UNIVERSITY OF CAMBRIDGE

NATURAL SCIENCES TRIPOS Part IA

Monday 11 June 2001 9 to 12

MATHEMATICS (1)

Before you begin read these instructions carefully:

You may submit answers to no more than ***six*** *questions. All questions carry the same number of marks.*

The approximate number of marks allocated to a part of a question will be indicated in the right hand margin.

Write on **one** *side of the paper only and begin each answer on a separate sheet.*

Questions marked with an asterisk (*) require a knowledge of B course material.

At the end of the examination:

Each question has a number and a letter (for example, ***3B****).*

Answers must be tied up in **separate** *bundles, marked* ***A, B, C, D, E*** *or* ***F*** *according to the letter affixed to each question.*

Do not join the bundles together*.*

For each bundle, a blue cover sheet **must** *be completed and attached to each bundle, with the appropriate letter written in the section box.*

A **separate** *yellow master cover sheet listing all the questions attempted* **must** *also be completed.*

Every cover sheet must bear your examination number and desk number.

1A The points A, B and C have position vectors (relative to the origin)

$$\mathbf{a} = 2\hat{\boldsymbol{\imath}} - \hat{\boldsymbol{\jmath}} - 2\hat{\boldsymbol{k}}$$
$$\mathbf{b} = -\hat{\boldsymbol{\imath}} - 2\hat{\boldsymbol{\jmath}} + \hat{\boldsymbol{k}}$$
$$\mathbf{c} = -\hat{\boldsymbol{\imath}} + \hat{\boldsymbol{\jmath}} \ .$$

(a) Find the equation of a straight line that passes through points A and B; [2]

(b) determine a unit vector perpendicular to the plane containing the vectors $\mathbf{a}$ and $\mathbf{b}$; [4]

(c) evaluate the area of the triangle with A, B and C at its vertices; [6]

(d) find an equation for the plane containing the triangle ABC and calculate the perpendicular distance of this plane from the origin. [8]

2A Sketch the even function

$$f(x) = \begin{cases} 1 + (x/\pi) & \text{for} \quad -\pi \leqslant x \leqslant 0 \\ 1 - (x/\pi) & \text{for} \quad 0 \leqslant x \leqslant \pi \ . \end{cases}$$

[3]

Find the cosine Fourier series of $f(x)$ in the range $-\pi \leqslant x \leqslant \pi$ and hence show that

$$\frac{\pi^2}{8} = \sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} \ .$$

[17]

3B Consider the matrix

$$A = \begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \ .$$

(a) Determine the eigenvalues of A. [7]

(b) Determine the corresponding eigenvectors. Normalise these eigenvectors and show they are orthogonal. [7]

(c) Determine the eigenvalues and corresponding eigenvectors of the matrix $B = A^2$. [6]

4B*

(a) The function $G(\theta, k)$ is defined as

$$G(\theta, k) = \int_0^\theta g(x, k) dx \ .$$

[4]

Give expressions for $\left(\frac{\partial G}{\partial \theta}\right)_k$ and $\left(\frac{\partial G}{\partial k}\right)_\theta$.

(b) The functions $E(\theta, k)$ and $F(\theta, k)$ are defined as

$$E(\theta, k) = \int_0^\theta \sqrt{1 - k^2 \sin^2 x} \ dx \ ,$$
$$F(\theta, k) = \int_0^\theta \frac{1}{\sqrt{1 - k^2 \sin^2 x}} \ dx \ .$$

[8]

Show that $\left(\frac{\partial E}{\partial k}\right)_\theta = \frac{E-F}{k}$.

(c) The function $I(\theta, k)$ is defined as

$$I(\theta, k) = \int_0^\theta E(x, k) \sqrt{1 - k^2 \sin^2 x} \ dx \ .$$

By differentiating $I(\theta, k)$ with respect to θ show that

$$I(\theta, k) = \tfrac{1}{2} E(\theta, k)^2 \ .$$

[8]

5C

(a) The functions

$$f(x) = x \ln x \qquad \text{and} \qquad g(x) = \frac{\ln x}{x}$$

are defined on the positive real line $0 < x < \infty$.

Find and classify their stationary points. State, without proof, how the functions behave as $x \to 0$ and as $x \to \infty$. Show that the graphs of f and g have a unique point of intersection, where they touch (i.e. have common gradient). Sketch the graphs of f and g on the same axes. [15]

(b) Using a change of variables, but without evaluating the integrals, show that

$$\int_1^\infty \frac{1}{x^2} \ln x \ dx = - \int_0^1 \ln x \ dx \ .$$

Show similarly that

$$\int_0^\infty \frac{1}{1 + x^2} \ln x \ dx = 0 \ .$$

[5]

6C Evaluate $\int_S \boldsymbol{F}.\,d\boldsymbol{S}$ in the following cases.

(a) $\boldsymbol{F} = (xy^2, yz^2, zx^2)$ and S is the boundary of the region $|x| \leqslant a,\ |y| \leqslant b, |z| \leqslant c$. [8]

(b) $\boldsymbol{F} = \boldsymbol{r} = (x, y, z)$ and S is the surface defined by

$$\boldsymbol{r} = (a\cos\phi\sin\theta,\ b\sin\phi\sin\theta,\ c\cos\theta)$$

with $0 \leqslant \theta \leqslant \pi$ and $0 \leqslant \phi \leqslant 2\pi$. [12]

[Use $d\boldsymbol{S} = \frac{\partial \boldsymbol{r}}{\partial\theta} \times \frac{\partial \boldsymbol{r}}{\partial\phi} d\theta d\phi$.]

7D

(a) Evaluate:

(i) $\begin{pmatrix} 2 & -9 & 12 \\ 6 & 1 & 4 \end{pmatrix} - \begin{pmatrix} 0 & 1 & -3 \\ 12 & 4 & 10 \end{pmatrix}$;

(ii) $\begin{pmatrix} 3 & 7 \\ 1 & 6 \end{pmatrix} \begin{pmatrix} 2 & 4 \\ 0 & 0 \end{pmatrix}$;

(iii) $\begin{pmatrix} 4 & 8 & 1 \end{pmatrix} \begin{pmatrix} 6 \\ 3 \\ 0 \end{pmatrix}$;

(iv) $\begin{pmatrix} 3 & 2 & 1 \\ 3 & 2 & 1 \\ 3 & 2 & 1 \end{pmatrix} \begin{pmatrix} 7 \\ 5 \\ 6 \end{pmatrix} \begin{pmatrix} -2 & 0 & 2 \end{pmatrix}$.

[8]

(b) Find the inverse of

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 4 & 9 \end{pmatrix} .$$

Hence, or otherwise, solve the following linear equations when $\lambda = 3$:

$$\begin{aligned} x + y + z &= 5 \\ x + 2y + \lambda z &= 13 \\ x + 4y + \lambda^2 z &= 35 \ . \end{aligned}$$

Find the values of λ for which these equations have no solution. [12]

8D* Let $\boldsymbol{r} = (x, y, z)$ be the Cartesian position vector, $\boldsymbol{p}$ a fixed vector, and $\boldsymbol{E} = |\boldsymbol{r}|^{-3}(3|\boldsymbol{r}|^{-2}(\boldsymbol{p}.\,\boldsymbol{r})\boldsymbol{r} - \boldsymbol{p})$. Use the divergence theorem to show that the surface integral

$$\int_A \boldsymbol{E}.\,d\boldsymbol{s}$$ [10]

vanishes for any closed surface A not enclosing the origin. Verify directly that the integral vanishes when $\boldsymbol{p} = (0, 0, 1)$ and A is the surface of the infinite cylinder $x^2 + y^2 = a^2$. [10]

9E Write down the Taylor expansion of $f(x)$ about the point $x = x_0$. [2]

Find, by any method, the first three non-zero terms in the Taylor expansion about $x = 0$ of the following functions, where a is a real constant:

(a) $\ln(1 + x)$ [4]

(b) $\frac{x}{\sqrt{x^2+a^2}}$ [4]

(c) $\exp\{-(x - a)^2\}$ [5]

(d) $\ln \frac{1-x}{1+2x^2}$. [5]

10E*

(a) If A, B, C are $n \times n$ matrices with components a_{ij}, b_{ij}, c_{ij}, using the summation convention write down an expression for:

(i) Trace(AB)

(ii) $(ABC)_{ij}$

(iii) Trace(I), where I is the $n \times n$ identity matrix. [10]

Show that Trace$(ABC) =$ Trace(CAB) and give an expression for Trace(ABA^{-1}).

(b) If

$$A = I + B$$

show

$$(A^2)_{ij} = I_{ij} + 2b_{ij} + b_{ik}b_{kj}.$$

If

$$B^2 = 0$$ [10]

show

$$(A^n)_{ij} = I_{ij} + nb_{ij}.$$

Verify this explicitly when

$$B = \begin{pmatrix} 0 & p \\ 0 & 0 \end{pmatrix},$$

and $p \neq 0$.

11F

(a) Find the real and imaginary parts of

(i)

$$\left(\frac{1+i}{1-i}\right)^2$$

(ii)

$$\ln\left(ie^{i\phi}\right) \ .$$ [5]

(b) Find all solutions of the equations

(i) $z^3 \ = \ 1$

(ii) $z^3 \ = \ -8$. [7]

(c) Let z be a solution of

$$z^2 + bz + b^2 \ = \ 0$$

where b is a real number. Find all values of b such that $|z| \leqslant 1$. [8]

12F x is a continuous real-valued random variable. The probability of x having a value in the range $(x, x + dx)$ is $P(x)dx$.

(a) Define the mean and variance of x. [3]

(b) Find the mean and variance of x in the case

$$P(x) = \begin{cases} 1 - |x| & -1 \leqslant x \leqslant 1 \\ 0 & \text{otherwise} \end{cases} .$$

[7]

(c) Suppose that x is now normally distributed with mean zero and variance 1, so that x has distribution function

$$P(x) \;=\; \frac{1}{\sqrt{2\pi}} \exp(-x^2/2).$$

[10]

Using integration by parts or otherwise

(i) show that x^3 has mean value zero;

(ii) derive the mean value of x^4.