

MAT1  
MATHEMATICAL TRIPOS Part IB

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Thursday, 11 June, 2026 2:00pm to 5:00pm

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**PAPER 3**

**Before you begin read these instructions carefully**

The examination paper is divided into two sections. Each question in Section II carries twice the number of marks of each question in Section I. Section II questions also carry an alpha or beta quality mark and Section I questions carry a beta quality mark.

Candidates may obtain credit from attempts on **at most four** questions from Section I and **at most six** questions from Section II.

Write on **one side** of the paper only and begin each answer on a separate sheet.

Write legibly; otherwise you place yourself at a grave disadvantage.

**At the end of the examination:**

Separate your answers to each question.

Complete a gold cover sheet **for each question** that you have attempted, and place it at the front of your answer to that question.

Complete a green main cover sheet listing **all the questions** that you have attempted.

**Every cover sheet must also show your Blind Grade Number and desk number.**

Tie up your answers and cover sheets into a **single bundle**, with the main cover sheet on the top, and then the cover sheet and answer for each question, in the numerical order of the questions.

**STATIONERY REQUIREMENTS**

Gold cover sheets

Green main cover sheet

Treasury tag

**You may not start to read the questions  
printed on the subsequent pages until  
instructed to do so by the Invigilator.**

## SECTION I

### 1E Groups, Rings and Modules

Let  $A$  be the abelian group generated by two elements  $x, y$ , subject to the relation  $6x + 9y = 0$ .

(i) Define  $A$  as an appropriate quotient of a free abelian group of rank 2.

(ii) Prove that  $A$  itself is not a free abelian group.

(iii) Determine, with justification, the exact structure of  $A$  as a direct sum of cyclic groups.

### 2F Topological Spaces

Let  $(X, \mathcal{T})$  be a topological space, and  $A \subset X$ . What is the *subspace topology* on  $A$ ? If  $Y = X/\sim$ , for  $\sim$  an equivalence relation on  $X$ , what is the *quotient topology* on  $Y$ ? If  $(Y, \mathcal{T}')$  is another topological space, what is a *homeomorphism* between  $X$  and  $Y$ ?

Consider  $\mathbb{R}^2$  with the Euclidean topology, and let  $S = [-1, 1]^2 \subset \mathbb{R}^2$ . Let  $\sim$  be the equivalence relation on  $S$  generated by  $(x, y) \sim (-x, y)$  for all  $(x, y) \in S$ , and let  $Y = S/\sim$ . Determine the quotient map  $q: S \rightarrow Y$ .

By first showing that  $Y$  is homeomorphic to  $S^+ = [0, 1] \times [-1, 1]$ , or otherwise, show that  $Y$  is homeomorphic to  $S$ . [You may assume results from lectures if carefully stated.]

### 3C Complex Methods

Consider the real integral

$$I = \int_0^{\infty} \frac{dx}{1+x^p},$$

where  $p > 1$  is a real number. Use a wedge-shaped contour (including the lines  $\arg z = 0$  and  $\arg z = 2\pi/p$ ) and the residue theorem to show that

$$I = \frac{\pi}{p} \operatorname{cosec} \left( \frac{\pi}{p} \right).$$

**4B Variational Principles**

Let  $C$  be the unit circle containing the unit disk  $D$  centred on the origin. Derive the Euler-Lagrange equation for the function  $u(x, y)$  which is fixed to be equal to 1 on  $C$  and gives a stationary value of

$$I[u] = \int_D L \left( x, y, u, \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y} \right) dx dy.$$

Find the Euler-Lagrange equation satisfied by the function  $u(x, y)$  which satisfies the different boundary condition  $u(x, y) = x^y$  on  $C$  and gives a stationary value of

$$\int_D \left[ u^2 + \frac{\partial u}{\partial x} \frac{\partial u}{\partial y} \right] dx dy.$$

**5A Methods**

The Fourier transform of a function  $f(x)$  on the real line is defined by

$$\tilde{f}(k) = \int_{-\infty}^{\infty} f(x) \exp(-ikx) dx.$$

(i) Compute the Fourier transform of the function

$$g_{\sigma}(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{x^2}{2\sigma^2}\right)$$

by obtaining a differential equation for  $\tilde{g}_{\sigma}(k)$  and solving it.

(ii) State without proof the convolution theorem for two functions  $f(x)$  and  $g(x)$ . Deduce that the convolution  $g_{\sigma_1}(x)$  and  $g_{\sigma_2}(x)$  is given by

$$\int_{-\infty}^{\infty} g_{\sigma_1}(y)g_{\sigma_2}(x-y) dy = g_{\sigma_3}(x),$$

where  $\sigma_3^2 = \sigma_1^2 + \sigma_2^2$ .

### 6B Quantum Mechanics

For three-dimensional space with indices  $a, b, c, d \in \{1, 2, 3\}$ , write the angular momentum operator  $L_a$  in terms of the position operator  $x_b$  and the momentum operator  $p_c$ .

State the commutation relations between  $x_a$  and  $p_b$ , between  $x_a$  and  $x_b$ , and between  $p_a$  and  $p_b$ .

Thus calculate the commutator between  $L_a$  and  $L_b$ , writing the right-hand side in terms of a linear function of the operators introduced above.

Find the commutator between  $L_a$  and  $p_d$ , writing the right-hand side as a linear function of the operators introduced above. Are the operators  $L_1$  and  $p_2$  simultaneously diagonalisable?

### 7A Fluid Dynamics

A spherical bubble of radius  $a$  translates with time-dependent velocity  $\mathbf{V}(t)$  in an inviscid fluid of constant density  $\rho$  in the absence of gravity. The resulting irrotational flow has the velocity potential

$$\phi(\mathbf{x}, t) = \frac{a^3}{2r^3} \mathbf{V}(t) \cdot \mathbf{x},$$

where  $\mathbf{x}$  is the position vector and  $r = |\mathbf{x}|$  is its modulus.

(i) Obtain the pressure acting on the bubble surface.

(ii) Hence obtain the total force  $\mathbf{F}$  acting on the bubble.

(iii) When an arbitrary time-dependent external force  $\mathbf{G}(t)$  acts on a bubble of mass  $m$ , show that the motion of the bubble satisfies  $m_* \mathbf{a} = \mathbf{G}(t)$ , where  $\mathbf{a}$  is the acceleration of the bubble and  $m_*$  is a constant. Obtain an expression for  $m_*$  and interpret the result.

**8H Markov Chains**

Suppose  $(X_n)_{n \geq 0}$  is a Markov chain with state space  $I$  and transition matrix  $P$ . What does it mean for  $C \subseteq I$  to be a *communicating class*? What does it mean for  $C$  to be *closed*?

Consider the Markov chain with state space  $\{1, \dots, 6\}$  and transition matrix

$$P = \begin{pmatrix} 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{6} & 0 & \frac{1}{6} \\ \frac{1}{3} & \frac{2}{3} & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{3} & \frac{1}{3} & 0 & \frac{1}{3} & 0 \\ 0 & 0 & 0 & \frac{1}{3} & \frac{2}{3} & 0 \\ 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 & 0 \end{pmatrix}.$$

Determine the communicating classes. Which of these are closed?

What is the probability that the chain ever reaches 2, given that it starts in 1?

## SECTION II

## 9G Linear Algebra

- (a) Let  $V$  be a finite-dimensional real vector space with inner product  $\langle \cdot, \cdot \rangle$ .
- (i) Define the *dual space*  $V^*$  and the *dual basis* associated with a basis  $\{v_1, \dots, v_n\}$  of  $V$ . Prove that the map  $\phi: V \rightarrow V^*$  given by  $\phi(u)(v) = \langle u, v \rangle$  is an isomorphism between  $V$  and  $V^*$ .
  - (ii) Prove that the map  $\psi: V \rightarrow V^{**}$  defined by  $\psi(v)(\phi) = \phi(v)$  is an isomorphism between  $V$  and  $V^{**}$ .
  - (iii) Define the *adjoint*  $T^*$  of a linear operator  $T: V \rightarrow V$ . Prove that, for every linear operator  $T$ , the adjoint  $T^*$  exists and is unique.
- (b) Let  $V$  be a finite-dimensional real vector space, and let  $T: V \rightarrow V$  be a linear operator.
- (i) Prove that if  $T$  is diagonalisable, then there exists an inner product on  $V$  with respect to which  $T$  is self-adjoint.
  - (ii) We say that a set  $S \subset V$  is  $T$ -invariant if  $T(S) \subset S$ . Prove that if  $T: V \rightarrow V$  is self-adjoint with respect to some inner product, then every  $T$ -invariant subspace has a  $T$ -invariant orthogonal complement.

### 10E Groups, Rings and Modules

In this question,  $R$  is a non-zero commutative ring with a multiplicative identity.

(a) Prove that an  $R$ -module is finitely-generated if and only if it is a quotient of a free module  $R^k$ , for some  $k > 0$ .

(b) Let  $M$  be a finitely-generated  $R$ -module, let  $I$  be an ideal of  $R$  and let  $\varphi$  be an  $R$ -module homomorphism from  $M$  to itself such that

$$\varphi(M) \subseteq I \cdot M = \{m \in M : m = rm' \text{ with } r \in I, m' \in M\}.$$

(i) Prove that  $\varphi$  satisfies an equation of the form

$$\varphi^n + a_{n-1}\varphi^{n-1} + \cdots + a_1\varphi + a_0 = 0,$$

where each  $a_i \in I$  and for some  $n$ . [You may assume that if  $A$  is a matrix over  $R$ , then the adjugate matrix satisfies  $\text{adj}(A)A = (\det A)E$ , with  $E$  the identity matrix.]

(ii) Deduce that if  $M$  satisfies  $I \cdot M = M$ , then there exists some  $x \in R$  such that  $x - 1 \in I$  and  $xM = 0$ .

Give an example of a finitely-generated  $\mathbb{Z}$ -module  $M$  and a proper ideal  $I$  of  $\mathbb{Z}$  satisfying the hypothesis  $I \cdot M = M$ , and for your example, give an explicit such element  $x$ .

(iii) Define  $J$  as the intersection of all the maximal ideals of  $R$ . Show that  $r \in J$  if and only if  $1 - rs$  is a unit in  $R$  for all  $s \in R$ . [You may assume that every non-unit of  $R$  is contained in a maximal ideal.]

Deduce from part (ii) that if  $M$  is a finitely-generated  $R$ -module and  $I$  is an ideal of  $R$  contained in  $J$ , then  $I \cdot M = M$  implies  $M = 0$ .

(iv) Suppose that  $M$  is a finitely-generated  $R$ -module. Let  $f : M \rightarrow M$  be a surjective  $R$ -homomorphism. Show that  $f$  is injective. [Hint: You may wish to view  $M$  as a  $R[X]$ -module via the rule  $p(X).m = p(f)(m)$  for  $p(X) \in R[X]$  and  $m \in M$  (you are not asked to verify this).]

### 11F Analysis II

(i) Let  $V$  be a vector space over the real numbers. What is a *norm* on  $V$ ? What does it mean for two norms to be *equivalent*?

(ii) Consider the space  $\mathcal{L}(\mathbb{R}^n, \mathbb{R}^n)$  of  $n \times n$  real matrices. Given a norm  $\|\cdot\|$  on  $\mathbb{R}^n$ , recall that the induced operator norm on  $\mathcal{L}(\mathbb{R}^n, \mathbb{R}^n)$  is defined as follows: for  $A \in \mathcal{L}(\mathbb{R}^n, \mathbb{R}^n)$ ,

$$\|A\|_{\text{op}} = \sup_{v \neq 0} \frac{\|Av\|}{\|v\|}.$$

Define  $\rho: \mathcal{L}(\mathbb{R}^n, \mathbb{R}^n) \rightarrow \mathbb{R}$  by

$$\rho(A) = \lim_{k \rightarrow \infty} \|A^k\|_{\text{op}}^{1/k}.$$

Show that  $\rho$  is independent of the choice of norm on  $\mathbb{R}^n$ . [You may assume the above limit exists. You may also assume results from lectures if properly stated.]

(iii) What is a *contraction* on a metric space? State the contraction mapping theorem.

Let  $A \in \mathcal{L}(\mathbb{R}^n, \mathbb{R}^n)$  and suppose  $A$  is a contraction with respect to some norm on  $\mathbb{R}^n$ . Show that  $I - A$  is invertible. By considering Picard iterates for the problem  $Ax + b = x$ , where  $x, b \in \mathbb{R}^n$ , obtain a formula for  $(I - A)^{-1}$ .

(iv) Let  $A \in \mathcal{L}(\mathbb{R}^n, \mathbb{R}^n)$  with  $\rho(A) < 1$ . Construct a norm  $\|\cdot\|_\alpha$  on  $\mathbb{R}^n$  that makes  $A$  a contraction. [Hint: Consider  $\|\cdot\|_\alpha := \sup_{m \in \mathbb{N} \cup \{0\}} \alpha^{-m} \|A^m \cdot\|$  for suitably chosen  $\alpha > 0$  and  $\|\cdot\|$  an arbitrary norm on  $\mathbb{R}^n$ .]

**12F Topological Spaces**

What is a *base* for a topology on a topological space?

Let  $X$  be a topological space whose topology is generated by a metric  $\rho$ . Fix  $k \in \mathbb{N}$ ,  $k \geq 2$  and let  $Y = \{1, \dots, k\}$ . Consider the product space  $X^k$ , whose elements are of the form  $(f(1), \dots, f(k))$  for some  $f: Y \rightarrow X$ . Determine the product topology on  $X^k$ .

Explain briefly why

$$d(f, g) = \sum_{n=1}^k \frac{1}{2^n} \min\{1, \rho(f(n), g(n))\},$$

where  $f, g: Y \rightarrow X$ , defines a metric on the space of functions from  $Y$  to  $X$ . Show that the product topology on  $X^k$  is the same as that induced by  $d$ . What are the convergent sequences with respect to this topology?

Let  $X^{\mathbb{N}} = X \times X \times \dots$  with a topology that has as base the family of finite intersections of sets of the form

$$\{\pi_n^{-1}(U_n) : n \in \mathbb{N} \text{ and } U_n \subset X \text{ open in } X\},$$

where  $\pi_n$  is the projection map from  $X^{\mathbb{N}}$  to the  $n$ th factor. Is this topology induced by a metric? Justify your answer. [*Hint: It may help to rewrite the base for the topology in terms of functions from  $\mathbb{N}$  to  $X$  as in the previous part.*]

**13G Complex Analysis**

(a) Let  $f$  be holomorphic in a domain  $D$  of the complex plane. Let  $f(x + iy) = u(x, y) + iv(x, y)$  for  $x + iy \in D$ , where  $u(x, y)$  and  $v(x, y)$  are real. Derive the Cauchy–Riemann equations for  $u$  and  $v$ .

(b) Let  $u(x, y) = e^x(x \cos y - y \sin y)$ . Find a holomorphic function  $f: \mathbb{C} \rightarrow \mathbb{C}$  such that  $f(0) = 0$  and  $\operatorname{Re}(f(x + iy)) = u(x, y)$  for all  $x, y \in \mathbb{R}$ . Prove that such a function  $f$  is unique.

### 14A Methods

(a) Verify that

$$\nabla^2 \hat{G}(\mathbf{x}, \mathbf{y}) = \delta(\mathbf{x} - \mathbf{y}),$$

where

$$\hat{G}(\mathbf{x}, \mathbf{y}) = -\frac{1}{4\pi|\mathbf{x} - \mathbf{y}|}, \quad \mathbf{x}, \mathbf{y} \in \mathbb{R}^3.$$

(b) Let  $\Omega = \{z > 0\}$  and suppose  $\varphi$  satisfies

$$\nabla^2 \varphi = 0 \text{ in } \Omega, \quad \varphi(x_1, x_2, 0) = h(x_1, x_2), \quad \varphi(\mathbf{x}) \rightarrow 0 \text{ as } |\mathbf{x}| \rightarrow \infty.$$

(i) State Green's second identity.

(ii) Let  $\mathcal{G}(\mathbf{x}, \mathbf{y})$  satisfy

$$\nabla^2 \mathcal{G} = \delta(\mathbf{x} - \mathbf{y}) \text{ in } \Omega, \quad \mathcal{G} = 0 \text{ on } z = 0.$$

Using Green's second identity, show that

$$\varphi(\mathbf{x}) = \int_{\mathbb{R}^2} h(y_1, y_2) \frac{\partial \mathcal{G}}{\partial n}(\mathbf{x}, \mathbf{y}) dy_1 dy_2.$$

(iii) Show that  $\mathcal{G}$  may be constructed from  $\hat{G}$  using the method of images and write  $\mathcal{G}$  explicitly.

(c) Suppose

$$h(y_1, y_2) = \frac{1}{2\pi\sigma^2} \exp\left(-\frac{y_1^2 + y_2^2}{2\sigma^2}\right), \quad \sigma > 0.$$

Show that on the axis  $x_1 = x_2 = 0$ ,

$$\varphi(0, 0, z) = \frac{z}{2\pi\sigma^2} \int_0^\infty \frac{e^{-u/(2\sigma^2)}}{(u + z^2)^{3/2}} du.$$

**15D Electromagnetism**

(a) A particle of mass  $m$  and charge  $q$  moves relativistically in the electromagnetic fields  $\mathbf{E}$  and  $\mathbf{B}$ . The covariant Lorentz force law is given by  $\frac{dP^\mu}{d\tau} = qF^{\mu\nu}U_\nu$ . Define all the quantities in this expression and show that it implies both

$$\frac{d\mathbf{p}}{dt} = q(\mathbf{E} + \mathbf{u} \times \mathbf{B}), \quad \text{and} \quad \frac{d}{dt} (\gamma(u)mc^2) = q\mathbf{E} \cdot \mathbf{u},$$

with  $u = |\mathbf{u}|$ . State the two Lorentz invariants that can be obtained from  $\mathbf{E}$  and  $\mathbf{B}$ .

(b) Suppose that the electric and magnetic fields are perpendicular in the laboratory frame and given by  $\mathbf{E} = E\hat{\mathbf{e}}_x$ ,  $\mathbf{B} = B\hat{\mathbf{e}}_z$ , with  $E < cB$ . Assume a Lorentz boost with velocity  $\mathbf{v}$  (with  $v = |\mathbf{v}|$  and  $\hat{\mathbf{v}} = \mathbf{v}/v$ ) of uniform electric  $\mathbf{E}$  and magnetic  $\mathbf{B}$  fields is given by

$$\begin{aligned} \mathbf{E}' &= \gamma(v)(\mathbf{E} + \mathbf{v} \times \mathbf{B}) - (\gamma(v) - 1)(\mathbf{E} \cdot \hat{\mathbf{v}})\hat{\mathbf{v}}, \\ \mathbf{B}' &= \gamma(v)\left(\mathbf{B} - \frac{1}{c^2}\mathbf{v} \times \mathbf{E}\right) - (\gamma(v) - 1)(\mathbf{B} \cdot \hat{\mathbf{v}})\hat{\mathbf{v}}. \end{aligned}$$

(i) Show that there exists an inertial frame in which the electric field vanishes  $\mathbf{E}' = 0$  and find the velocity  $\mathbf{v}$  of that frame relative to the laboratory frame. Determine the transformed magnetic field  $\mathbf{B}'$ .

(ii) In the frame where  $\mathbf{E}' = 0$ , assume that the particle velocity is perpendicular to the uniform magnetic field  $\mathbf{B}'$ . Show that this results in a circular trajectory and find the angular frequency and radius.

(iii) Transform this circular trajectory back to the laboratory frame and show that the particle undergoes a superposition of circular motion and a uniform drift with velocity approximately (for small  $|\mathbf{v}| \ll c$ ) given by

$$\mathbf{u}_d \approx \frac{\mathbf{E} \times \mathbf{B}}{B^2}.$$

Briefly explain why this drift is independent of the particle's mass and charge. How is this solution distinct from the helical trajectory possible in the  $\mathbf{E}' = 0$  and non-zero  $\mathbf{B}'$  frame?

### 16A Fluid Dynamics

Consider the incompressible, 2D parallel viscous flow of a Newtonian fluid of constant density  $\rho$  and dynamic viscosity  $\mu > 0$  in a long, straight, horizontal channel of height  $H > 0$ . Use Cartesian coordinates  $(x, y, z)$  with channel walls located at  $y = 0$  and  $y = H$ . The flow is driven by a pressure gradient in the  $x$ -direction and has Cartesian components

$$\mathbf{u} = (u(y, t), 0, 0).$$

- (i) By considering the balance of momentum in an elementary volume, derive the governing equation

$$\rho \frac{\partial u}{\partial t} = -\frac{\partial p}{\partial x} + \mu \frac{\partial^2 u}{\partial y^2},$$

for the rate of change of momentum in the  $x$ -direction.

- (ii) Consider steady flow. Show that  $\partial p / \partial x$  must be independent of  $x$ , and hence can be expressed as  $\partial p / \partial x = -G$ , for a constant  $G$  which may be assumed to be positive. Solve for the velocity profile  $u(y)$  subject to the no-slip boundary conditions at  $y = 0$  and  $y = H$ .
- (iii) Determine the volumetric flow rate per unit width

$$Q = \int_0^H u(y) dy$$

in terms of  $G$ ,  $H$ , and  $\mu$ .

- (iv) Now suppose that at time  $t = 0$  the pressure gradient is suddenly removed, so that  $\partial p / \partial x = 0$  for  $t > 0$ . Solve for the time-dependent velocity  $u(y, t)$ , assuming that the initial condition is the steady solution found in part (ii). You may assume a solution in the form of a Fourier sine series.

**17C Numerical Analysis**

Consider the  $s$ -step,  $s$ -order backward differentiation formula (BDF)

$$\sum_{l=0}^s \rho_l \mathbf{y}_{n+l} = h \sigma_s \mathbf{f}(t_{n+s}, \mathbf{y}_{n+s}), \quad \rho_s = 1$$

as a method for solving the ordinary differential equation  $d\mathbf{y}/dt = \mathbf{f}(t, \mathbf{y})$ , where  $h$  is the step size such that  $t_n = nh$ . Let

$$\rho(w) = \sum_{l=0}^s \rho_l w^l, \quad \sigma(w) = \sigma_s w^s.$$

(i) Show that to satisfy the order condition

$$\rho(e^z) - z\sigma(e^z) = \mathcal{O}(z^{s+1}) \quad \text{as } z \rightarrow 0,$$

we require

$$\rho(w) = \sigma_s \sum_{l=1}^s \frac{1}{l} w^{s-l} (w-1)^l, \quad \sigma_s = \left( \sum_{l=1}^s \frac{1}{l} \right)^{-1}.$$

(ii) Determine  $\rho(w)$  and  $\sigma_s$  explicitly for the BDF methods with  $s = 2$  and  $s = 3$ . Use the Dahlquist equivalence theorem to show that these two methods are convergent.

(iii) Show that the BDF method with  $s = 2$  is A-stable. [*Hint: Find the boundary of the linear stability domain.*]

**18H Statistics**

- (a) (i) Suppose a random variable  $X$  has unknown density  $p$ . State the Neyman–Pearson lemma for testing the null  $H_0 : p = f$  against the alternative  $H_1 : p = g$ , where  $f$  and  $g$  are continuous densities with the same support.
- (ii) Suppose  $X_1, \dots, X_n \stackrel{\text{i.i.d.}}{\sim} N(\mu, 1)$ . Find, with justification, a uniformly most powerful test for testing

$$H_0 : \mu = 0, \quad \text{against} \quad H_1 : \mu > 0,$$

that has size  $\alpha \in (0, 1)$ . Express the critical region in terms of the cumulative distribution function  $\Phi$  of the  $N(0, 1)$  distribution. Argue moreover that your test is also a uniformly most powerful size  $\alpha$  test for testing

$$H_0 : \mu \leq 0, \quad \text{against} \quad H_1 : \mu > 0.$$

- (b) (i) Consider the setting of part (a) (i) but where we modify the null hypothesis  $H_0$  to be that  $p \in \{f_1, f_2\}$ , with the alternative unchanged (again  $f_1, f_2$  are continuous with the same support as  $g$ ). Given  $\pi \in [0, 1]$ , let  $\phi$  be the size  $\alpha$  likelihood ratio test for the null  $H_{0,\pi}$  that  $p = \pi f_1 + (1 - \pi)f_2$ , against  $H_1$ . Suppose that in fact  $\phi$  has size  $\alpha$  for the null hypothesis  $H_0$ . Explain why  $\phi$  must be uniformly most powerful for  $H_0$  against  $H_1$ .
- (ii) A certain drug requires the logarithm  $\mu$  of the relative concentrations of two chemicals to be within a range  $[-\mu_0, \mu_0]$ , for  $\mu_0 > 0$ . A new cheaper method for manufacturing the drug has been proposed for which it is claimed that  $\mu = 0$ , but there is concern that  $\mu \notin [-\mu_0, \mu_0]$ . The value of  $\mu$  cannot be measured directly, but a single measurement  $X \sim N(\mu, 1)$  is available. Consider therefore testing

$$H_0 : \mu \notin [-\mu_0, \mu_0], \quad \text{against} \quad H_1 : \mu = 0,$$

based on  $X$ . Given  $\alpha \in (0, 1)$ , let  $z > 0$  be such that

$$\Phi(z - \mu_0) - \Phi(-z - \mu_0) = \alpha.$$

Show that the test which rejects  $H_0$  when  $X \in [-z, z]$  is uniformly most powerful.

### 19H Optimisation

Let  $f : \mathbb{R}^d \rightarrow \mathbb{R}$  be a twice differentiable convex function.

(i) Describe the procedure of *gradient descent* for minimising  $f$  with step-size  $\eta > 0$ .

(ii) Suppose in what follows that  $\nabla^2 f(x) \preceq \beta I$  for  $\beta > 0$ , where  $\nabla^2 f(x)$  denotes the Hessian of  $f$  at  $x$ , and for symmetric matrices  $A, B \in \mathbb{R}^{d \times d}$ ,  $A \preceq B$  means that  $B - A$  is positive semi-definite. Show that

$$f(z) - f(y) \leq \nabla f(y)^\top (z - y) + \frac{\beta}{2} \|z - y\|^2.$$

(iii) Show further that

$$f(x) - f(y) \leq \nabla f(x)^\top (x - y) - \frac{1}{2\beta} \|\nabla f(x) - \nabla f(y)\|^2.$$

[Hint: Consider  $z = y - \frac{1}{\beta}(\nabla f(y) - \nabla f(x))$  and recall that  $f(x) - f(z) \leq \nabla f(x)^\top (x - z)$ .]

Conclude that

$$0 \leq (\nabla f(x) - \nabla f(y))^\top (x - y) - \frac{1}{\beta} \|\nabla f(x) - \nabla f(y)\|^2.$$

(iv) Suppose in what follows that additionally  $\nabla^2 f(x) \succeq \alpha I$  for  $\beta > \alpha > 0$ . Writing  $\varphi(x) = f(x) - \frac{\alpha}{2} \|x\|^2$ , explain briefly why  $\varphi$  is convex and why  $\nabla^2 \varphi(x) \preceq (\beta - \alpha)I$ . Using part (iii), show that

$$(\nabla f(x) - \nabla f(y))^\top (x - y) \geq \frac{\alpha\beta}{\beta + \alpha} \|x - y\|^2 + \frac{1}{\beta + \alpha} \|\nabla f(x) - \nabla f(y)\|^2.$$

(v) Let  $x^*$  be the minimiser of  $f$  and let  $x_t$  be the  $t$ th iterate of the gradient descent procedure from part (i). Show that when  $\eta = \frac{2}{\alpha + \beta}$ , we have

$$\|x_{t+1} - x^*\|^2 \leq \left( \frac{\beta - \alpha}{\beta + \alpha} \right)^{2t} \|x_1 - x^*\|^2.$$

[Hint: Use part (iv) with  $y = x^*$ .]

**END OF PAPER**