1C Vectors and Matrices

For $z, a \in \mathbb{C}$ define the *principal value* of $\log z$ and hence of z^a.

Let $z = \mathrm{r}e^{i\theta}$. *Then* $\log z = \log r + i\theta$, *where* r *is the modulus and* θ *is*
[2] *the argument of* z, *with* $-\pi < \theta \leqslant \pi$.

[1] *Write* $z^a = \mathrm{e}^{a \log z}$ *and use the principal value of* $\log$.

Hence find all solutions to (i) $z^{\mathrm{i}} = 1$

Let $z = \mathrm{r}e^{i\theta}$. *Then* $r^i \mathrm{e}^{-\theta} = 1$ *and* $r^i = e^{i \log r}$ *so we need* $e^{-\theta} = 1$ *and*
[2] $\log r = 2n\pi$ *for some integer* n. *So* $z = e^{2\pi n}$.

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and (ii) $z^{\mathrm{i}} + \overline{z}^{\,\mathrm{i}} = 2\mathrm{i}$,

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Let $z = re^{i\theta}$. *Then* $r^i e^{-\theta} + r^i e^{\theta} = 2i$ *so* $e^{i \log r}(e^{\theta} + e^{-\theta}) = 2i$. *So*
[2] $\log r = \pi/2 + 2n\pi$ *and* $\theta = 0$. *This gives* $z = e^{\pi/2 + 2n\pi}$

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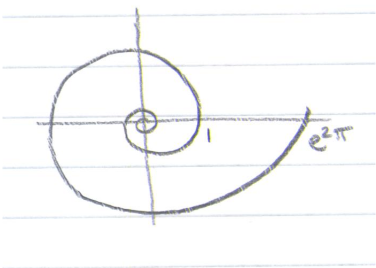
and sketch the curve $|z^{\mathrm{i}+1}| = 1$.

$|z^{i+1}| = |r^{i+1} e^{-\theta + i\theta}| = |e^{i \log r + \log r - \theta + i\theta}| = e^{\log r - \theta}$. *Need* $\log r - \theta = 0$
[2] *so* $r = e^{\theta}$.



[1]

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5C Vectors and Matrices

Explain why each of the equations

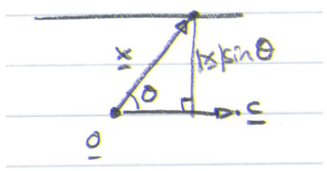
$$\mathbf{x} = \mathbf{a} + \lambda \mathbf{b} \tag{1}$$

$$\mathbf{x} \times \mathbf{c} = \mathbf{d} \tag{2}$$

describes a straight line, where $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ and $\mathbf{d}$ are constant vectors in $\mathbb{R}^3$, $\mathbf{b}$ and $\mathbf{c}$ are non-zero, $\mathbf{c} \cdot \mathbf{d} = 0$ and λ is a real parameter. Describe the geometrical relationship of $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ and $\mathbf{d}$ to the relevant line, assuming that $\mathbf{d} \neq \mathbf{0}$.

$\mathbf{x} = \mathbf{a} + \lambda \mathbf{b}$ *has* $(\mathbf{x} - \mathbf{a}) = \lambda \mathbf{b}$ *so the point* $\mathbf{x}$ *lies in the direction of* $\mathbf{b}$ *from* $\mathbf{a}$. *So* $\mathbf{a}$ *is a point on the line and* $\mathbf{b}$ *is the direction of the line.*

If $\mathbf{x}\times\mathbf{c}=\mathbf{d}$ *then* $\mathbf{x}\cdot\mathbf{d}=\mathbf{x}\cdot(\mathbf{x}\times\mathbf{c})=0$ *so* $\mathbf{x}$ *is orthogonal to* $\mathbf{d}$. *On the plane* $\mathbf{x}\cdot\mathbf{d}=0$, $|\mathbf{x}||\mathbf{c}|\sin\theta=|\mathbf{d}|$ *so* $|\mathbf{x}|\sin\theta$ *is a constant. This is a line in the direction of* $\mathbf{c}$.



Then $\mathbf{d}$ *is the vector orthogonal to the plane through the origin and the*
[5] *line.*

Show that the solutions of (2) satisfy an equation of the form (1), defining $\mathbf{a}$, $\mathbf{b}$ and $\lambda(\mathbf{x})$ in terms of $\mathbf{c}$ and $\mathbf{d}$ such that $\mathbf{a}\cdot\mathbf{b}=0$ and $|\mathbf{b}|=|\mathbf{c}|$. Deduce that the conditions on $\mathbf{c}$ and $\mathbf{d}$ are sufficient for (2) to have solutions.

If $\mathbf{x}\times\mathbf{c}=\mathbf{d}$ *then let* $\mathbf{b}=\mathbf{c}$ *and find an* $\mathbf{a}$ *on the line with* $\mathbf{a}\cdot\mathbf{b}=0$.

Take $(\mathbf{c}\times\mathbf{d})\times\mathbf{c}=\mathbf{d}(\mathbf{c}\cdot\mathbf{c})-\mathbf{c}(\mathbf{c}\cdot\mathbf{d})=\mathbf{d}(\mathbf{c}\cdot\mathbf{c})$ *so* $\mathbf{a}=(\mathbf{c}\times\mathbf{d})/(\mathbf{c}\cdot\mathbf{c})$
[4] *is on the line and has* $\mathbf{a}\cdot\mathbf{b}=0$. *Now* $\lambda(\mathbf{x})=(\mathbf{x}-\mathbf{a})\cdot\mathbf{b}/(\mathbf{b}\cdot\mathbf{b})$

[1] *So there is a solution.*

For each of the lines described by (1) and (2), find the point $\mathbf{x}$ that is closest to a given fixed point $\mathbf{y}$.

For a line of the form $\mathbf{x}=\mathbf{a}+\lambda\mathbf{b}$, *find the value of* λ *that gives* $(\mathbf{y}-\mathbf{x})\cdot\mathbf{b}=0$. *This has* $\mathbf{y}\cdot\mathbf{b}-\mathbf{a}\cdot\mathbf{b}-\lambda\mathbf{b}\cdot\mathbf{b}=0$ *so* $\lambda=(\mathbf{y}\cdot\mathbf{b}-\mathbf{a}\cdot\mathbf{b})/\mathbf{b}\cdot\mathbf{b}$.

For a line of the form $\mathbf{x}\times\mathbf{c}=\mathbf{d}$, *use the previous result. Take* $\mathbf{a}=(\mathbf{c}\times\mathbf{d})/(\mathbf{c}\cdot\mathbf{c})$ *and* $\mathbf{b}=\mathbf{c}$ *in*

[5] $$\mathbf{x}=\mathbf{a}+\frac{(\mathbf{y}\cdot\mathbf{b}-\mathbf{a}\cdot\mathbf{b})}{\mathbf{b}\cdot\mathbf{b}}\mathbf{b}=\frac{\mathbf{c}\times\mathbf{d}}{\mathbf{c}\cdot\mathbf{c}}+\frac{\mathbf{y}\cdot\mathbf{c}}{\mathbf{c}\cdot\mathbf{c}}\mathbf{c}$$

Find the line of intersection of the two planes $\mathbf{x}\cdot\mathbf{m}=\mu$ and $\mathbf{x}\cdot\mathbf{n}=\nu$, where $\mathbf{m}$ and $\mathbf{n}$ are constant unit vectors, $\mathbf{m}\times\mathbf{n}\neq\mathbf{0}$ and μ and ν are constants. Express your answer in each of the forms (1) and (2), giving both $\mathbf{a}$ and $\mathbf{d}$ as linear combinations of $\mathbf{m}$ and $\mathbf{n}$.

Suppose $\mathbf{a}=\alpha\mathbf{m}+\beta\mathbf{n}$. *Then for* $\mathbf{a}$ *to be on the line, we need* $\mathbf{a}\cdot\mathbf{m}=\mu$ *and* $\mathbf{x}\cdot\mathbf{n}=\nu$ *so*

[2] $$\begin{pmatrix}1 & \mathbf{m}\cdot\mathbf{n}\\ \mathbf{m}\cdot\mathbf{n} & 1\end{pmatrix}\begin{pmatrix}\alpha\\ \beta\end{pmatrix}=\begin{pmatrix}\mu\\ \nu\end{pmatrix}\quad \textit{so}\quad \begin{pmatrix}\alpha\\ \beta\end{pmatrix}=\frac{1}{1-(\mathbf{m}\cdot\mathbf{n})^2}\begin{pmatrix}1 & -\mathbf{m}\cdot\mathbf{n}\\ -\mathbf{m}\cdot\mathbf{n} & 1\end{pmatrix}\begin{pmatrix}\mu\\ \nu\end{pmatrix}$$

Then the direction of the line is orthogonal to both $\mathbf{m}$ *and* $\mathbf{n}$ *so*
[1] *take* $\mathbf{b}=\mathbf{m}\times\mathbf{n}$

For lines of the form $\mathbf{x}\times\mathbf{c}=\mathbf{d}$, *take* $\mathbf{c}=\mathbf{m}\times\mathbf{n}$. *Then*
[2] $\mathbf{d}=\mathbf{x}\times(\mathbf{m}\times\mathbf{n})=\mathbf{m}(\mathbf{x}\cdot\mathbf{n})-\mathbf{n}(\mathbf{x}\cdot\mathbf{m})=\nu\mathbf{m}-\mu\mathbf{n}$

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8B Vectors and Matrices

(a) Let M be a real symmetric $n \times n$ matrix. Prove the following.

(i) Each eigenvalue of M is real.

Suppose $M\mathbf{x} = \lambda\mathbf{x}$. *Then* $\mathbf{x}^\dagger M^\dagger = \mathbf{x}^\dagger \lambda^*$ *so* $(\mathbf{x}^\dagger M^\dagger)\mathbf{x} = \mathbf{x}^\dagger \lambda^* \mathbf{x}$ *and* $\mathbf{x}^\dagger M^\dagger \mathbf{x} = \mathbf{x}^\dagger M\mathbf{x} = \mathbf{x}^\dagger \lambda \mathbf{x}$, *so* $\lambda^* = \lambda$ *and so* λ *is real.*

(ii) Each eigenvector can be chosen to be real.

Suppose $M\mathbf{x} = \lambda\mathbf{x}$. *Then* $M\mathbf{x}^* = \lambda\mathbf{x}^*$ *so choose the eigenvectors* $\mathbf{x}+\mathbf{x}^*$ *and* $i(\mathbf{x}-\mathbf{x}^*)$.

(iii) Eigenvectors with different eigenvalues are orthogonal.

Suppose $M\mathbf{x} = \lambda_1\mathbf{x}$ *and* $M\mathbf{y} = \lambda_2\mathbf{y}$. *Then* $(\mathbf{y}^\dagger M^\dagger)\mathbf{x} = \lambda_2 \mathbf{y}^\dagger \mathbf{x}$ *but also* $(\mathbf{y}^\dagger M^\dagger)\mathbf{x} = \mathbf{y}^\dagger M\mathbf{x} = \lambda_1 \mathbf{y}^\dagger \mathbf{x}$. *If* $\lambda_1 \neq \lambda_2$ *then* $\mathbf{y}^\dagger \mathbf{x} = 0$ *so the*
[10] *eigenvectors are orthogonal.*

(b) Let A be a real antisymmetric $n \times n$ matrix. Prove that each eigenvalue of A^2 is real and is less than or equal to zero.

Suppose $A^2\mathbf{x} = \lambda\mathbf{x}$. *Then* $\mathbf{x}^\dagger A^2 \mathbf{x} = \lambda \mathbf{x}^\dagger \mathbf{x} = \lambda |\mathbf{x}|^2$ *but also* $\mathbf{x}^\dagger A^2 \mathbf{x} =$
[4] $-\mathbf{x}^\dagger A^\dagger A\mathbf{x} = -|A\mathbf{x}|^2$, *and so* λ *is real and negative.*

If $-\lambda^2$ and $-\mu^2$ are distinct, non-zero eigenvalues of A^2, show that there exist orthonormal vectors $\mathbf{u}$, $\mathbf{u}'$, $\mathbf{w}$, $\mathbf{w}'$ with

$$A\mathbf{u} = \lambda\mathbf{u}', \qquad A\mathbf{w} = \mu\mathbf{w}',$$
$$A\mathbf{u}' = -\lambda\mathbf{u}, \qquad A\mathbf{w}' = -\mu\mathbf{w}.$$

Let $\mathbf{x}$ *and* $\mathbf{y}$ *be unit eigenvectors of* A^2 *corresponding to* $-\lambda^2$ *and* $-\mu^2$ *respectively. Then let* $\mathbf{u} = \mathbf{x}$, $\mathbf{w} = \mathbf{y}$, $\mathbf{u}' = (A\mathbf{x})/\lambda$ *and* $\mathbf{w}' = (A\mathbf{y})/\mu$. *Since* $A^2\mathbf{x} = -\lambda^2\mathbf{x}$, *we must have* $A\mathbf{u}' = -\lambda\mathbf{u}$ *and similarly for* $\mathbf{w}'$. *So we just need to check that the vectors* $\mathbf{u}$, $\mathbf{u}'$, $\mathbf{w}$, $\mathbf{w}'$ *are orthogonal and that the vectors* $\mathbf{u}'$ *and* $\mathbf{w}'$ *are unit vectors. First note that* A^2 *is real and symmetric as* $(A^2)^T = A^T A^T = (-A)(-A) = A^2$, *so the eigenvectors of* A^2 *corresponding to different eigenvalues are orthogonal. Now note that* $\mathbf{u}^T A\mathbf{u} = -(\mathbf{u}^T A^T)\mathbf{u} = \left(-(\mathbf{u}^T A^T)\mathbf{u}\right)^T = -\mathbf{u}^T(A\mathbf{u})$ *so* $\mathbf{u}^T A\mathbf{u} = 0$. *Finally, check that* $\mathbf{u}'$ *is a unit vector with* $(\mathbf{u}^T A^T/\lambda)(A\mathbf{u}/\lambda) = -\mathbf{u}^T A^2 \mathbf{u}/\lambda^2 = \mathbf{u}^T\mathbf{u}$.*As* $\mathbf{u}$ *is a unit vector, so is* $\mathbf{u}'$
[6] *(and similarly for* $\mathbf{w}'$*).*

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