UNIVERSITY OF CAMBRIDGE

NATURAL SCIENCES TRIPOS Part IA

Monday 10 June 2002 9 to 12

MATHEMATICS (1)

Before you begin read these instructions carefully:

You may submit answers to no more than ***six*** *questions. All questions carry the same number of marks.*

The approximate number of marks allocated to a part of a question is indicated in the right hand margin.

Write on ***one*** *side of the paper only and begin each answer on a separate sheet.*

Questions marked with an asterisk (*) require a knowledge of B course material.

At the end of the examination:

Each question has a number and a letter (for example, ***3B****).*

Answers must be tied up in ***separate*** *bundles, marked* ***A, B, C, D, E*** *or* ***F*** *according to the letter affixed to each question.* ***Do not join the bundles together****. For each bundle, a blue cover sheet* ***must*** *be completed and attached to each bundle, with the appropriate letter written in the section box.*

A ***separate*** *yellow master cover sheet listing all the questions attempted* ***must*** *also be completed.*

Every cover sheet must bear your examination number and desk number.

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1A

(a) Show that the equation

$$\mathbf{r} \cdot \mathbf{n} = \mu \qquad (*)$$

describes a plane, where $\mathbf{r}$ is the variable position vector of a point in the plane, $\mathbf{n}$ is a constant unit vector, and μ is a constant scalar. [5]

(b) Three distinct points A, B, C have position vectors $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ relative to an origin O, with $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) \neq 0$. Derive equations of the form $(*)$ for each of the planes specified as follows:

(i) passing through A and perpendicular to OB; [5]

(ii) passing through A, B and C; [5]

(iii) passing through A and B and parallel to OC. [5]

2A Find the cosine Fourier series of the even function defined by $f(x) = x^2$ for $-\pi \leqslant x \leqslant \pi$ (and by periodicity for all other values of x). [14]

Hence show that

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6} \quad \text{and} \quad \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2} = \frac{\pi^2}{12} .$$

[6]

3B

(a) State the condition for a function $f(x)$ to have a stationary point at $x = a$ and give criteria for deciding whether it is a maximum, minimum or point of inflexion. Find the positions and natures of the stationary points of the function $f(x) = x^3 - 5x^2 + 3x - 1$. [8]

(b) Find the equations of the lines tangent to the following curves at the specified points:

(i) $xy^3 - yx^3 - 6 = 0$ at $(x, y) = (1, 2)$;

(ii) $e^{xy} + y \ln x = x^2$ at $(x, y) = (1, 0)$. [12]

4B* Explain, without proof, a method for finding the stationary points of a function $f(x,y,z)$ subject to simultaneous constraints $g(x,y,z) = h(x,y,z) = 0$. [4]

A point is constrained to lie on the plane $x - y + z = 0$ and also on the ellipsoid $x^2 + \frac{1}{4}y^2 + \frac{1}{4}z^2 = 1$. Find the minimum and maximum distances of this point from the origin, by considering the function $f(x,y,z) = x^2 + y^2 + z^2$. [16]

5C

(a) Evaluate the definite integrals

$$\int_0^\infty e^{-x^2} dx \ , \qquad \int_0^\infty x^2 e^{-x^2} dx \ ,$$

as well as the indefinite integrals

$$\int x\, e^{-x^2} dx \ , \qquad \int x^3 e^{-x^2} dx \ .$$

[10]

(b) Sketch the region R in the positive quadrant of the xy plane which is enclosed by the lines $y = 0$, $x = 2$, $y = x$ and by the curve $xy = 1$. Evaluate

$$\int\int_R x^2 e^{-x^2} dx dy \ .$$

[10]

6C Let

$$A = \begin{pmatrix} 3 & 2 \\ 2 & 3 \end{pmatrix} \ , \quad \mathbf{u} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \ , \quad \mathbf{v} = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \ .$$

(a) Verify that

$$A^2 - (\mathrm{Tr}A)A + (\mathrm{Det}A)I = 0 \ ,$$

where I is the 2×2 unit matrix. Calculate A^{-1}.

[10]

(b) Calculate $\mathbf{u}^{\mathrm{T}}\mathbf{u}$, $\mathbf{v}^{\mathrm{T}}\mathbf{v}$, $\mathbf{u}\mathbf{u}^{\mathrm{T}}$ and $\mathbf{v}\mathbf{v}^{\mathrm{T}}$. Find constants λ and μ such that

$$A = \frac{\lambda}{2}\mathbf{u}\mathbf{u}^{\mathrm{T}} + \frac{\mu}{2}\mathbf{v}\mathbf{v}^{\mathrm{T}}$$

and verify that

$$A^{-1} = \frac{1}{2\lambda}\mathbf{u}\mathbf{u}^{\mathrm{T}} + \frac{1}{2\mu}\mathbf{v}\mathbf{v}^{\mathrm{T}} \ .$$

[10]

7D

(a) Find the general solutions of the differential equations

(i) $\frac{dy}{dx} = \frac{y+1}{x^2-1}$; [6]

(ii) $\cos x \frac{dy}{dx} + y = \sec x + \tan x$. [6]

[You may use $\int \sec x \; dx = \ln(\sec x + \tan x)$ and $\int \sec x \tan x \; dx = \sec x$.]

(b) By substituting $y = z^{-1}$, solve the differential equation

$$\frac{dy}{dx} + 2y = x^2 y^2$$

subject to $y = 1$ when $x = 0$. [8]

8D* Let $\mathbf{a}$ and $\mathbf{b}$ be orthogonal vectors in the plane, with components a_i and b_i, respectively $(i = 1, 2)$.

(a) The 2×2 matrix L has components

$$L_{ij} = \delta_{ij} + a_i b_j \ .$$

Evaluate $\mathrm{Tr}L$ and $\mathrm{Det}L$. [6]

(b) The 2×2 matrix M has components

$$M_{ij} = \delta_{ij} - b_i a_j \ .$$

Show, by calculating its components, that LM^{T} is the identity matrix. [6]

(c) Find the matrices L and M when $\mathbf{a} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$. Verify that they are related by

$$M = S^{\mathrm{T}} L S \ , \qquad \text{where} \qquad S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \ .$$

[8]

9E Consider the forces defined by the vector fields

$$\mathbf{F} = \mathbf{a} \times \mathbf{r} \ , \qquad \mathbf{G} = \mathbf{r}/(r^2+1)^{3/2} \ ,$$

where $\mathbf{r} = (x, y, z)$ and $\mathbf{a} = (0, 1, 1)$. Find, by calculating their curls, whether either of these fields is conservative. [7]

Check your answers by computing the line integral of each field (i.e., the work done by each force) along the following paths: (i) the straight line directly from (1,0,0) to (0,1,0); (ii) the path consisting of two straight line segments, the first from (1,0,0) to the origin, and the second from the origin to (0,1,0). [13]

10E The function $\theta(x, t)$ obeys the diffusion equation

$$\frac{\partial \theta}{\partial t} = \frac{\partial^2 \theta}{\partial x^2} \ .$$

Find, by substitution, solutions of the form

$$\theta(x, t) = f(t) \exp[-(x+a)^2/4(t+b)] \ ,$$

where a and b are arbitrary constants and the function f is to be determined. [12]

Hence find a solution which satisfies the initial condition

$$\theta(x, 0) = \exp[-(x-2)^2] - \exp[-(x+2)^2]$$

and sketch its behaviour for $t \geqslant 0$. [8]

11F

(a) Let $z = 2e^{i\phi}$ where $0 < \phi < \pi/2$. Express z^*, z^{-1} and $-z$ in the form $re^{i\theta}$ where $r > 0$ and $0 < \theta < 2\pi$. Draw a diagram showing the location of all four complex numbers in the Argand plane. [6]

(b) Write down formulae for $\cos\theta$ and $\sin\theta$ in terms of complex exponentials. Use these to derive formulae for $\cos 2\theta$ and $\sin 2\theta$ in terms of $\cos\theta$ and $\sin\theta$. [5]

(c) If $z = 2e^{i\phi}$ where $0 < \phi < \pi$, calculate the real and imaginary parts of $w = (z-2)/(z+2)$. Hence calculate $\ln w$ and deduce that this complex number always lies on a line which is parallel to the real axis in the Argand plane. [9]

12F* Write down the hyperbolic solutions $h_k(x)$ and the trigonometric solutions $g_k(x)$ of the differential equations

$$\frac{d^2 h_k}{dx^2} = k^2 h_k \ , \qquad \frac{d^2 g_k}{dx^2} = -k^2 g_k \ .$$

Hence find separable solutions of

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0$$

and determine which of these satisfy:

(i) $f = 0$ when $x = 0$, $0 \leqslant y \leqslant \pi$, and $f = 0$ when $y = 0$, $0 \leqslant x \leqslant \pi$. [11]

Show that there are infinitely many solutions $f_n(x, y)$ (with n a positive integer) which satisfy in addition the constraint:

(ii) $f = 0$ when $y = \pi$.

Now impose a final condition:

(iii) $f = 1$ when $x = \pi$, $0 < y < \pi$.

Express $f(x, y)$ satisfying conditions (i), (ii), (iii) as a sum over the solutions $f_n(x, y)$, by using the identity

$$1 = \sum_{m=0}^{\infty} \frac{4}{(2m+1)\pi} \sin(2m+1)y \ , \qquad 0 < y < \pi \ .$$

[9]