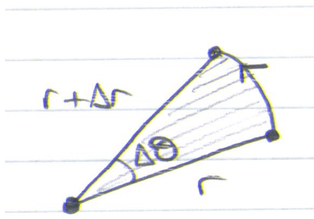
3B Dynamics and Relativity

The motion of a planet in the gravitational field of a star of mass M obeys

$$\frac{\mathrm{d}^2 r}{\mathrm{d}t^2} - \frac{h^2}{r^3} = -\frac{GM}{r^2}\ , \qquad r^2\frac{\mathrm{d}\theta}{\mathrm{d}t} = h\ ,$$

where $r(t)$ and $\theta(t)$ are polar coordinates in a plane and h is a constant. Explain one of Kepler's Laws by giving a geometrical interpretation of h.

The area swept out by a line from the star to the planet grows at constant rate, because this area $A \propto r^2 \mathrm{d}\theta$ and $r^2\frac{\mathrm{d}\theta}{\mathrm{d}t} = h$.



[2]

Show that circular orbits are possible, and derive another of Kepler's Laws relating the radius a and the period T of such an orbit.

Circular orbits with radius a satisfy

$$-\frac{h^2}{a^3} = -\frac{GM}{a^2} \quad \textit{and} \quad a^2\frac{\mathrm{d}\theta}{\mathrm{d}t} = h$$

so $a = h^2/GM$ and $\frac{\mathrm{d}\theta}{\mathrm{d}t} = G^2M^2/h^3$. The period is $T = 2\pi/\frac{\mathrm{d}\theta}{\mathrm{d}t} = 2\pi h^3/G^2M^2 \propto a^{3/2}$. $T \propto a^{3/2}$ is another of Kepler's laws.

[3]

Show that any circular orbit is stable under small perturbations that leave h unchanged.

Let $r = a + \epsilon(t)$. Then if $\epsilon \ll a$, expand to $O(\epsilon)$:

$$\epsilon'' - \frac{h^2}{a^3}\left(1 - \frac{3\epsilon}{a}\right) = -\frac{GM}{a^2}\left(1 - \frac{2\epsilon}{a}\right)$$

so $\epsilon'' + (h^2/a^4)\epsilon = 0$. Since $h^2/a^4 > 0$, the orbit is stable.

[5]

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[10]

10B Dynamics and Relativity

The trajectory of a particle $\mathbf{r}(t)$ is observed in a frame S which rotates with constant angular velocity $\boldsymbol{\omega}$ relative to an inertial frame I. Given that the time derivative in I of any vector $\mathbf{u}$ is

$$\left(\frac{\mathrm{d}\mathbf{u}}{\mathrm{d}t}\right)_I = \dot{\mathbf{u}} + \boldsymbol{\omega}\times\mathbf{u}\ ,$$

where a dot denotes a time derivative in S, show that

$$m\ddot{\mathbf{r}} = \mathbf{F} - 2m\boldsymbol{\omega}\times\dot{\mathbf{r}} - m\boldsymbol{\omega}\times(\boldsymbol{\omega}\times\mathbf{r})\ ,$$

where $\mathbf{F}$ is the force on the particle and m is its mass.

$$\begin{aligned}\mathbf{F} = m\left(\frac{\mathrm{d}^2\mathbf{r}}{\mathrm{d}t^2}\right)_I &= m\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)_I(\dot{\mathbf{r}} + \omega\times\mathbf{r}) \\ &= m\left(\ddot{\mathbf{r}} + \boldsymbol{\omega}\times\dot{\mathbf{r}} + \boldsymbol{\omega}\times(\dot{\mathbf{r}} + \boldsymbol{\omega}\times\mathbf{r})\right) \\ &= m\ddot{\mathbf{r}} + 2m\boldsymbol{\omega}\times\dot{\mathbf{r}} + m\boldsymbol{\omega}\times(\boldsymbol{\omega}\times\mathbf{r})\end{aligned}$$

[2]

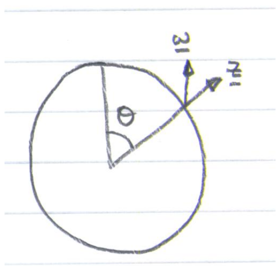
Let S be the frame that rotates with the Earth. Assume that the Earth is a sphere of radius R. Let P be a point on its surface at latitude $\pi/2 - \theta$, and define vertical to be the direction normal to the Earth's surface at P.

(a) A particle at P is released from rest in S and is acted on only by gravity. Show that its initial acceleration makes an angle with the vertical of approximately

$$\frac{\omega^2 R}{g}\sin\theta\cos\theta\ ,$$

working to lowest non-trivial order in ω.

Let $\mathbf{e}_z$ be the unit vector normal to the Earth's surface at P and $\mathbf{e}_N$ be the unit vector in the North direction.



$\boldsymbol{\omega}\times\dot{\mathbf{r}}$ is initially small. $\boldsymbol{\omega}\times\mathbf{r}$ is $\omega R\sin\theta\mathbf{e}_E$ where $\mathbf{e}_E$ is the unit vector in the East direction. Take components in the North and vertical directions;

$$\ddot{\mathbf{r}} = -g\mathbf{e}_z + \sin\theta(\omega^2 R\sin\theta)\mathbf{e}_z - \cos\theta(\omega^2 R\sin\theta)\mathbf{e}_N$$

Then the angle from the vertical satisfies

[8]

$$\tan\phi = \frac{-\omega^2 R\sin\theta\cos\theta}{-g + \sin\theta\omega^2 R\sin\theta} \approx \frac{\omega^2 R\sin\theta\cos\theta}{g}\ \textit{and}\ \tan\phi\approx\phi$$

(b) Now consider a particle fired vertically upwards from P with speed v. Assuming that terms of order ω^2 and higher can be neglected, show that it falls back to Earth under gravity at a distance

$$\frac{4}{3}\frac{\omega v^3}{g^2}\sin\theta$$

from P. [*You may neglect the curvature of the Earth's surface and the vertical variation of gravity.*]

Neglect $\boldsymbol{\omega}\times(\boldsymbol{\omega}\times\mathbf{r})$ term. Velocity is approximately vertical $u(t)\mathbf{e}_z$. Vertical component of forces gives $\ddot{z}=-g$ so $z=-gt^2/2+vt$. East component gives

$$\ddot{x}=-2\omega(-gt+v)\sin\theta \quad \textit{so} \quad x=-2\omega\left(\frac{-gt^3}{6}+\frac{vt^2}{2}\right)\sin\theta$$

Particle lands at time $t=2v/g$, so distance East is $-2\omega\left(\frac{-8v^3}{6g^2}+\frac{4v^3}{2g^3}\right)=$

[10] $-\frac{4}{3}\omega\sin\theta v^3/g^2$

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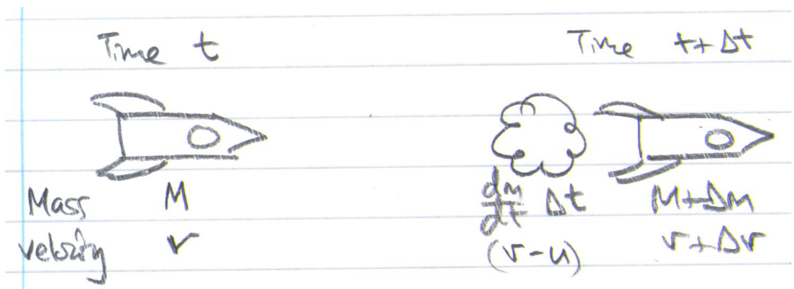
[20]

11B Dynamics and Relativity

A rocket carries equipment to collect samples from a stationary cloud of cosmic dust. The rocket moves in a straight line, burning fuel and ejecting gas at constant speed u relative to itself. Let $v(t)$ be the speed of the rocket, $M(t)$ its total mass, including fuel and any dust collected, and $m(t)$ the total mass of gas that has been ejected. Show that

$$M\frac{\mathrm{d}v}{\mathrm{d}t}+v\frac{\mathrm{d}M}{\mathrm{d}t}+(v-u)\frac{\mathrm{d}m}{\mathrm{d}t}=0\ ,$$

assuming that all external forces are negligible.



Conservation of momentum:

$$Mv=(M+\Delta M)(v+\Delta v)+\frac{\mathrm{d}m}{\mathrm{d}t}\Delta t(v-u)$$
$$0=v\Delta M+M\Delta v+\frac{\mathrm{d}m}{\mathrm{d}t}\Delta t(v-u)$$

[5] *neglecting quadratically small terms. Divide by Δt for result.*

(a) If no dust is collected and the rocket starts from rest with mass M_0, deduce that

$$v = u \log(M_0/M) .$$

With no dust, $M + m = const. \Rightarrow \frac{dM}{dt} = -\frac{dm}{dt}$, *so*

$$M\frac{dv}{dt} = -u\frac{dM}{dt}$$

$$\frac{1}{u}\frac{dv}{dt} = -\frac{1}{M}\frac{dM}{dt} \quad \Rightarrow \quad \frac{v}{u} = -\log M + c$$

[4] *and* $M = M_0$ *when* $v = 0$ *so* $c = \log M_0$.

(b) If cosmic dust is collected at a constant rate of α units of mass per unit time and fuel is consumed at a constant rate $dm/dt = \beta$, show that, with the same initial conditions as in (a),

$$v = \frac{u\beta}{\alpha}\left(1 - (M/M_0)^{\alpha/(\beta-\alpha)}\right) .$$

Now $\frac{dm}{dt} = \beta$ *and* $\frac{dM}{dt} = \alpha - \beta$. *So* $M = (\alpha-\beta)t + M_0$ *and*

$$[(\alpha-\beta)t + M_0]\frac{dv}{dt} + v(\alpha-\beta) + \beta(v-u) = 0$$

$$[(\alpha-\beta)t + M_0]^{\alpha/(\alpha-\beta)}\frac{dv}{dt} + \alpha\,[(\alpha-\beta)t + M_0]^{\alpha/(\alpha-\beta)-1}\,v = \beta u\,[(\alpha-\beta)t + M_0]^{\alpha/(\alpha-\beta)-1}$$

$$[(\alpha-\beta)t + M_0]^{\alpha/(\alpha-\beta)}\,v = \frac{\beta u}{\alpha}\,[(\alpha-\beta)t + M_0]^{\alpha/(\alpha-\beta)} + c$$

and at $t = 0$, $v = 0$ *so* $c = -\beta u M_0^{\alpha/(\alpha-\beta)}/\alpha$.

$$v = \frac{\beta u}{\alpha} - \frac{\beta u}{\alpha}\frac{M_0^{\alpha/(\alpha-\beta)}}{M^{\alpha/(\alpha-\beta)}}$$

[7]

Verify that the solution in (a) is recovered in the limit $\alpha \to 0$.

In the limit $\alpha \to 0$, $(M/M_0)^{\alpha/(\beta-\alpha)} = e^{\log(M/M_0)\alpha/(\beta-\alpha)} \sim 1 + \log(M/M_0)\alpha/(\beta-\alpha)$ *so*

$$v \sim \frac{u\beta}{\alpha}\left(-\frac{\alpha}{\beta-\alpha}\log\left(\frac{M}{M_0}\right)\right) \sim u\log\left(\frac{M_0}{M}\right)$$

[4] *as above.*

[20]