UNIVERSITY OF CAMBRIDGE

MAT1
MATHEMATICAL TRIPOS Part IB

Friday, 12 June, 2026 2:00pm to 5:00pm

PAPER 4

Before you begin read these instructions carefully

The examination paper is divided into two sections. Each question in Section II carries twice the number of marks of each question in Section I. Section II questions also carry an alpha or beta quality mark and Section I questions carry a beta quality mark.

Candidates may obtain credit from attempts on **at most four** questions from Section I and **at most six** questions from Section II.

Write on **one side** of the paper only and begin each answer on a separate sheet.

Write legibly; otherwise you place yourself at a grave disadvantage.

At the end of the examination:

Separate your answers to each question.

Complete a gold cover sheet **for each question** that you have attempted, and place it at the front of your answer to that question.

Complete a green main cover sheet listing **all the questions** that you have attempted.

Every cover sheet must also show your Blind Grade Number and desk number.

Tie up your answers and cover sheets into **a single bundle**, with the main cover sheet on the top, and then the cover sheet and answer for each question, in the numerical order of the questions.

STATIONERY REQUIREMENTS
Gold cover sheets
Green main cover sheet
Treasury tag

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

SECTION I

1E Linear Algebra

Let $\mathbb{F}_p$ be the field of integers modulo p, where p is a prime number. Let V be a vector space of dimension n over $\mathbb{F}_p$.

How many vectors are there in V? How many (ordered) bases does V have? How many bijective endomorphisms of V are there? How many k-dimensional subspaces are there in V? Justify your answers.

2F Analysis II

(a) Let $\Omega \subset \mathbb{R}^n$ and $x_0 \in \Omega$. Let $d\colon \Omega \times \Omega \to \mathbb{R}$ be defined by

$$d(x,y) = \begin{cases} 0\,, & \text{if } x = y\,, \\ \|x - x_0\| + \|y - x_0\|\,, & \text{if } x \neq y\,, \end{cases}$$

where $\|\cdot\|$ denotes the usual Euclidean norm on $\mathbb{R}^n$. Is d a metric on Ω? Justify your answer briefly.

(b) What does it mean to say $X \subset \mathbb{R}^n$ is *path-connected*?

We say $\Omega \subset \mathbb{R}^n$ is star-shaped with star-centre $x_0 \in \Omega$ if, for every $x \in \Omega$, the line segment from x_0 to x lies in Ω, i.e. $\{(1-t)x_0 + tx \colon t \in [0,1]\} \subset \Omega$. Show that star-shaped sets are path-connected.

(c) Let $f\colon \mathbb{R}^n \to \mathbb{R}^m$. Assume that f is C^1, and that there exists a finite $M \geqslant 0$ such that $\sup_{z\in\mathbb{R}^n} \|Df(z)\| \leqslant M$. Show that, for $\Omega \subset \mathbb{R}^n$ star-shaped with star-centre x_0,

$$\|f(x) - f(y)\| \leqslant M d(x,y) \qquad \forall\, x, y \in \Omega,$$

for d as defined in part (a). Deduce that if Df vanishes on Ω, then f is constant in Ω.

3G Complex Analysis

(i) State the argument principle on a domain bounded by a simple closed curve γ. Using the argument principle, or otherwise, state and prove Rouché's theorem.

(ii) Determine the number of zeros of the polynomial $p(z) = z^6 + 15z + 1$ in the annulus $\{z \in \mathbb{C} \colon 1 < |z| < 2\}$.

(iii) Let $U \subset \mathbb{C}$ be an open set. Let $f_n \colon U \to \mathbb{C}$ be injective holomorphic functions, and suppose that $f_n \to f$ uniformly as $n \to \infty$. Prove that the function $f\colon U \to \mathbb{C}$ is either constant or injective. [*Hint: Argue by contradiction and use Rouché's theorem.*]

4B Quantum Mechanics

Consider the Gaussian wavepacket wavefunction

$$\psi(x,t) = \frac{A}{\sqrt{1 + i\hbar t/m}} \exp\left(-\frac{x^2}{2(1 + i\hbar t/m)} \right),$$

where A is a normalising constant which you may take to be real. Show that $\psi(x,t)$ satisfies the time-dependent Schrödinger equation. What is the form of the potential energy?

Find A if $\psi(x,t)$ is normalised.

Calculate the uncertainty in the measurement of the particle position $\Delta x = \sqrt{\langle x^2 \rangle - \langle x \rangle^2}$ as a function of t. What does this imply about whether the state is stationary or not?

[*Hint: You may assume that, for real* $\alpha > 0$, $\int_{-\infty}^{\infty} dx\, e^{-\alpha x^2} = \sqrt{\pi/\alpha}$.]

5D Electromagnetism

A linearly polarized electromagnetic plane wave, described by the real part of the fields

$$\mathbf{E}_{\text{inc}}(\mathbf{x},t) = E_0\, \hat{\mathbf{e}}_x\, e^{i(kz+\omega t)}, \qquad \mathbf{B}_{\text{inc}}(\mathbf{x},t) = -\frac{E_0}{c}\, \hat{\mathbf{e}}_y\, e^{i(kz+\omega t)}, \quad (\omega, k > 0),$$

propagates through vacuum for $z > 0$ and is incident normal to a perfect conductor occupying the region $z \leqslant 0$.

What is the dispersion relation required for this wave to satisfy Maxwell's equations?

Write down the boundary conditions that the electric and magnetic fields must satisfy at the surface of the conductor.

Assuming that a reflected plane wave is produced in the region $z > 0$, use the boundary conditions to determine the reflected electric $\mathbf{E}_{\text{ref}}$ and magnetic $\mathbf{B}_{\text{ref}}$ fields.

What is the resulting surface current induced by the wave?

6C Numerical Analysis

(i) Write down the form of a Givens rotation matrix $\Omega^{[p,q]}$.

(ii) Let A be an $n \times n$ matrix. For any i and j such that $1 \leqslant j < i \leqslant n$, explain how to choose p, q and the angle of the Givens rotation such that $\left(\Omega^{[p,q]} A\right)_{i,j} = 0$. Describe the effect of the Givens rotation on the rows of A.

(iii) Outline the Givens algorithm to obtain a QR factorization of a 3×3 matrix A, showing symbolically the structure of the matrix at each stage of the algorithm.

7H Markov Chains

After each minute, a viewer watching television either continues to watch the same channel with probability $0 < 1 - \alpha < 1$, or picks randomly from any of the other $m - 1$ channels available. You observe the viewer at a time when they are watching the BBC1 channel. Determine the probability that they are watching the BBC2 channel (one of the other channels available) in an hour's time.

SECTION II

8G Linear Algebra

(a) Let A be a complex Hermitian matrix. Prove that all the eigenvalues of A are real.

(b) Let C be a positive definite Hermitian matrix. Prove that there exists a positive definite Hermitian matrix B such that $B^2 = C$.

(c) Let

$$H = \begin{pmatrix} 2 & i \\ -i & 2 \end{pmatrix}.$$

Find a unitary matrix U such that $U^{-1}HU$ is diagonal.

(d) Let T be a self-adjoint operator on a finite-dimensional inner product space $(V, \langle \cdot\,, \cdot \rangle)$. Let $\lambda_{\max}$ be the largest eigenvalue of T. Prove that

$$\lambda_{\max} = \sup_{v \in V \setminus \{0\}} \frac{\langle Tv, v\rangle}{\langle v, v\rangle}.$$

9E Groups, Rings and Modules

In this question rings are commutative with a multiplicative identity.

(a) What is a *Noetherian* ring? You should state your definition using two equivalent statements about ideals.

(i) Show that the image of a Noetherian ring under a ring homomorphism is always Noetherian. Use this to give an example of a Noetherian integral domain that is not a UFD. Is every UFD Noetherian? In both cases give brief justification of your answer.

(ii) Let R be a Noetherian ring. Show that each surjective ring homomorphism $R \to R$ is an injection (and thus is an isomorphism). Give an example of a non-Noetherian ring S and a surjective homomorphism $S \to S$ which is not injective. Is it always true that every injective ring homomorphism from a Noetherian ring to itself is an isomorphism? In both cases give brief justification of your answer.

(b) Let $\mathrm{Int}(\mathbb{Z}) = \{f \in \mathbb{Q}[X] : f(\mathbb{Z}) \subseteq \mathbb{Z}\}$ be the subring of $\mathbb{Q}[X]$ consisting of integer-valued polynomials. You may assume that $\mathrm{Int}(\mathbb{Z})$ is also an (additive) abelian group. Let $f_0(X) = 1$. For any $n \in \mathbb{N}$ define the polynomial

$$f_n(X) = \frac{X(X-1)\cdots(X-n+1)}{n!}.$$

(i) Verify that $f_n(X) \in \mathrm{Int}(\mathbb{Z})$.

(ii) Show that $\{f_n(X) : n \geqslant 0\}$ is a $\mathbb{Z}$-basis of $\mathrm{Int}(\mathbb{Z})$, and hence that $\mathrm{Int}(\mathbb{Z})$ is a free $\mathbb{Z}$-module.

(iii) Is $\mathrm{Int}(\mathbb{Z})$ Noetherian? Justify your answer. [*Hint: You may wish to consider polynomials of the form $f_p(X)$ where p is any prime number.*]

10F Analysis II

What does it mean for a metric space to be *bounded*? What does it mean for a metric space to be *sequentially compact*?

Let (X, d) and (Y, ρ) be metric spaces, with Y having at least two elements. Let $f \colon X \to Y$ be a continuous function.

(i) Show that, if X is sequentially compact, then f is uniformly continuous.

(ii) Show that, if ρ is the discrete metric, then f must be locally constant, i.e. for each $x \in X$ there exists $\delta > 0$ such that

$$d(x, y) < \delta \implies f(x) = f(y).$$

(iii) Show that, if X is sequentially compact *and* ρ is the discrete metric, then f must be uniformly locally constant, i.e. there exists $\delta > 0$ such that

$$d(x, y) < \delta \implies f(x) = f(y) \quad \forall x, y \in X.$$

Consider the Cantor set $C \subseteq [0, 1]$ which is constructed by iteratively deleting the open middle third from a set of line segments:

$$C = \bigcap_{n=0}^{\infty} C_n, \quad \text{where} \quad C_n = \begin{cases} [0,1], & n = 0, \\ \dfrac{C_{n-1}}{3} \cup \left(\dfrac{2}{3} + \dfrac{C_{n-1}}{3} \right), & n \geqslant 1. \end{cases}$$

Figure: C_0 through C_5.

Show that C is sequentially compact with respect to the induced Euclidean metric. [You may assume results from lectures if carefully stated.]

Let $f \colon C \to \{0, 1\}$, where $\{0, 1\}$ is endowed with the discrete metric, be a continuous function. Show that there exists a finite N such that, for each $x \in C$, $f(x)$ is entirely determined by the first N digits of x in base 2.

If we replace C with $C \setminus \{0\}$, is the above conclusion still true? Justify your answer. [*Hint: Consider the function g that, for each $x \in C \setminus \{0\}$, returns the binary digit immediately following the first 1 in the binary expansion of x, e.g. $g(0 \ldots 0100 \ldots) = 0$ and $g(0 \ldots 0110 \ldots) = 1$.*]

11F Topological Spaces

For $c = (c_1, c_2) \in \mathbb{R}^2$ and $r > 0$, let $S_r(c) = (c_1 - \frac{r}{2}, c_1 + \frac{r}{2}) \times (c_2 - \frac{r}{2}, c_2 + \frac{r}{2})$ be the open square of centre c and side length r. Consider the space

$$P = S_2((0,0)) \setminus \left(\overline{S_{\frac{1}{4}}((-\tfrac{1}{2},0))} \cup \overline{S_{\frac{1}{4}}((\tfrac{1}{2},0))} \right) \subset \mathbb{R}^2$$

endowed with the subspace topology.

(i) Define the *interior* and *closure* of a subset of a topological space. Identify the topological boundary of P inside $\mathbb{R}^2$, which is defined as $\partial P = \overline{P} \setminus \mathring{P}$.

(ii) Construct a polyhedron homeomorphic to $\overline{P}$. How many vertices, edges and triangles does it have?

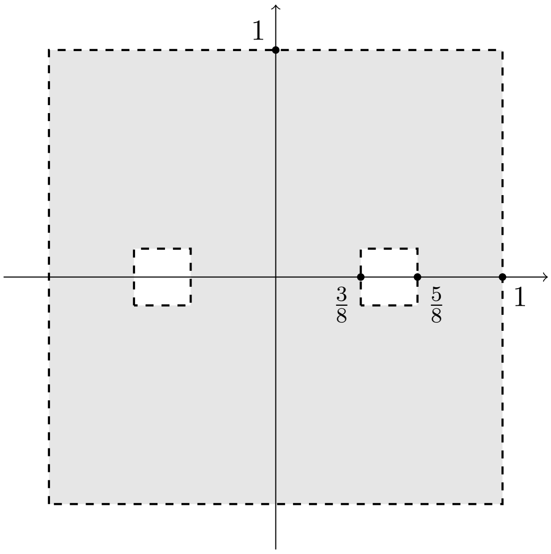

Figure: the set P.

Define $\sim$ to be the smallest equivalence relation on $(0 \times \overline{P}) \sqcup (1 \times \overline{P})$ such that

$$(0, x) \sim (1, x) \quad \text{for all} \quad x \in \partial P.$$

Consider the space $D = ((0 \times \overline{P}) \sqcup (1 \times \overline{P}))/\sim$ endowed with the quotient topology.

(iii) Show carefully that D is compact and connected. [You may assume results from lectures if clearly stated.]

(iv) What does it mean for a topological space to be *locally Euclidean*? What is a *topological manifold*? Explain briefly why D is locally Euclidean. [You do not need to construct explicit homeomorphisms.]

(v) Define the *Euler characteristic* of a topological surface. What is the image of the triangulation you found in part (ii) under the quotient map? Is it a triangulation of D? Justify your answer.

(vi) State the classification theorem for topological surfaces. Assume that D is homeomorphic to a connected sum of k tori. Write down, without proof, the value of k.

12C Complex Methods

(i) Define the Laplace transform $Y(s)$ of $y(t)$. Find the Laplace transforms of $\dot{y}(t)$ and $\ddot{y}(t)$, the first and second derivatives of $y(t)$, in terms of $Y(s)$. Find the Laplace transforms of $e^{\lambda t}$ (where λ is a complex constant) and of the Dirac distribution $\delta(t)$. [In the latter case you may assume that the lower limit of the integral defining the Laplace transform tends to 0 from below.]

(ii) Consider the ordinary differential equation

$$\ddot{y}(t) + 2\gamma\dot{y}(t) + \omega^2 y(t) = f(t)$$

for the unknown function $y(t)$ in $t \geqslant 0$, with given initial values for $y(0)$ and $\dot{y}(0)$, where γ and ω are positive constants and $f(t)$ is a given function with Laplace transform $F(s)$.

Show that the Laplace transform $Y(s)$ of $y(t)$ is given by

$$Y(s) = G(s)H(s)\,,$$

where

$$G(s) = \frac{1}{s^2 + 2\gamma s + \omega^2}$$

and $H(s)$ is a function that you should determine.

(iii) Use the convolution theorem to express $y(t)$ in terms of the functions $g(t)$ and $h(t)$, of which $G(s)$ and $H(s)$ are the Laplace transforms, respectively.

(iv) Determine the (manifestly real) forms of $g(t)$ for $t \geqslant 0$ in the three cases $\gamma > \omega$, $\gamma < \omega$ and $\gamma = \omega$. Also determine $h(t)$ in the case $y(0) = 0$ and deduce the contribution of the initial value $\dot{y}(0)$ to the solution $y(t)$.

13B Variational Principles

Consider the integral

$$I = \int_{t_0}^{t_1} dt\; g(x, \dot{x}, \ddot{x}, t)$$

where $x(t_0)$, $x(t_1)$, $\dot{x}(t_0)$, $\dot{x}(t_1)$ are all fixed. Derive the Euler-Lagrange equation which extremises I.

Find $x(t)$ for which $x(t) \to 0$ and $\dot{x}(t) \to 0$ when $t \to \infty$, $x(0) = \dot{x}(0) = 1$ and the integral

$$\int_0^{\infty} dt\; \left(16x^2 + 8(\dot{x})^2 + (\ddot{x})^2 + \pi^e \dot{x}\ddot{x}\right)$$

is stationary.

14A Methods

Consider the first-order quasilinear PDE

$$a(x,y)u_x + b(x,y)u_y = c(x,y,u).$$

(a) Briefly describe the method of characteristics for such an equation. Write down the associated characteristic system and state what initial data must be prescribed to determine a local solution.

(b) Suppose that

$$\frac{\partial a}{\partial x} + \frac{\partial b}{\partial y} = 0\,.$$

so that, locally, there exists a function $H(x,y)$ such that

$$(a,b) = (H_y, -H_x)\,.$$

(i) Show that the characteristic curves satisfy

$$\dot{x} = H_y\,, \qquad \dot{y} = -H_x\,.$$

(ii) Prove that $H(x,y)$ is constant along characteristics.

(c) Consider the PDE

$$(2xy)\,u_x + (x^2 - y^2)\,u_y = 0\,.$$

(i) Verify that the vector field determining the characteristic has zero divergence.

(ii) Find a function $H(x,y)$ such that

$$(2xy,\ x^2 - y^2) = (H_y, -H_x)\,.$$

(iii) Determine and sketch the characteristic curves.

(iv) Hence obtain the general solution of the PDE.

15B Quantum Mechanics

Consider a wavefunction $\psi(x,t)$ describing a particle of mass m moving in one dimension.

(i) Write down expressions for the probability density ρ and current density j.

(ii) Starting from the time-dependent Schrödinger equation, show that ρ and j satisfy

$$\frac{\partial \rho}{\partial t} + \frac{\partial j}{\partial x} = 0.$$

For

$$\psi(x,t) = \Psi(x) e^{-iEt/\hbar},$$

show that the current is a constant in x.

(iii) A particle of mass m moving in one dimension is incident from the left with energy $E = 3U_0/2$ on a potential well $U(x)$ given by

$$U(x) = \begin{cases} 0\,, & 0 \leqslant x \leqslant a, \\ U_0\,, & x < 0,\ x > a\,, \end{cases}$$

where U_0 is a positive constant. Write down the conditions that should be imposed on the wavefunction at $x = 0$ and $x = a$. Use these conditions to determine that the probability of transmission to the right is

$$\frac{1}{1 + F \sin^2(G\sqrt{U_0})},$$

where F and G are constants which you should determine.

(iv) Keeping $E = 3U_0/2$, for what values of U_0 is the probability of transmission minimised?

16A Fluid Dynamics

A layer of inviscid fluid of constant density ρ and equilibrium depth H lies on a horizontal surface in a frame rotating with angular velocity $\Omega\hat{\mathbf{z}}$. Gravity acts vertically downward with acceleration g. Assume the motion is hydrostatic and independent of the vertical coordinate.

(a) Starting from the Euler equations in a rotating frame,

$$\rho\left(\frac{D\mathbf{u}}{Dt} + 2\boldsymbol{\Omega}\times\mathbf{u}\right) = -\nabla p + \rho\mathbf{g},$$

derive the shallow-water equations in the form

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u}\cdot\nabla\mathbf{u} + f\hat{\mathbf{z}}\times\mathbf{u} = -g\nabla h, \qquad \frac{\partial h}{\partial t} + \nabla\cdot(h\mathbf{u}) = 0,$$

where $h(x, y, t)$ is the free-surface height and $f = 2\Omega$.

(b) Define the potential vorticity

$$q = \frac{\zeta + f}{h}, \qquad \zeta = \hat{\mathbf{z}}\cdot(\nabla\times\mathbf{u}).$$

Show that q satisfies

$$\frac{Dq}{Dt} = 0.$$

(c) The shallow-water equations are inviscid and unforced.

(i) Show that the quantity

$$E = \frac{1}{2}\rho h|\mathbf{u}|^2 + \frac{1}{2}\rho g h^2$$

satisfies a local conservation law of the form

$$\frac{\partial E}{\partial t} + \nabla\cdot\mathbf{J} = 0,$$

for some flux $\mathbf{J}$ to be determined. Interpret the quantity E.

(ii) Hence show that for a domain A, on whose boundary $\mathbf{u}\cdot\mathbf{n} = 0$, the total amount of E,

$$\int_A \left(\frac{1}{2}\rho h|\mathbf{u}|^2 + \frac{1}{2}\rho g h^2\right) dA,$$

is conserved.

(iii) Consider an initial state consisting of a localized height perturbation in an otherwise motionless fluid. Explain why the system cannot evolve to a state of rest with uniform depth if the total amount of E is conserved.

17H Statistics

(a) Suppose $X_1, \ldots, X_n \overset{\text{i.i.d.}}{\sim} \text{Exp}(\theta)$ where the rate parameter $\theta > 0$ is unknown. Find the maximum likelihood estimator $\hat{\theta}$ of θ.

(b) Show that $\sum_{i=1}^n X_i \sim \text{Gamma}(n, \theta)$. If $Y \sim \text{Gamma}(\beta, \lambda)$, where $\beta, \lambda > 0$, what is the distribution of γY for $\gamma > 0$?

(c) Let $\alpha \in (0,1)$. Find, with justification, a $(1-\alpha)$-level confidence interval for θ of the form $[l\hat{\theta}, u\hat{\theta}]$, where l and u are deterministic quantities depending only on n and α.

(d) Now suppose a Bayesian approach is taken, and we use a $\text{Gamma}(\beta, \lambda)$ prior for θ. Find the posterior distribution of θ and the posterior mean $\widetilde{\theta}$.

(e) Find deterministic quantities l' and u' depending only on β, n and α such that the posterior probability that θ lies in $[\,l'\,\widetilde{\theta},\, u'\,\widetilde{\theta}\,]$ is $1-\alpha$.

[*Hint: A* $\text{Gamma}(\beta, \lambda)$ *distribution has density* $f(y) = \lambda^\beta y^{\beta-1} e^{-\lambda y}/\Gamma(\beta)$ *and mean* β/λ. *Moreover if* $Y \sim \text{Gamma}(\beta, \lambda)$ *then* $\mathbb{E}e^{tY} = \{\lambda/(\lambda - t)\}^\beta$ *if* $t < \lambda$ *and* ∞ *otherwise.*]

18H Optimisation

(i) What does it mean for a point x to be an *extreme point* of a convex set $C \subset \mathbb{R}^d$?

(ii) Consider the optimisation problem

$$\min_{x \in \mathbb{R}^n} \sum_{i=1}^n |x_i| \qquad \text{subject to } Ax = b, \tag{$*$}$$

where $A \in \mathbb{R}^{m \times n}$ and $b \in \mathbb{R}^m$. Show how this may be converted to a linear program in standard form: that is, find a linear program of the form

$$\min_{y \geqslant 0} \; c^\top y \qquad \text{subject to } My = a, \tag{$**$}$$

where c, M and a are related to (A, b) in ways that you should specify, and where an optimal y^* for $(**)$ may be converted to an optimal x^* for $(*)$. [*Hint: Consider decomposing* $x = x^+ - x^-$ *where* $x^+, x^- \geqslant 0$.]

(iii) Suppose there exists an optimal solution to $(*)$. Show that there must exist an optimal solution with no more than m non-zero components.

(iv) For $\gamma > 0$ and some $f : \mathbb{R}^m \to \mathbb{R}$, consider the optimisation problem

$$\min_{x \in \mathbb{R}^n} \left\{ f(Ax) + \gamma \sum_{i=1}^n |x_i| \right\}.$$

Assuming an optimal solution exists, show that there must also exist an optimal solution with no more than m non-zero components.

[*You may assume that if a linear program of the form* $(**)$ *has an optimal solution, then there must also exist an optimal solution that is an extreme point of the feasible set.*]

END OF PAPER