UNIVERSITY OF CAMBRIDGE

NATURAL SCIENCES TRIPOS Part IB & II (General)

Friday 1 June 2001 9.00 to 12.00

MATHEMATICS (2)

Before you begin read these instructions carefully:

You may submit answers to no more than **seven** *questions. All questions carry the same number of marks.*

The approximate number of marks allocated to a part of a question will be indicated in the right hand margin.

Write on **one** *side of the paper only and begin each answer on a separate sheet.*

At the end of the examination:

Each question has a number and a letter (for example, ***6B****).*

Answers must be tied up in **separate** *bundles, marked* ***A, B or C*** *according to the letter affixed to each question.*

Do not join the bundles together.

For each bundle, a blue cover sheet must be completed and attached to the bundle.

A **separate** *yellow master cover sheet listing all the questions attempted* **must** *also be completed.*

Every cover sheet must bear your examination number and desk number.

1A In plane polar coordinates (r, θ) Laplace's equation is

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \phi}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2 \phi}{\partial \theta^2}=0.$$

In the region between two concentric cylinders of radii a and b $(b > a > 0)$, $\phi(r, \theta)$ satisfies $\nabla^2\phi = 0$. The inner cylinder is held at $\phi = 0$ and the outer at $\phi = V \cos 2\theta$. By separation of variables or otherwise find ϕ in the region $a < r < b$. [12]

Find ϕ if the inner cylinder is held at $\phi = -V$ and the outer at $\phi = V \cos 2\theta$. [8]

2A From the divergence theorem establish the Green's identity

$$\int_V (u\nabla^2 v + \nabla u .\, \nabla v)\, dV = \int_S u\frac{\partial v}{\partial n}\, dS,$$

for a scalar function $u(\boldsymbol{r})$ in a simply-connected volume V bounded by a surface S. [5]

Show that the Klein-Gordon equation $\nabla^2\Phi - m^2\Phi = 0$ $(m \neq 0)$ has a unique solution when subject to either Dirichlet or Neumann boundary conditions. [6]

What difference is introduced if $m = 0$? [2]

The scalar function $v(\boldsymbol{r})$ satisfies Laplace's equation in V and on the surface S takes the same values as an arbitrary scalar function $w(\boldsymbol{r})$ defined throughout V. Show, by considering $v - w$, that

$$\int_V (\nabla w)^2\, dV \geq \int_V (\nabla v)^2\, dV.$$

[7]

3A Use the Cauchy–Riemann relations to show that the real and imaginary parts of an analytic function $f(z)$ satisfy Laplace's equation in two dimensions. [4]

Describe how to make the function

$$f(z) = \log_{\mathrm{e}}\left\{\frac{1+z}{1-z}\right\} \qquad (*)$$

single-valued in the complex plane. [4]

The cross-section of an infinitely long hollow circular cylinder of unit radius is described in plane polar coordinates (r, θ). One half at $r = 1$ and $0 < \theta < \pi$ is held at potential V_0 while the other half at $r = 1$ and $\pi < \theta < 2\pi$ is held at $-V_0$. By considering the imaginary part of $(*)$ in the complex plane $z = r\mathrm{e}^{\mathrm{i}\theta}$, find the potential inside the cylinder as a function of r and θ. [12]

4A What is meant by a pole of a complex function $f(z)$ and what is meant by its residue? [2]

If $h(z)$ has a simple zero at $z = z_0$ and $g(z)$ is analytic and non-zero at $z = z_0$ show that $g(z)/h(z)$ has a simple pole at $z = z_0$ with residue $g(z_0)/h'(z_0)$. [5]

Find the poles and residues of

$$f(z) = \frac{z^a}{1+z^3},$$

if $a > -1$. [5]

Use contour integration to show that

$$\int_0^\infty \frac{x^a}{1+x^3}\,dx = \frac{\pi}{3\sin\left[\frac{1}{3}\pi(a+1)\right]},$$

being careful to determine the range of a for which the integral converges. [8]

5A Take the Laplace transform of the equation

$$y' + y = \delta(t-1), \qquad (*)$$

with $y = 0$ for $t \leq 0$, and solve for $Y(p)$, the Laplace transform of $y(t)$. [5]

Write down the Bromwich integral and use it to invert $Y(p)$ and obtain $y(t)$. [5]

Now take the Fourier transform of $(*)$ and solve for $\tilde{y}(k)$, the Fourier transform of $y(t)$. [5]

Write down the formula for the inverse Fourier transform and evaluate it to find the same solution $y(t)$ to $(*)$. [5]

[You may use Jordan's Lemma without proof.]

6B A magnetic field H_i applied to a crystal induces a magnetic moment vector $M_i = \chi_{ij} H_j$ where χ_{ij} is a property of the material known as the magnetic susceptibility. Why is the matrix χ_{ij} a tensor? [4]

In suitable units the magnetic susceptibility χ_{ij} is given, in a particular frame of reference, by the matrix:

$$\begin{pmatrix} 1 & 1 & 0 \\ 1 & 5 & 2 \\ 0 & 2 & 1 \end{pmatrix}.$$

Find the directions in which the magnetic moment is parallel to the magnetic field and the corresponding proportionality factor for each direction. [6]

The dissipated energy density is defined by $\mathcal{E} = M_i H_i$. Show that for a fixed magnitude of the magnetic field, the minimum and maximum value of the dissipated energy density occur when H_i and M_i are parallel. Find these extrema for the magnetic susceptibility given above. [6]

Sketch the shape of a surface of constant energy density in terms of the components of the applied magnetic field. [4]

[Summation over repeated indices is assumed in this question]

7B A system with N degrees of freedom described by means of generalised coordinates $q_1, q_2, \cdots, q_N$ is subject to small oscillations. The Lagrangian is:

$$L = \sum_i \frac{1}{2} m_i \dot{q}_i^2 - \sum_{i,j} \frac{1}{2} V_{ij}\, q_i q_j, \quad \text{where } V_{ij} = V_{ji}.$$

Use the corresponding equations of motion to define the *normal modes, normal frequencies and normal coordinates.* [4]

Consider three objects constrained to move on a horizontal line. The object at the centre (mass $2m$) is connected to the object on its left (mass m) by means of a spring with force constant k and to the object on its right (mass $3m$) by a different spring with force constant $3k$. Neglecting frictional forces set up the equations of motion for the three objects. [4]

Determine the normal frequencies of oscillation. [4]

Find the normal modes of oscillation. What is the orthogonality condition for the normal modes? [5]

Find the normal coordinates. [3]

8B Given a finite group G of order $|G|$ and a subgroup H of order $|H|$, define the term *left coset of* H *in* G. State Lagrange's theorem relating the order of a group and its subgroups. [4]

Prove that the set of matrices of the form:

$$\boldsymbol{A} = \begin{pmatrix} 1 & 0 & 0 \\ a & 1 & 0 \\ b & c & 1 \end{pmatrix}$$

where a, b, c are integers mod 4 (for instance 7 mod 4 $=$ 3) form a finite group G under matrix multiplication. Show that G has 64 elements. [5]

Show that the subset given by $a = c$ defines an Abelian subgroup H. Find the order of H and verify Lagrange's theorem. How many distinct left cosets of H are in G? [5]

Find all the elements of G whose square is the 3×3 identity matrix; is it a subgroup of G? Is the subset of G defined by $a \neq 0 \neq b$ a subgroup of G? (You may assume Lagrange's theorem.) [6]

9B Given two arbitrary groups G_1 and G_2, define the terms *homomorphism*, *isomorphism* and *kernel*. [5]

If G_2 is homomorphic to G_1 show that the kernel K of the mapping from G_1 to G_2 is a subgroup of G_1. [5]

A normal subgroup H of G_1 is defined as follows: if $h \in H$ then $ghg^{-1} \in H$ for arbitrary $g \in G_1$. Show that the kernel K is a normal subgroup of G_1. [5]

Hence verify that the cosets of K in G_1 also form a group. [5]

10B Define the terms *representation* and *irreducible representation* of a group G. [3]

Consider the following set of matrices:

$$\boldsymbol{I} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}; \quad \boldsymbol{A} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}; \quad \boldsymbol{B} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}; \quad \boldsymbol{C} = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$$

Construct their multiplication table and verify that they form a group V under matrix multiplication. Is this group abelian? [4]

From the knowledge of the order of the group, show that these matrices do not form an irreducible representation of V. What are the dimensions of all the irreducible representations of V? Find all the conjugacy classes for V and compute the character of each conjugacy class in the 2×2 matrix representation defined above. [4] [3]

Construct the table of characters for the irreducible representations and use it to decompose the 2×2 matrix representation above as a direct sum of the irreducible representations of the group and express the matrices in diagonal form. [6]

[The orthogonality relation for characters may be assumed as well as the relation $\sum_\alpha n_\alpha^2 = |G|$ where n_α are the dimensions of the irreducible representations.]

11C Show that the second derivative of the function $u(x)$ can be expanded as

$$\frac{d^2u}{dx^2} = \frac{1}{12h^2}\Big[-u(x+2h) + 16u(x+h) - 30u(x) + 16u(x-h) - u(x-2h)\Big] + \mathcal{O}(h^4)\,. \qquad (*)$$

[8]

We wish to solve the diffusion equation

$$\frac{\partial \psi}{\partial t} = \frac{\partial^2 \psi}{\partial x^2}\,, \qquad 0 < x < 1, \qquad t > 0\,,$$

given initial conditions $\psi(x,0) = f(x)$ and boundary conditions

$$\frac{\partial \psi}{\partial x}(0,t) = 0\,, \qquad \frac{\partial \psi}{\partial x}(1,t) = 0\,.$$

Using the result $(*)$ for the spatial derivative and Euler's method for the time derivative, write down a finite difference scheme to solve the diffusion equation which has a local truncation error $\mathcal{O}(\delta t^2) + \mathcal{O}(\delta x^4 \delta t)$. [6]

Briefly discuss how the boundary conditions might be implemented for your scheme. What are the advantages and disadvantages of replacing the Euler scheme with an implicit method such as the Crank-Nicholson scheme? [6]