MATHEMATICS (1)

Before you begin read these instructions carefully:

You may submit answers to no more than six questions. All questions carry the same number of marks.

The approximate number of marks allocated to a part of a question will be indicated in the right hand margin.

Write on one side of the paper only and begin each answer on a separate sheet.

At the end of the examination:

Each question has a number and a letter (for example, 6C).

Answers must be tied up in separate bundles, marked A, B or C according to the letter affixed to each question.

Do not join the bundles together.

For each bundle, a blue cover sheet must be completed and attached to the bundle.

A separate yellow master cover sheet listing all the questions attempted must also be completed.

Every cover sheet must bear your examination number and desk number.
1A  In a curvilinear system a point $P$ has coordinates $(u, v, w)$ such that

$$u^2 = \frac{1}{2}(s + x), \quad v^2 = \frac{1}{2}(s - x) \quad \text{and} \quad w = z,$$

where $(x, y, z)$ are the rectangular Cartesian coordinates of $P$ and $s = \sqrt{x^2 + y^2}$ is its distance from the $z$-axis. Describe the surfaces $u = \text{const}$, $v = \text{const}$ and $w = \text{const}$ and sketch the loci of intersections of $u^2 = 0, 1, 2$ and $v^2 = 0, 1, 2$ with the $x, y$ plane. \[5\]

Express $x$, $y$ and $z$ explicitly in terms of $u$, $v$ and $w$ in such a way that, when $u$ is defined so that $u \geq 0$, $y$ and $v$ have the same sign and the point $P$ is uniquely determined by $u$, $v$ and $w$. \[4\]

Show that $u$, $v$ and $w$ are orthogonal curvilinear coordinates and find the coefficients $h_u$, $h_v$ and $h_w$ such that $dl$, the distance between points $(u, v, w)$ and $(u + du, v + dv, w + dw)$, is given by

$$dl^2 = h_u^2 du^2 + h_v^2 dv^2 + h_w^2 dw^2$$

in the limit $dl \to 0$. \[7\]

If $\phi = \phi(u, v)$ only express $\nabla^2 \phi$ in terms of derivatives with respect to $u$ and $v$. \[4\]

[You may use the following formulae

$$\nabla \phi = \frac{1}{h_u} \frac{\partial \phi}{\partial u} e_u + \frac{1}{h_v} \frac{\partial \phi}{\partial v} e_v + \frac{1}{h_w} \frac{\partial \phi}{\partial w} e_w$$

and

$$\nabla \cdot A = \frac{1}{h_u h_v h_w} \left\{ \frac{\partial}{\partial u} (h_v h_w A_u) + \frac{\partial}{\partial v} (h_w h_u A_v) + \frac{\partial}{\partial w} (h_u h_v A_w) \right\}.$$

]
2A  The temperature in an insulated silver rod placed along the x-axis from \( x = 0 \) to \( x = l \) obeys the diffusion equation

\[
\frac{\partial T}{\partial t} = \nu \frac{\partial^2 T}{\partial x^2}.
\]

At \( t = 0 \) the rod is uniformly of temperature \( T = 0 \). For \( t > 0 \) the end at \( x = 0 \) is held at \( T = 0 \) while the end at \( x = l \) is held at \( T = T_0 \).

What is the steady state temperature distribution along the rod at large \( t \)? [5]

By looking for separable solutions of the form \( T(x,t) = \xi(x) \theta(t) \) find the temperature distribution for any \( t > 0 \). [10]

[Hint: You can use the steady state solution as the boundary condition when \( t \to \infty \) to determine the spatial eigenfunctions.]

The heat capacity of the bar is \( \gamma \) per unit length so that the total heat in the bar is

\[
H = \int_0^l \gamma T \, dx.
\]

Find the rate at which heat is absorbed by the bar at time \( t > 0 \). If your answer is an infinite series ensure that it converges. [5]

3A  Define the Fourier transform \( \tilde{f}(k) \) of a function \( f(x) \) and write down its inverse. [3]

Show that the Fourier transform of \( \frac{d^2 f}{dx^2} \) is

\[
-k^2 \tilde{f}(k).
\] [4]

By taking the Fourier transform with respect to \( x \) find the function \( \phi(x,y) \), \(-\infty < x < \infty, 0 \leq y < \infty \) that satisfies Laplace’s equation

\[
\nabla^2 \phi = \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0,
\]

with \( \phi(x,0) = \delta(x) \) and \( \phi(x,y) \to 0 \) as \( y \to \infty \). [9]

Verify your solution satisfies \( \nabla^2 \phi = 0 \). [4]
4B  Suppose $A$ is an $n \times n$ matrix:

Prove that $A^n = 0$ if and only if all eigenvalues of $A$ vanish.

[You may use the fact that a matrix satisfies its own characteristic equation]

By considering the quadratic form $Q = x^T A x$, or otherwise, prove that if the matrix $A$ is symmetric then $A^n = 0$ implies $A = 0$.

For the following matrix

$$ T = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & \alpha \\ 1 & 1 & \beta \end{pmatrix} $$

find the values of $\alpha$ and $\beta$ for which all the eigenvalues of $T$ vanish and verify that in that case $T^3 = 0$.

Consider the transformation $x' = Tx$ for three dimensional vectors $x$ and $x'$. Show that all $x'$ are confined to a single plane for any given $x$.

What is the effect of $T^2$ and $T^3$ operating on $x$?

5B  (a) Let $A$ and $B$ be $n \times n$ Hermitian matrices ($A = A^\dagger$, $B = B^\dagger$) with distinct eigenvalues. Show that:

(i) $H = i(AB - BA)$ is Hermitian, 

(ii) the eigenvectors of $A$ and $B$ are identical if and only if $AB = BA$, 

(iii) the matrix $N = A + iB$ can be diagonalised if and only if $NN^\dagger = N^\dagger N$.

(b) If $C$ is a unitary matrix and $A$ is Hermitian show that $(C^{-1}AC)^n$ has real eigenvalues if $n$ is a positive integer.

[You may quote the properties of the eigenvalue and eigenvectors of Hermitian matrices without proof.]
6C  The differential equation

\[(1 - x^2)y'' - xy' + m^2 y = 0, \quad (*)\]

has two linearly independent solutions about the origin of the form \( y = x^\sigma \sum_{n=0}^{\infty} a_n x^n \) (with \( a_0 \neq 0 \)).

(a) Is the origin \( x = 0 \) an ordinary or singular point of this differential equation? Determine the two appropriate values of \( \sigma \) and find recurrence relations between the \( a_n \)'s for the two cases. \[8\]

(b) Show that if \( m \) is an integer then there always exists a polynomial solution, denoted by \( T_m(x) \). How many non-zero terms do these polynomials contain? Find the first four polynomials \( T_m(x) \) (i.e. \( m = 0, 1, 2, 3 \)) given the normalization \( T_m(1) = 1 \). \[7\]

(c) Use the ratio test to discuss the convergence of the non-polynomial solutions on the interval \( -1 \leq x \leq 1 \). Comment on the relationship of the radius of convergence \( R \) to the location of the singular points of the differential equation \((*)\). \[5\]

7C  (a) Consider the general eigenvalue equation

\[ y'' - b(x)y' + c(x)y = -\lambda d(x)y, \]

subject to homogeneous boundary conditions \( y(0) = y(1) = 0 \). Find suitable functions \( p(x), q(x) \) and \( w(x) \) which enable this equation to be re-expressed in Sturm-Liouville form

\[- (p(x)y')' + q(x)y = \lambda w(x)y. \quad [4]\]

(b) Find eigenfunctions and eigenvalues for the equation

\[ y'' + \lambda y = 0, \quad (\dagger) \]

subject to the boundary conditions \( y(0) = 0 \) and \( y'(\pi/2) = 0 \). Determine an appropriate normalization for these eigenfunctions. \[5\]

Use these to obtain an eigenfunction expansion as a solution of the inhomogeneous equation

\[ y'' + \kappa y = x, \]

subject to the same boundary conditions as in \((\dagger)\) and where \( \kappa \) is a constant (not an eigenvalue). \[7\]

Hence, by choosing appropriate values for \( \kappa \) and \( x \) (or otherwise), show that

\[ \frac{\pi^4}{96} = 1 + \frac{1}{3^4} + \frac{1}{5^4} + \frac{1}{7^4} + \ldots \]

\[[4]\]
Consider the inhomogeneous differential equation
\[ y'' - \frac{1}{x}y' + \frac{1}{x^2}y = f(x), \] (†)
subject to the boundary conditions \( y(1) = y(e) = 0 \) where \( e = 2.718... \)

(a) Given that one solution of the homogeneous equation is \( y = x \), substitute \( y(x) = xv(x) \) (or otherwise) to find the second homogeneous solution. [4]

(b) Construct the Green’s function \( G(x, \xi) \) using \( f(x) = \delta(x - \xi) \) in the inhomogeneous equation (†) and subject to the same boundary conditions. [10]

(c) Hence show that the solution of (†) is given by
\[ y = x \ln x \int_x^e (\ln \xi - 1)f(\xi)d\xi + (x \ln x - x) \int_1^x \ln \xi f(\xi)d\xi. \]
Find \( y \) explicitly for \( f(x) = 1/x \) and verify your solution. [6]
Consider a set of eigenfunctions \( y_n(x) \) (with \( n = 0, 1, 2, \ldots \)) and corresponding eigenvalues \( \lambda_n \) which satisfy the Sturm-Liouville equation
\[
Ly \equiv -[p(x) y']' + q(x)y = \lambda w(x)y,
\]
with boundary conditions \( y(0) = y(1) = 0 \). Assume that the eigenfunctions are unit normalized, that is, \( \int_0^1 y_n^2 w \, dx = 1 \).

(a) Given an arbitrary function \( \tilde{y} \) satisfying the same boundary conditions, we can define
\[
F_n(x) \equiv L\tilde{y} - \lambda_n w(x)\tilde{y}.
\]
Show that for every \( n \) we must have
\[
\int_0^1 y_n(x)F_n(x) \, dx = 0.
\]

(b) Consider an approximate trial function \( \tilde{y} \) (with \( \tilde{y}(0) = \tilde{y}(1) = 0 \)) which is close to the lowest eigenfunction \( y_0 \) so that we can expand the difference in terms of the remaining eigenfunctions as
\[
\tilde{y} = y_0 + \sum_{n=1}^{\infty} a_n y_n.
\]
where the \( a_n \)'s can be assumed small. Use the Rayleigh-Ritz method with the trial function \( \tilde{y} \) to show that the lowest eigenvalue \( \lambda_0 \) can be approximated as
\[
\tilde{\lambda}_0 \approx \lambda_0 + \sum_{n=1}^{\infty} a_n^2 \lambda_n \approx \lambda_0 + \sum_{n=1}^{\infty} a_n^2 (\lambda_n - \lambda_0),
\]
where terms of order \( \mathcal{O}(a_n^4) \) have been neglected.

Discuss the relative error in the approximation \( \tilde{y} \) to the lowest eigenfunction \( y_0 \) compared with the error in the approximation \( \tilde{\lambda}_0 \) to the lowest eigenvalue \( \lambda_0 \). Is \( \tilde{\lambda}_0 \) lower or higher than \( \lambda_0 \)?
10C  (a) Derive the Euler-Lagrange equation satisfied by the function $y(x)$ which makes the integral

$$I = \int_a^b F[x, y(x), y'(x)]dx$$

stationary subject to the boundary conditions $y(a) = y_1$ and $y(b) = y_2$. [7]

(b) Efficient international airline routes must minimize the distance between two locations on the globe.

Show that the path length between two points $A(\phi_1, \theta_1)$ and $B(\phi_2, \theta_2)$ on a unit sphere can be expressed in polar coordinates $(\phi, \theta)$ as

$$S = \int_{\theta_1}^{\theta_2} \sqrt{1 + \sin^2 \theta \phi'^2} \, d\theta,$$

where $\phi' = \frac{d\phi}{d\theta}$. [4]

Hence, use the Euler-Lagrange equation to show that a stationary path satisfies

$$\phi(\theta) = \pm \int_{\theta_1}^{\theta_2} \frac{d\theta}{\sin \theta (k^2 \sin^2 \theta - 1)^{1/2}},$$

where $k$ is a constant. [4]

Integrate this expression to find the extremal solution. Use this to specify the shortest route for a flight setting out from Rome ($\theta \approx 45^\circ$, $\phi \approx 15^\circ$) and heading for Singapore ($\theta \approx 90^\circ$, $\phi \approx 105^\circ$).

[Hint: In the integration consider the substitution $t = \cot \theta$.] [5]