3F Probability

Let X be a random variable taking non-negative integer values and let Y be a random variable taking real values.

(a) Define the probability-generating function $G_X(s)$. Calculate it explicitly for a Poisson random variable with mean $\lambda > 0$.

$G_X(s) = \mathbb{E}\big(s^X\big) = \sum_{n=0}^{\infty} s^n \mathbb{P}(X = n)$. For $X \sim Po(\lambda)$,

$$G_X(s) = \sum_{n=0}^{\infty} s^n e^{-\lambda} \frac{\lambda^n}{n!} = e^{-\lambda} \sum_{n=0}^{\infty} \frac{(s\lambda)^n}{n!} = e^{\lambda(s-1)}.$$

[3]

(b) Define the moment-generating function $M_Y(t)$. Calculate it explicitly for a normal random variable $\mathrm{N}(0, 1)$.

$M_Y(t) = \mathbb{E}\big(e^{tX}\big)$. For $Y \sim N(0, 1)$,

$$\begin{aligned} M_Y(t) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{tx} e^{-x^2/2}\, \mathrm{d}x \\ &= \frac{e^{t^2/2}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-(x-t)^2/2}\, \mathrm{d}x = e^{t^2/2}, \end{aligned}$$

as the last integral gives $\sqrt{2\pi}$ (since $e^{-(x-t)^2/2}\big/\sqrt{2\pi}$ is the PDF for
[3] *$N(t, 1)$).*

(c) By considering a random sum of independent copies of Y, prove that, for general X and Y, $G_X\big(M_Y(t)\big)$ is the moment-generating function of some random variable.

Let $X, Y_1, Y_2, \ldots$ be independent reandom variables with $Y_n \sim Y$ for all n. Define the random variable $Z = \sum_{n=1}^{X} Y_n$.

We then have (for all t)

$$\begin{aligned} M_Z(t) = \mathbb{E}\big(e^{tZ}\big) &= \sum_{n=0}^{\infty} \mathbb{P}(X = n) \mathbb{E}\big(e^{tZ} \big| X = n\big) \\ &= \sum_{n=0}^{\infty} \mathbb{P}(X = n) \mathbb{E}\Big(e^{t(Y_1 + \cdots + Y_n)}\Big) \\ &= \sum_{n=0}^{\infty} \mathbb{P}(X = n) \prod_{i=1}^{n} \mathbb{E}\big(e^{tY_i}\big) \quad \textit{since the } Y_i \textit{ are indep} \\ &= \sum_{n=0}^{\infty} \mathbb{P}(X = n) \mathbb{E}\big(e^{tY}\big)^n = G_X\big(M_Y(t)\big). \end{aligned}$$

[4]

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[10]

9F Probability

(a) Let $B_1, \ldots, B_n$ be pairwise disjoint events such that their union $B_1 \cup B_2 \cup \ldots \cup B_n$ gives the whole set of outcomes, with $\mathbb{P}(B_i) > 0$ for $1 \leqslant i \leqslant n$. Prove that for any event A with $\mathbb{P}(A) > 0$ and for any i

$$\mathbb{P}(B_i|A) = \frac{\mathbb{P}(A|B_i)\mathbb{P}(B_i)}{\sum\limits_{1 \leqslant j \leqslant n} \mathbb{P}(A|B_j)\mathbb{P}(B_j)} .$$

By definition of conditional probability, for each i

$$\mathbb{P}(B_i|A) = \frac{\mathbb{P}(B_i \cap A)}{\mathbb{P}(A)} = \frac{\mathbb{P}(A|B_i)\mathbb{P}(B_i)}{\mathbb{P}(A)}. \qquad (1)$$

Since the B_j are pairwise disjoint and cover all outcomes,

$$\mathbb{P}(A) = \mathbb{P}\Big(\bigcup_j (A \cap B_j)\Big) = \sum_j \mathbb{P}(A \cap B_j).$$

The RHS can be rewritten as

$$\sum_j \mathbb{P}(A|B_j)\mathbb{P}(B_j),$$

[5] *and subbing into (1) for $\mathbb{P}(A)$ gives the result.*

(b) A prince is equally likely to sleep on any number of mattresses from six to eight; on half the nights a pea is placed beneath the lowest mattress. With only six mattresses his sleep is always disturbed by the presence of a pea; with seven a pea, if present, is unnoticed in one night out of five; and with eight his sleep is undisturbed despite an offending pea in two nights out of five.

What is the probability that, on a given night, the prince's sleep was undisturbed?

Let U be the event that he slept undisturbed, P be the event that a pea is placed (with complement P^c), and N be the number of mattresses. Assuming P and N are indep, we have

$$\begin{aligned}
\mathbb{P}(U) &= \mathbb{P}(U \cap P^c) + \sum_{n=6}^{8} \mathbb{P}(U \cap P \cap \{N = n\}) \\
&= \frac{1}{2} + \sum_{n=6}^{8} \mathbb{P}(U|P \cap \{N = n\}) \times \mathbb{P}(P) \times \mathbb{P}(N = n) \\
&= \frac{1}{2} + \sum_{n=6}^{8} \frac{n-6}{5} \times \frac{1}{2} \times \frac{1}{3} = \frac{1}{2} + \frac{1}{6}\left(0 + \frac{1}{5} + \frac{2}{5}\right) = \frac{3}{5}.
\end{aligned}$$

[7]

On the morning of his wedding day, he announces that he has just spent the most peaceful and undisturbed of nights. What is the expected number of mattresses on which he slept the previous night?

(c) *For each* $n \in \{6, 7, 8\}$ *we have*

$$\mathbb{P}(N = n|U) = \frac{\mathbb{P}(U|N = n)\mathbb{P}(N = n)}{\mathbb{P}(U)} = \frac{5}{9} \times \mathbb{P}(U|N = n).$$

And

$$\begin{aligned}\mathbb{P}(U|N = n) &= \mathbb{P}(P)\mathbb{P}(U|P \cap \{N = n\}) + \mathbb{P}(P^c)\mathbb{P}(U|P^c \cap \{N = n\}) \\ &= \frac{n-6}{10} + \frac{1}{2} = \frac{n-1}{10}.\end{aligned}$$

So

$$\mathbb{P}(N = n|U) = \frac{n-1}{18}$$

and hence

$$\begin{aligned}\mathbb{E}(N|U) &= \sum_{n=6}^{8} n\mathbb{P}(N = n|U) \\ &= \sum_{n=6}^{8} \frac{n(n-1)}{18} = \frac{1}{18}(30 + 42 + 56) = \frac{128}{18} = \frac{64}{9}.\end{aligned}$$

[8]

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[20]

12F Probability

A circular island has a volcano at its central point. During an eruption, lava flows from the mouth of the volcano and covers a sector with random angle Φ (measured in radians), whose line of symmetry makes a random angle Θ with some fixed compass bearing.

The variables Θ and Φ are independent. The probability density function of Θ is constant on $(0, 2\pi)$ and the probability density function of Φ is of the form $A(\pi - \phi/2)$ where $0 < \phi < 2\pi$, and A is a constant.

(a) Find the value of A. Calculate the expected value and the variance of the sector angle Φ. Explain briefly how you would simulate the random variable Φ using a uniformly distributed random variable U.

By normalisation of the PDF, $\int_0^{2\pi} A(\pi - \phi/2)\, \mathrm{d}\phi = 1$

Hence $1 = A\big[\pi\phi - \phi^2/4\big]_0^{2\pi} = \pi^2 A$. *Hence* $A = \dfrac{1}{\pi^2}$.

Then

$$\mathbb{E}(\Phi) = \int_0^{2\pi} A\phi(\pi - \phi/2)\, \mathrm{d}\phi = A\big[\pi\phi^2/2 - \phi^3/6\big]_0^{2\pi} = A(2\pi^3 - 4\pi^3/3) = \frac{2\pi}{3}.$$

and

$$\mathbb{E}(\Phi^2) = \int_0^{2\pi} A\phi^2(\pi - \phi/2)\ \mathrm{d}\phi = A\big[\pi\phi^3/3 - \phi^4/8\big]_0^{2\pi}$$
$$= A(8\pi^4/3 - 2\pi^4) = \frac{2\pi^2}{3}$$

so

$$\mathrm{Var}(\Phi) = \mathbb{E}(\Phi^2) - \mathbb{E}(\Phi)^2 = \frac{2\pi^2}{3} - \frac{4\pi^2}{9} = \frac{2\pi^2}{9}.$$

Φ *has distribution function*

$$F_\Phi(\phi) = \mathbb{P}(\Phi \leqslant \phi) = \begin{cases} 0 & \text{for } \phi \leqslant 0 \\ \phi/\pi - \phi^2/4\pi^2 & \text{for } \phi \in [0, 2\pi] \\ 1 & \text{for } \phi \geqslant 2\pi. \end{cases}$$

For $x \in (0,1)$ *the inverse* $F_\Phi^{-1}(x)$ *is a solution to*

$$\frac{\phi^2}{4\pi^2} - \frac{\phi}{\pi} + x = 0$$

and is therefore

$$\frac{1/\pi \pm \sqrt{1/\pi^2 - x/\pi^2}}{1/2\pi^2} = 2\pi\big(1 \pm \sqrt{1-x}\big).$$

Since $\Phi \leqslant 2\pi$ *we must have the minus in* $\pm$*. Hence if* $U \sim U(0,1)$ *then*

$$F_\Phi^{-1}(U) = 2\pi\big(1 - \sqrt{1-U}\big)$$

[7] *simulates* Φ.

(b) H_1 and H_2 are two houses on the island which are collinear with the mouth of the volcano, but on different sides of it. Find

(i) the probability that H_1 is hit by the lava;

WLOG let H_1 *and* H_2 *be at angles* $\theta = 0$ *and* π *from the volcano mouth, w.r.t the given fixed bearing. Then*

$$\mathbb{P}(H_1\ \mathit{hit}) = \int_0^{2\pi}\int_0^{2\pi} f_\Theta(\theta) f_\Phi(\phi) \mathbb{1}_{\{\theta<\phi/2\}\cup\{2\pi-\phi/2<\theta\}}\ \mathrm{d}\theta\ \mathrm{d}\phi$$
$$= \int_0^{2\pi} \frac{\phi}{2\pi} f_\Phi(\phi)\ \mathrm{d}\phi = \mathbb{E}(\Phi)/2\pi = \frac{1}{3}.$$

[4]

(ii) the probability that both H_1 and H_2 are hit by the lava;

$$\begin{aligned}
\mathbb{P}(\textit{Both hit}) &= \int_0^{2\pi}\int_0^{2\pi} f_\Theta(\theta) f_\Phi(\phi) \mathbb{1}_{\{\pi-\phi/2<\theta<\phi/2\}\cup\{2\pi-\phi/2<\theta<\pi+\phi/2\}} \, \mathrm{d}\theta \, \mathrm{d}\phi \\
&= \int_\pi^{2\pi} \frac{2(\phi-\pi)}{2\pi} f_\Phi(\phi) \, \mathrm{d}\phi \\
&= \frac{1}{\pi^3}\int_\pi^{2\pi} (\phi-\pi)(\pi-\phi/2) \, \mathrm{d}\phi \\
&= \frac{1}{2\pi^3}\int_0^{\pi} \psi(\pi-\psi) \, \mathrm{d}\psi \quad \textit{by substitution} \\
&= \frac{1}{2\pi^3}\left[\pi\psi^2/2 - \psi^3/3\right]_0^\pi = \frac{1}{2\pi^3} \times \frac{\pi^3}{6} = \frac{1}{12}.
\end{aligned}$$

[5]

(iii) the probability that H_2 is not hit by the lava given that H_1 is hit.

$$\begin{aligned}
\mathbb{P}(H_2 \textit{ not hit} | H_1 \textit{ hit}) &= 1 - \mathbb{P}(H_2 \textit{ hit} | H_1 \textit{ hit}) \\
&= 1 - \frac{\mathbb{P}(\textit{Both hit})}{\mathbb{P}(H_1 \textit{ hit})} = 1 - \frac{1/12}{1/3} = \frac{3}{4}.
\end{aligned}$$

[4]

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[20]