UNIVERSITY OF CAMBRIDGE

NATURAL SCIENCES TRIPOS Part IA

Wednesday 13 June 2001 9 to 12

MATHEMATICS (2)

Before you begin read these instructions carefully:

You may submit answers to no more than ***six*** *questions. All questions carry the same number of marks.*

The approximate number of marks allocated to a part of a question will be indicated in the right hand margin.

Write on **one** *side of the paper only and begin each answer on a separate sheet.*

Questions marked with an asterisk (*) require a knowledge of B course material.

At the end of the examination:

Each question has a number and a letter (for example ***3B****).*

Answers must be tied up in **separate** *bundles, marked* ***A, B, C, D, E*** *or* ***F*** *according to the letter affixed to each question.*

Do not join the bundles together.

For each bundle, a blue cover sheet **must** *be completed and attached to the bundle, with the appropriate letter written in the section box.*

A ***separate*** *yellow master cover sheet listing all the questions attempted* ***must*** *also be completed.*

Every cover sheet must bear your examination number and desk number.

1A Solve the following differential equations:

(a)

$$\frac{dy}{dx} = \frac{2x + xy^2}{x^2 y - 3y}$$ [7]

such that when $x = 3,\ y = 4$;

(b)

$$\frac{dy}{dx}\sin x + y\cos x = \tfrac{1}{2}\sin 2x$$ [5]

such that when $x = \frac{\pi}{6},\ y = \frac{1}{4}$;

(c)

$$\left(2ye^{y/x} + x\right)\frac{dy}{dx} - 2x - y = 0\ .$$ [8]

2A* By mathematical induction or otherwise prove Leibnitz's formula

$$(uv)^{(n)} = \sum_{r=0}^{n} \frac{n!}{r!(n-r)!}\ v^{(n-r)}\ u^{(r)}$$

where $u^{(n)} = \frac{d^n u}{dx^n}$, i.e. the n^{th} derivative of u with respect to x. [7]

Use this formula to find the n^{th} derivative of

$$y = x(x+1)e^{2x}$$ [5]

and show that the n^{th} derivative of $y = x^2 \sin x$ is

$$\left\{x^2 - n(n-1)\right\}\sin\left(x + \frac{n\pi}{2}\right) - 2nx\cos\left(x + \frac{n\pi}{2}\right)\ .$$ [8]

(a) Write down the Cartesian co-ordinates (x, y, z) of a point with position vector $\boldsymbol{r}$ in terms of its cylindrical polar co-ordinates (ρ, ϕ, z). [3]

(b) The square of the infinitesimal element of distance in cylindrical polar co-ordinates is given by

$$ds^2 = d\rho^2 + \rho^2 d\phi^2 + dz^2 \ . \qquad (*)$$

Two points, $\boldsymbol{r}_0$ and $\boldsymbol{r}_1$, on the surface of a cylinder of radius R have cylindrical polar co-ordinates $(R, 0, 0)$ and (R, ϕ_1, z_1). Use $(*)$ to obtain an expression for the length of the shortest path on the surface of the cylinder between $\boldsymbol{r}_0$ and $\boldsymbol{r}_1$. [10]

(c) Sketch the length of the shortest path as a function of ϕ_1 where $0 \leqslant \phi_1 \leqslant 2\pi$, when (i) $R = 1, z_1 = 0$ and (ii) $R = 1, z_1 = 1$. [7]

3B

4B Briefly explain the relationship between the order of an ordinary linear differential equation and the number of boundary conditions required to determine a unique solution. [2]

Find the general solutions of:

(a) $\frac{d^2y}{dx^2} + 3\frac{dy}{dx} + 2y = x - e^{-x}$; [9]

(b) $\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + y = x^2 + 2e^{-x}$. [9]

5C Using Cartesian co-ordinates $\boldsymbol{r} = (x, y, z)$, consider the vector field

$$\boldsymbol{v}(\boldsymbol{r}) = \left(e^x y,\ e^x + 2yz^3,\ 3y^2z^2\right) \ .$$

(a) Calculate $\int_C \boldsymbol{v}.\, d\boldsymbol{r}$ directly when C is:

(i) the straight line from (0,0,0) to (1,1,1);

(ii) the curve joining these same points but defined by $\boldsymbol{r} = (t, t^2, t^3)$ with $0 \leqslant t \leqslant 1$. [10]

(b) Compute $\nabla \times \boldsymbol{v}$. [4]

(c) Find a function $f(x, y, z)$ such that $\boldsymbol{v} = \nabla f$ and hence confirm your answers to parts (a) and (b). [6]

6C* A right circular cylinder of radius r and height h has volume $V = \pi r^2 h$ and surface area $A = 2\pi r(r+h)$. Use Lagrange multipliers to do the following.

(a) Show that the maximum volume for a given area is

$$V = \frac{1}{3}\sqrt{\frac{A^3}{6\pi}} \ .$$

[10]

(b) Find the value of h/r which maximizes the area for a cylinder inscribed in the unit sphere, so that $r^2 + h^2/4 = 1$. [10]

[You need not show that suitable extrema you find are actually maxima.]

7D

(a) Evaluate:

(i) $\begin{vmatrix} 8 & 1 \\ 5 & 3 \end{vmatrix}$;

(ii) $\begin{vmatrix} 4 & 1 & -2 \\ 3 & 0 & 4 \\ 10 & 1 & -4 \end{vmatrix}$;

(iii) $\begin{vmatrix} 4 & 1 & -2 \\ 3 & 0 & 4 \\ 8 & 2 & -4 \end{vmatrix}$.

[6]

(b) Find the values of λ for which the following set of linear equations has non-zero solutions, and give the solutions in each case:

$$\begin{aligned} x - 3z &= 0 \\ x + \lambda y - \lambda z &= 0 \\ x + 2y + \lambda z &= 0 \ . \end{aligned}$$

[10]

(c) For matrices A and B such that $|AB| = -2$, $|B| > 0$, and $B = B^{-1}$, find $|A|$. [4]

8D

(a) Draw a diagram to show the area over which the following integral is evaluated:

$$\int_{y=0}^{1}\int_{x=y}^{2-y}(2x^2+y)dxdy\ .$$

Evaluate the integral. [10]

(b) The equation $r=(1+\cos\theta)$ defines a closed surface in three dimensions. Find the volume enclosed. [10]

[(r,θ,ϕ) are spherical polar co-ordinates.]

9E

(a) $x, y, z,$ and u are variables related by two conditions so that any two can be regarded as functions of the other two. The notation $\left(\frac{\partial x}{\partial y}\right)_z$ means that x is to be regarded as a function of y and z, and the derivative is to be taken keeping z fixed.

If

$$z^2=2x^2+y^2$$

and

$$u^2=x^2-y^2$$

show that

(i)

$$\left(\frac{\partial x}{\partial z}\right)_u\left(\frac{\partial z}{\partial x}\right)_y+\left(\frac{\partial y}{\partial z}\right)_u\left(\frac{\partial z}{\partial y}\right)_x=1$$

[7]

and

(ii)

$$\left(\frac{\partial x}{\partial y}\right)_z\left(\frac{\partial y}{\partial z}\right)_u+\left(\frac{\partial x}{\partial u}\right)_z\left(\frac{\partial u}{\partial z}\right)_y=0.$$

[7]

(b) If now (x,y) and (z,u) are regarded as two sets of independent variables, and

$$\phi(x,y)=\psi(z,u),$$

show that

$$\frac{1}{x}\left(\frac{\partial\phi}{\partial x}\right)_y+\frac{1}{y}\left(\frac{\partial\phi}{\partial y}\right)_x=\frac{3}{z}\left(\frac{\partial\psi}{\partial z}\right)_u\ .$$

[6]

10E Give a necessary condition for the expression

$$P(x,y)dx + Q(x,y)dy$$ [2]

to be an exact differential.

For the thermodynamics of a gas, the internal energy U can be regarded as a function of the entropy S and the volume V. It is given that:

$$dU = TdS - pdV$$

where T is the temperature and p the pressure. By considering the function

$$A = U - TS$$

or by some other method, show that

$$\left(\frac{\partial S}{\partial V}\right)_T = \left(\frac{\partial p}{\partial T}\right)_V .$$ [4]

Now, considering U as a function of T and V show that

$$\left(\frac{\partial U}{\partial V}\right)_T = T\left(\frac{\partial S}{\partial V}\right)_T - p.$$ [4]

Given

$$p = \frac{nRT}{V - nb}\ \exp\left\{\frac{-an}{VRT}\right\}$$ [6]

where a, b, n, R are constants, find $(\frac{\partial U}{\partial V})_T$.

If, instead

$$p = \frac{nRT}{V}$$ [4]

and $\left(\frac{\partial U}{\partial T}\right)_V = C_V$ where C_V is constant, find an expression for U.

11F A bag contains 2 red and 5 green counters.

(a) In a trial, counters are repeatedly drawn from the bag and replaced each time. Find the probability that a red counter is drawn on the n-th draw for the first time. [6]

(b) In another trial counters are now drawn without being replaced. Let E_1 be the event that the first drawn is red, and E_2 the event that the second drawn is red. If $P(E)$ denotes the probability of event E, find the following probabilities:

(i) $P(E_1)$;

(ii) $P(E_2)$;

(iii) $P(E_1 \cap E_2)$.

Hence or otherwise find

(iv) $P(E_1 \cup E_2)$;

(v) $P(E_1|E_2)$;

where $E_1|E_2$ denotes the event "E_1 given E_2". [14]

12F*

(a) In each of the following cases state whether the function has a finite limit as x tends to zero, and if so find its value:

(i) $\frac{1}{x}\sin 2x$;

(ii) $x \cos\frac{1}{x}$

(iii) $\dfrac{x}{1-\exp(-x)}$. [7]

(b) Explain what is meant by the statement that a series $\sum u_n$ is

(i) convergent; [6]

(ii) absolutely convergent.

(c) Show whether or not the series $\sum u_n$ is convergent when

$$u_n = \frac{n^4}{2^n} \ .$$ [7]