differential equation

$$y'' - \frac{1}{x}y' + \frac{1}{x^2}y = f(x)\,, \qquad (\ddagger)$$

subject to the boundary conditions $y(1) = y(e) = 0$ where $e = 2.718...$

(a) Given that one solution of the homogeneous equation is $y = x$, substitute $y(x) = x\,v(x)$ (or otherwise) to find the second homogeneous solution. [4]

(b) Construct the Green's function $G(x, \xi)$ using $f(x) = \delta(x - \xi)$ in the inhomogeneous equation ($\ddagger$) and subject to the same boundary conditions. [10]

(c) Hence show that the solution of ($\ddagger$) is given by

$$y = x \ln x \int_x^e (\ln \xi - 1) f(\xi) d\xi + (x \ln x - x) \int_1^x \ln \xi \, f(\xi) d\xi\,.$$

Find y explicitly for $f(x) = 1/x$ and verify your solution. [6]

9C Consider a set of eigenfunctions $y_n(x)$, (with $n = 0, 1, 2 \ldots$) and corresponding eigenvalues λ_n which satisfy the Sturm-Liouville equation

$$\mathcal{L}y \equiv -\left[p(x)\, y'\right]' + q(x)y = \lambda w(x)y\,,$$

with boundary conditions $y(0) = y(1) = 0$. Assume that the eigenfunctions are unit normalized, that is, $\int_0^1 y_n^2 w dx = 1$.

(a) Given an arbitrary function $\tilde{y}$ satisfying the same boundary conditions, we can define

$$F_n(x) \equiv \mathcal{L}\tilde{y} - \lambda_n w(x)\tilde{y}\,.$$

Show that for every n we must have

$$\int_0^1 y_n(x) F_n(x) dx = 0\,.$$

[6]

(b) Consider an approximate trial function $\tilde{y}$ (with $\tilde{y}(0) = \tilde{y}(1) = 0$) which is close to the lowest eigenfunction y_0 so that we can expand the difference in terms of the remaining eigenfunctions as

$$\tilde{y} = y_0 + \sum_{n=1}^{\infty} a_n y_n\,.$$

where the a_n's can be assumed small. Use the Rayleigh-Ritz method with the trial function $\tilde{y}$ to show that the lowest eigenvalue λ_0 can be approximated as

$$\tilde{\lambda}_0 = \frac{\lambda_0 + \sum_{n=1}^{\infty} a_n^2 \lambda_n}{1 + \sum_{n=1}^{\infty} a_n^2} \;\approx\; \lambda_0 + \sum_{n=1}^{\infty} a_n^2 (\lambda_n - \lambda_0)\,,$$

where terms of order $\mathcal{O}(a_n^4)$ have been neglected. [10]

Discuss the relative error in the approximation $\tilde{y}$ to the lowest eigenfunction y_0 compared with the error in the approximation $\tilde{\lambda}_0$ to the lowest eigenvalue λ_0. Is $\tilde{\lambda}_0$ lower or higher than λ_0? [4]

10C (a) Derive the Euler-Lagrange equation satisfied by the function $y(x)$ which makes the integral

$$I = \int_a^b F[x, y(x), y'(x)]dx$$

stationary subject to the boundary conditions $y(a) = y_1$ and $y(b) = y_2$. [7]

(b) Efficient international airline routes must minimize the distance between two locations on the globe.

Show that the path length between two points $A(\phi_1, \theta_1)$ and $B(\phi_2, \theta_2)$ on a unit sphere can be expressed in polar coordinates (ϕ, θ) as

$$S = \int_{\theta_1}^{\theta_2} \sqrt{1 + \sin^2\theta\, \phi'^2}\, d\theta\,.$$

where $\phi' = \frac{d\phi}{d\theta}$. [4]

Hence, use the Euler-Lagrange equation to show that a stationary path satisfies

$$\phi(\theta) = \pm \int_{\theta_1}^{\theta_2} \frac{d\theta}{\sin\theta \left(k^2 \sin^2\theta - 1\right)^{1/2}}\,,$$

where k is a constant. [4]

Integrate this expression to find the extremal solution. Use this to specify the shortest route for a flight setting out from Rome ($\theta \approx 45°$, $\phi \approx 15°$) and heading for Singapore ($\theta \approx 90°$, $\phi \approx 105°$).

[*Hint: In the integration consider the substitution* $t = \cot\theta$.] [5]