4C Vector Calculus

State the value of $\partial x_i/\partial x_j$ and find $\partial r/\partial x_j$, where $r = |\mathbf{x}|$.

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[1]
$$\frac{\partial x_i}{\partial x_j} = \delta_{ij}$$

[1]
$$\frac{\partial r}{\partial x_j} = \frac{\partial}{\partial x_j}(x_i x_i)^{1/2} = \frac{1}{2} 2x_j (x_i x_i)^{-1/2} = \frac{x_j}{r}$$

Vector fields $\mathbf{u}$ and $\mathbf{v}$ in $\mathbb{R}^3$ are given by $\mathbf{u} = r^\alpha \mathbf{x}$ and $\mathbf{v} = \mathbf{k} \times \mathbf{u}$, where α is a constant and $\mathbf{k}$ is a constant vector. Calculate the second-rank tensor $d_{ij} = \partial u_i/\partial x_j$, and deduce that $\boldsymbol{\nabla} \times \mathbf{u} = \mathbf{0}$ and $\boldsymbol{\nabla} \cdot \mathbf{v} = 0$.

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[2]
$$\frac{\partial u_i}{\partial x_j} = \frac{\partial}{\partial x_j}(r^\alpha x_i) = \alpha x_j r^{\alpha-2} x_i + r^\alpha \delta_{ij}$$

$$(\nabla \times \mathbf{u})_i = \epsilon_{ijk} \frac{\partial x_k}{\partial x_j} = \epsilon_{ijk} \left(\alpha x_j x_k r^{\alpha-2} + r^\alpha \delta_{jk}\right) = 0$$

[2] *as ϵ_{ijk} is antisymmetric in j and k while the term in brackets is symmetric.*

$$\nabla \cdot \mathbf{v} = \frac{\partial}{\partial x_j}(\epsilon_{jkl} k_k u_l) = \epsilon_{jkl} k_k \left(\alpha r^{\alpha-2} x_l x_j + r^\alpha \delta_{jl}\right) = 0$$

[2] *as ϵ_{jkl} is antisymmetric in j and l while the term in brackets is symmetric.*

When $\alpha = -3$, show that $\boldsymbol{\nabla} \cdot \mathbf{u} = 0$ and

$$\boldsymbol{\nabla} \times \mathbf{v} = \frac{3(\mathbf{k} \cdot \mathbf{x})\mathbf{x} - \mathbf{k} r^2}{r^5} \quad .$$

$$\nabla \cdot \mathbf{u} = \alpha r^\alpha + r^\alpha \delta_{ii} = -3r^{-3} + 3r^{-3} = 0 \text{ } if \text{ } \alpha = -3$$

$$\begin{aligned}
(\nabla \times \mathbf{v})_i =& \epsilon_{ijk} \frac{\partial}{\partial x_j}(\epsilon_{klm} k_l u_m) \\
=& (\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}) k_l \left(-3r^{-5} x_m x_j + r^{-3}\delta_{mj}\right) \\
=& -3k_i r^{-5} r^2 + 3r^{-5} k_j x_i x_j + 3r^{-3} k_i - k_j r^{-3} \delta_{ij} \\
=& \left(3r^{-5}(\mathbf{k} \cdot \mathbf{x})\mathbf{x} - \mathbf{k} r^{-3}\right)_i
\end{aligned}$$

[2]

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[10]

9C Vector Calculus

Write down the most general isotropic tensors of rank 2 and 3. Use the tensor transformation law to show that they are, indeed, isotropic.

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[2] *$\lambda\delta_{ij}$ and $\mu\epsilon_{ijk}$.*

Check that $R_{li}R_{mj}\delta_{ij} = R_{li}R_{mi} = (R^TR)_{lm} = \delta_{mn}$

[4] *and $R_{li}R_{mj}R_{nk}\epsilon_{ijk} = \epsilon_{lmn}\det R = \epsilon_{lmn}$*

Let V be the sphere $0 \leqslant r \leqslant a$. Explain briefly why

$$T_{i_1 \ldots i_n} = \int_V x_{i_1} \ldots x_{i_n}\,\mathrm{d}V$$

is an isotropic tensor for any n.

A rotation of the basis is equivalent to a backward rotation of the sphere.
[2] *Spheres have no preferred direction, so the integral will still be T.*

Hence show that

$$\int_V x_i x_j\,\mathrm{d}V = \alpha\delta_{ij}, \quad \int_V x_i x_j x_k\,\mathrm{d}V = 0 \quad \text{and} \quad \int_V x_i x_j x_k x_l\,\mathrm{d}V = \beta(\delta_{ij}\delta_{kl} + \delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})$$

for some scalars α and β, which should be determined using suitable contractions of the indices or otherwise.

[*You may assume that the most general isotropic tensor of rank 4 is*

$$\lambda\delta_{ij}\delta_{kl} + \mu\delta_{ik}\delta_{jl} + \nu\delta_{il}\delta_{jk}\ ,$$

where λ, μ and ν are scalars.]

These are all isotropic and symmetric in the indicies. So $\int_V x_i x_j\,\mathrm{d}V$ must be $\alpha\,\delta_{ij}$. Contract $i = j$: $3\alpha = \int_V x_i x_i\,\mathrm{d}V = \int_V r^4 \sin\theta\,\mathrm{d}r\mathrm{d}\theta\mathrm{d}\phi =$
[3] *$4\pi a^5/5$ so $\alpha = 4\pi a^5/15$*

By isotropy, $\int_V x_i x_j x_k \mathrm{d}V = \mu\epsilon_{ijk}$. But the RHS is antisymmetric, so
[1] *$\mu = 0$ and $\int_V x_i x_j x_k \mathrm{d}V = 0$.*

By isotropy and hint, the integral must be $\lambda\delta_{ij}\delta_{kl} + \mu\delta_{ik}\delta_{jl} + \nu\delta_{il}\delta_{jk}$. Check symmetry $i \leftrightarrow j$ swaps $\mu \leftrightarrow \nu$ so must have $\mu = \nu$ and similarly
[2] *$\mu = \lambda$. So $\int_V x_i x_j x_k x_l \mathrm{d}V = \beta(\delta_{ij}\delta_{kl} + \delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})$ for some β.*

Contract $i = j$ and $k = l$: $x_i x_i x_k x_k = r^4$

$$\int_V r^4 r^2 \sin\theta \mathrm{d}r\mathrm{d}\theta\mathrm{d}\phi = \frac{4\pi a^7}{7} \quad \textit{and} \quad \beta(\delta_{ii}\delta_{kk} + \delta_{ik}\delta_{ik} + \delta_{ik}\delta_{ik}) = \beta(9 + 3 + 3)$$

[4] *so $\beta = 4\pi a^7/105$.*

Deduce the value of

$$\int_V \mathbf{x} \times (\boldsymbol{\Omega} \times \mathbf{x})\,\mathrm{d}V\ ,$$

where $\mathbf{\Omega}$ is a constant vector.

$$\int_V \epsilon_{ijk}x_j\epsilon_{klm}\Omega_l x_m \mathrm{d}V = (\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl})\Omega_l\left(\frac{4\pi a^5}{15}\delta_{jm}\right)$$
$$= (3\delta_{il} - \delta_{il})\Omega_l\left(\frac{4\pi a^5}{15}\right) = \frac{\Omega_i 8\pi a^5}{15}$$

[2] *so* $\int_V \mathbf{x}\times(\mathbf{\Omega}\times\mathbf{x})\,\mathrm{d}V = 8\pi a^5\mathbf{\Omega}/15$

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[20]

11C Vector Calculus

The electric field $\mathbf{E}(\mathbf{x})$ due to a static charge distribution with density $\rho(\mathbf{x})$ satisfies

$$\mathbf{E} = -\nabla\phi\,, \qquad \nabla\cdot\mathbf{E} = \frac{\rho}{\varepsilon_0}\,, \tag{1}$$

where $\phi(\mathbf{x})$ is the corresponding electrostatic potential and ε_0 is a constant.

(a) Show that the total charge Q contained within a closed surface S is given by Gauss' law

$$Q = \varepsilon_0\int_S \mathbf{E}\cdot\mathrm{d}\mathbf{S}\,.$$

Assuming spherical symmetry, deduce the electric field and potential due to a point charge q at the origin i.e. for $\rho(\mathbf{x}) = q\,\delta(\mathbf{x})$.

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[2] $Q = \int_V \rho\,\mathrm{d}V = \epsilon_0\int_V \nabla\cdot\mathbf{E}\,\mathrm{d}V = \epsilon_0\int_S \mathbf{n}\cdot\mathbf{E}\,\mathrm{d}S$ *by divergence theorem.*

$\rho(\mathbf{x}) = q\delta(\mathbf{x})$ *and* $\mathbf{E} = E(r)\mathbf{n}$. *Hence by Gauss' law* $q = 4\pi r^2\epsilon_0 E(r)$.

[4] *So* $E(r) = q/4\pi\epsilon_0 r^2$ *and* $\phi = q/4\pi\epsilon_0 r$.

(b) Let $\mathbf{E}_1$ and $\mathbf{E}_2$, with potentials ϕ_1 and ϕ_2 respectively, be the solutions to (1) arising from two different charge distributions with densities ρ_1 and ρ_2. Show that

$$\frac{1}{\varepsilon_0}\int_V \phi_1\rho_2\,\mathrm{d}V + \int_{\partial V}\phi_1\nabla\phi_2\cdot\mathrm{d}\mathbf{S} = \frac{1}{\varepsilon_0}\int_V \phi_2\rho_1\,\mathrm{d}V + \int_{\partial V}\phi_2\nabla\phi_1\cdot\mathrm{d}\mathbf{S} \tag{2}$$

for any region V with boundary ∂V, where $\mathrm{d}\mathbf{S}$ points out of V.

$\nabla\cdot(\phi_2\nabla\phi_1 - \phi_1\nabla\phi_2) = \nabla\phi_2\cdot\nabla\phi_1 + \phi_2\nabla^2\phi_1 - \nabla\phi_1\cdot\nabla\phi_2 - \phi_1\nabla^2\phi_2$
$= \phi_2\nabla^2\phi_1 - \phi_1\nabla^2\phi_2$ *and (1) gives* $\nabla^2\phi = -\rho/\varepsilon_0$. *Hence*

[5] $$\int_S(\phi_2\nabla\phi_1 - \phi_1\nabla\phi_2)\cdot\mathbf{n}\,\mathrm{d}S = \int_V -\phi_2\frac{\rho_1}{\varepsilon_0} + \phi_1\frac{\rho_2}{\varepsilon_0}\,\mathrm{d}V \quad\Rightarrow\quad \textit{answer}$$

(c) Suppose that $\rho_1(\mathbf{x}) = 0$ for $|\mathbf{x}| \leqslant a$ and that $\phi_1(\mathbf{x}) = \Phi$, a constant, on $|\mathbf{x}| = a$. Use the results of (a) and (b) to show that

$$\Phi = \frac{1}{4\pi\varepsilon_0}\int_{r>a}\frac{\rho_1(\mathbf{x})}{r}\,\mathrm{d}V\,.$$

[*You may assume that* $\phi_1 \to 0$ *as* $|\mathbf{x}| \to \infty$ *sufficiently rapidly that any integrals over the 'sphere at infinity' in* (2) *are zero.*]

Take $V = \{r > a\}$ *and* $\rho_2 = q\delta(\mathbf{x})$, *so that* $\phi_2 = q/4\pi\varepsilon_0 r$. *Now part* (b) *gives*

$$\frac{1}{\varepsilon_0}\int_V \phi_1\rho_2 \mathrm{d}V + \int_{r=a} \Phi\left(\frac{q}{4\pi\varepsilon_0 a^2}\right)\mathrm{d}S = \frac{1}{\varepsilon_0}\int_V \frac{q}{4\pi\varepsilon_0 r}\rho_1 \mathrm{d}V + \int_{r=a}\frac{q}{4\pi\epsilon_0 a}\boldsymbol{\nabla}\phi_1\cdot\mathbf{n}\mathrm{d}S$$

But $\rho_2 = 0$ *in* V, *and* $\int_{r=a}\boldsymbol{\nabla}\phi_1\cdot\mathbf{n}\,\mathrm{d}S = Q_1/\varepsilon_0 = 0$ *by part* (a) *so*

$$\frac{\Phi q}{\varepsilon_0} = \frac{q}{4\pi\varepsilon_0^2}\int_V \frac{1}{r}\rho_1 \mathrm{d}V \quad \Rightarrow \quad \textit{answer}$$

[9]

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[20]