MAT1

MATHEMATICAL TRIPOS Part IB

Wednesday 10 June, 2026 9:00am to 12:00pm

PAPER 2

Before you begin read these instructions carefully

The examination paper is divided into two sections. Each question in Section II carries twice the number of marks of each question in Section I. Section II questions also carry an alpha or beta quality mark and Section I questions carry a beta quality mark.

Candidates may obtain credit from attempts on **at most four** questions from Section I and **at most six** questions from Section II.

If you attempt the joint Complex Analysis and Complex Methods question, you may submit an answer to at most one of the two sub-questions.

Write on **one side** of the paper only and begin each answer on a separate sheet.

Write legibly; otherwise you place yourself at a grave disadvantage.

At the end of the examination:

Separate your answers to each question.

Complete a gold cover sheet **for each question** that you have attempted, and place it at the front of your answer to that question.

Complete a green main cover sheet listing **all the questions** that you have attempted.

Every cover sheet must also show your Blind Grade Number and desk number.

Tie up your answers and cover sheets into **a single bundle**, with the main cover sheet on the top, and then the cover sheet and answer for each question, in the numerical order of the questions.

STATIONERY REQUIREMENTS
Gold cover sheets
Green main cover sheet
Treasury tag

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

SECTION I

1E Groups, Rings and Modules

(a) Let $n \geqslant 3$. Let $R = \mathbb{Z}[\sqrt{-n}]$. By factorising n or $n+1$, as appropriate, show that R is not a UFD.

(b) Can the ideal $I = (4, 2X, X^2)$ of $\mathbb{Z}[X]$ be generated by two elements? Justify your answer.

For each $n \geqslant 3$, write down (without proof) an ideal of $\mathbb{Z}[X]$ that is generated by n elements but not by $n-1$ elements.

2F Analysis II

Consider a function $f\colon \mathbb{R}^2 \to \mathbb{R}$. What does it mean for f to be *differentiable* at a point $p \in \mathbb{R}^2$, and what is its *derivative* at p? What are the *partial derivatives* of f at p?

Show that, if the partial derivatives of f exist in a ball about a point $p \in \mathbb{R}^2$ and are continuous at p, then f is differentiable at p.

Suppose $f\colon \mathbb{R}^2 \to \mathbb{R}$ is given by

$$f(x,y) = \begin{cases} x^2 \sin(1/x) + y^2 \sin(1/y) & \text{if } xy \neq 0 \\ x^2 \sin(1/x) & \text{if } x \neq 0,\ y = 0 \\ y^2 \sin(1/y) & \text{if } y \neq 0,\ x = 0 \\ 0 & \text{if } x = y = 0 \end{cases}.$$

At which points $(x,y) \in \mathbb{R}^2$ is f differentiable? At which points is it C^1? Justify your answers.

3D Methods

The Fourier series for the real-valued function $f(x)$ on $(-1,1)$ is given by

$$f(x) \;=\; \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos n\pi x + b_n \sin n\pi x\right).$$

(i) Derive Parseval's identity, assuming the standard orthogonality relations

$$\int_{-1}^{1} \cos n\pi x \, \cos m\pi x \, dx \;=\; \delta_{mn}\,, \qquad \int_{-1}^{1} \sin n\pi x \, \sin m\pi x \, dx \;=\; \delta_{mn}\,,$$

$$\int_{-1}^{1} \sin n\pi x \, \cos m\pi x \, dx \;=\; 0\,,$$

where n, m are non-zero integers.

(ii) The Fourier series for $f(x) = x^2$ is given by

$$x^2 = \frac{1}{3} + \frac{4}{\pi^2} \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \cos n\pi x, \qquad -1 < x < 1.$$

Use Parseval's identity in this case to evaluate the series,

$$\sum_{n=1}^{\infty} \frac{1}{n^4}\,.$$

4D Electromagnetism

A point charge q is placed at position $(0,0,d)$ above a conducting region $z \leqslant 0$ held at zero potential $\phi = 0$. Use the method of images to find the electric field $\mathbf{E}$ for $z > 0$ by placing another charge $-\,q$ at position $(0,0,-d)$. Show that the surface charge density on the conductor is given by

$$\sigma(r) = -\frac{q}{2\pi} \frac{d}{(r^2 + d^2)^{3/2}}\,,$$

where $r^2 = x^2 + y^2$. Integrate to find the total surface charge induced on the conductor.

5A Fluid Dynamics

Consider the 2D flow with velocity field

$$\boldsymbol{u}(x, y, t) = \big(\alpha y + yt,\ 1\big),$$

where $\alpha \geqslant 0$ is a parameter, x and y are spatial coordinates, and $t \geqslant 0$ is time.

(i) Compute the streamfunction $\psi(x, y, t)$ for this flow.

(ii) Sketch the approximate streamlines at $t = 0$ and $t = 1$.

(iii) Explain why the flow is approximately steady when $0 \leqslant t \ll \alpha$ and determine the streamlines in this limit.

6H Statistics

Suppose $X_1, \ldots, X_n \overset{\text{i.i.d.}}{\sim} N(\mu, 1)$. Find the maximum likelihood estimator $\hat{\mu}$ of μ, and write down its mean squared error.

Now adopt a Bayesian perspective and take a $N(0, \tau^{-2})$ prior for μ, with $\tau > 0$. Find the posterior distribution. What is the posterior mean $\tilde{\mu}$?

Show that the mean squared error of $\tilde{\mu}$ is strictly smaller than that of $\hat{\mu}$ if and only if

$$\mu^2 \ < \ 2\tau^{-2} + \frac{1}{n}.$$

7H Optimisation

What is the *maximum flow problem* in a network?

Describe the Ford–Fulkerson algorithm.

Suppose all edge capacities are rational numbers. Show that the algorithm is guaranteed to terminate in a finite number of steps.

SECTION II

8E Linear Algebra

In this question V is a finite-dimensional vector space over $\mathbb{C}$.

(a) Suppose $\alpha : V \to V$ is an endomorphism and let W be a subspace of V. We say that W is α-stable if $\alpha(W) \leqslant W$. Assume that W is α-stable.

(i) Let $m_\alpha(X)$ be the minimal polynomial of α. Show that the induced linear maps $\alpha_W : W \to W$ and $\alpha_{V/W} : V/W \to V/W$ have minimal polynomials such that their least common multiple is a factor of $m_\alpha(X)$ and their product is divisible by $m_\alpha(X)$.

(ii) Write down an example to show that $m_\alpha(X)$ is not always equal to the least common multiple. Write down an example to show that $m_\alpha(X)$ is not always equal to the product.

(iii) Deduce that if α is diagonalisable, then α_W and $\alpha_{V/W}$ are diagonalisable. Is the converse of this statement true? Briefly justify your answer.

(b) Let $r \geqslant 2$. Suppose that $\alpha_1, \ldots, \alpha_r$ are (pairwise) commuting endomorphisms on V and that each α_i is diagonalisable. Using induction on r, or otherwise, show that they are simultaneously diagonalisable, namely, there is a basis of V consisting of simultaneous eigenvectors for all the α_i. [*Hint: You may wish to begin by showing that certain eigenspaces are α_i-stable for all i.*]

(c) Let $\mathcal{A} = \{\alpha_i\}_i$ be a (not necessarily finite) set of (pairwise) commuting endomorphisms on V. Suppose that each α_i is diagonalisable on V. Are the members of $\mathcal{A}$ simultaneously diagonalisable? Justify your answer.

9E Groups, Rings and Modules

In this question, p is a prime number.

(a) State Sylow's first theorem. State and prove Sylow's second theorem.

(b) Let G be a finite group.

(i) Suppose that p^d divides $|G|$ for some $d > 0$. Show that each subgroup of G with order p^{d-1} has index p in a subgroup of G. [*Hint: Cauchy's theorem may be assumed. If* $d > 1$*, let* H *be a subgroup of* G *of order* p^{d-1} *and consider the left multiplication action of* H *on* G/H.]

(ii) Deduce that if p^d divides $|G|$ then there is a subgroup of G of order p^d.

(iii) Deduce further that if H is a p-subgroup of G with normaliser $N(H)$, then the indices satisfy

$$|G : H| \equiv |N(H) : H| \pmod{p}.$$

(c) Let G be a finite group and let p be a divisor of $|G|$. Is it true that p-subgroups of a common non-maximal order are all conjugate? Briefly justify your answer.

10F Analysis II

Carefully state the inverse function theorem.

Let Ω be a bounded open subset of $\mathbb{R}^n$; denote its closure in $\mathbb{R}^n$ by $\overline{\Omega}$ and its boundary in $\mathbb{R}^n$ by $\partial\Omega = \overline{\Omega}\setminus\Omega$. Let $f\colon \overline{\Omega} \to \mathbb{R}^n$ be continuous in $\overline{\Omega}$ and C^1 in Ω. Suppose $0 \notin f(\partial\Omega)$ and that $\det Df(x) \neq 0$ for every $x \in f^{-1}(0)$.

(i) Show that the zeros of f are isolated, i.e. that

$$\forall x \in f^{-1}(0)\ \exists r > 0 \ \text{ such that } \ y \in B_r(x)\setminus\{x\} \implies f(y) \neq 0.$$

(ii) Recall that Ω is bounded and $0 \notin f(\partial\Omega)$. Show that the zero set of f is finite and contained in Ω, i.e. $f^{-1}(0) = \{x_1, \ldots, x_N\}$ for some $N < \infty$ and $x_1, \ldots, x_N \in \Omega$. [You may assume results from lectures if properly stated.]

Deduce that there exist $r_1, \ldots, r_N > 0$ such that

$$\sup_{x \in B_{r_k}(x_k)} \|[Df(x_k)]^{-1}[Df(x) - Df(x_k)]\|_{\mathrm{op}} \leqslant \frac{1}{4} \quad \forall k = 1, \ldots, N,$$

where $\|\cdot\|_{\mathrm{op}}$ is the operator norm on $\mathbb{R}^{n\times n}$ induced by a choice of norm $\|\cdot\|$ on $\mathbb{R}^n$.

(iii) Define $N(f) = \sum_{x\in f^{-1}(0)} \operatorname{sign}(\det Df(x))$. Here, as usual, $\operatorname{sign}\colon \mathbb{R}\setminus\{0\} \to \{-1, 1\}$ takes the value $+1$ for positive real numbers and -1 for negative real numbers.

Show that there exists $\delta > 0$ sufficiently small such that, for all $g\colon \overline{\Omega} \to \mathbb{R}^n$ continuous in $\overline{\Omega}$ and C^1 in Ω,

$$\sup_{\Omega}\|g - f\| + \sup_{\Omega}\|Dg - Df\|_{\mathrm{op}} < \delta \implies N(g) = N(f).$$

[You may assume the following statement. Let $r > 0$, $x \in \mathbb{R}^n$ and $h\colon \overline{B_r(x)} \to \mathbb{R}^n$ be continuous in $\overline{B_r(x)}$ and C^1 in $B_r(x)$. If both $\sup_{B_r(x)}\|A^{-1}(Dh - A)\|_{\mathrm{op}} \leqslant \frac{1}{2}$ and $\sup_{B_r(x)}\|A^{-1}h\| \leqslant \frac{r}{2}$ for some invertible $A \in \mathbb{R}^{n\times n}$, then h has a unique zero in $B_r(x)$ and $\operatorname{sign}(\det Dh(y)) = \operatorname{sign}(\det A)$ for all $y \in B_r(x)$.]

11F Topological Spaces

(a) What does it mean for a topological space to be *connected*?

Consider $\mathbb{R}$ with the usual Euclidean topology. Let $f\colon \mathbb{R} \to \mathbb{R}$ be a bijection with the property that, for every connected $X \subset \mathbb{R}$, $f(X)$ is connected. Show that f must be monotone. [You may assume the following fact: $g\colon \mathbb{R} \to \mathbb{R}$ is monotone $\iff$ for every triple (a, b, c) with $a < b < c$, $g(b)$ lies between $g(a)$ and $g(c)$.]

Show that f must be continuous.

(b) What does it mean for a topological space to be *path-connected*? Show that a path-connected space is connected.

Consider the unit square $Q = [0,1] \times [0,1]$ with the usual Euclidean topology. The boundary of Q is $\partial Q = ([0,1] \times [0,1]) \setminus ((0,1) \times (0,1))$. Let γ and δ be two continuous paths on ∂Q starting at $(0,0)$ and ending at $(1,1)$ that only intersect at these two points.

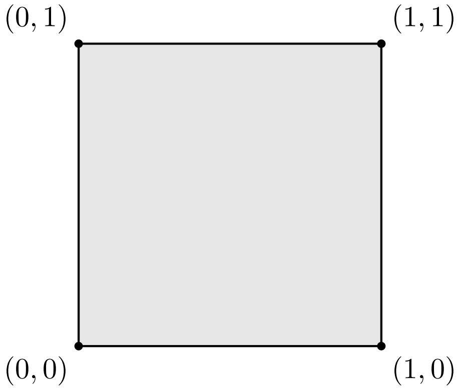


Figure: the unit square Q with its four corners.

(i) Give an explicit example of one possible such pair (γ, δ).

(ii) Let $\alpha, \beta\colon [0,1] \to Q$ be continuous paths on Q satisfying

$$\begin{aligned} \alpha(0) &= (0,0), \quad \alpha(1) = (1,1), \\ \beta(0) &= (0,1), \quad \beta(1) = (1,0). \end{aligned}$$

Show that the images of α and β must intersect in Q. [You may assume the following fact: there is no continuous $F : Q \to S^1$ such that the images of $F \circ \gamma$ and of $F \circ \delta$ are opposite hemispheres of the unit circle S^1.]

(iii) Suppose that $A \subset Q$ is path-connected, contains $(0,0)$ and $(1,1)$ and does not contain the other two corners of Q. Deduce that $(0,1)$ and $(1,0)$ must lie in different path components of $Q \backslash A$.

[*You may assume results from the lectures about intervals in* $\mathbb{R}$ *if stated clearly.*]

12 Complex Analysis OR Complex Methods

This is the joint question for Complex Analysis/Complex Methods. Attempt only ONE of the sub-questions. On your answer sheet, specify the question number as either "12.1G" or "12.2C".

(12.1G) Complex Analysis

(a) Let $0 < r < R$ and $z_0 \in \mathbb{C}$. Let f be a holomorphic function on the open disc $D(z_0, R)$. Let γ be the circle of radius r centred at z_0, oriented anticlockwise. State and prove Cauchy's integral formula expressing $f(w)$ for any $w \in D(z_0, r)$ in terms of an integral along γ. State, without proof, a similar formula for $f'(w)$.

[You may assume Cauchy's theorem in the following form. If U is a star-shaped domain and $g\colon U \to \mathbb{C}$ is continuous everywhere and holomorphic at all but finitely many points, then for any closed piecewise C^1 path γ in U we have $\int_\gamma g(z)\,\mathrm{d}z = 0$.]

(b) Prove that if a holomorphic function $f\colon \mathbb{C} \to \mathbb{C}$ is bounded, then f is constant.

(c) Evaluate the integral

$$\int_{-\infty}^{\infty} \frac{\cos^2 x}{x^2+1}\,\mathrm{d}x.$$

(12.2C) Complex Methods

(i) Let $c > 1$ be a real number. Show that the function

$$f(z) = \frac{\sqrt{z^2-1}}{z+c}$$

of a complex variable $z = x + iy$ can be made single-valued by introducing a branch cut along the real axis from -1 to 1, such that $f(z) \to 1$ as $z \to \infty$. Express the limiting values of $f(z)$ above and below the branch cut in terms of x.

(ii) Let

$$J = \int_C f(z)\,dz\,,$$

where C is a closed curve that encloses the branch cut once in a positive sense but does not enclose the point $z = -c$. Using the substitution $z = 1/\zeta$, show that

$$J = \int_{C'} g(\zeta)\,d\zeta\,,$$

where

$$g(\zeta) = -\frac{1}{\zeta^2}\frac{\sqrt{1-\zeta^2}}{1+c\zeta}$$

and C' is the image of C under the map $z \mapsto \zeta$. Describe how the branch cut of $g(\zeta)$ relates to that of $f(z)$. Explain why C' encloses the points $\zeta = 0$ and $\zeta = -1/c$ once in a negative sense.

[QUESTION CONTINUES ON THE NEXT PAGE]

(iii) Use the residue theorem to show that

$$J = -2\pi i\left(c - \sqrt{c^2-1}\right)$$

and deduce the value of the real integral

$$I = \int_{-1}^{1} \frac{\sqrt{1-x^2}}{x+c}\,dx\,.$$

13B Variational Principles

Use Lagrange multipliers to do the following:

(a) A solid cylinder of radius a and length l has volume V and total surface area A.

(i) Show that, for fixed volume V, the extremal surface area is

$$A = B(2\pi)^{1/3}V^{2/3},$$

determining the integer B in the process.

(ii) The cylinder is inscribed within a sphere of fixed radius. The ratio l/a is chosen to maximise the surface area of the cylinder. Show that

$$A = 2\pi\sqrt{C}a^2,$$

where C is an integer which you should determine.

(b) Find all stationary points of

$$f(x,y,z) = -2x^3 + y^2 + 3z^2$$

on the plane $x = -y - z$.

[You need *not* identify the nature of any extrema.]

14D Methods

The wave equation in polar coordinates for a stretched membrane $u(r,\theta,t)$ on the unit disc is given by

$$\frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2},$$

with Dirichlet boundary conditions $u(1,\theta,t)=0$ and initial conditions given by $u(r,\theta,0)=\phi(r,\theta)$, and $\frac{\partial u}{\partial t}(r,\theta,0)=\psi(r,\theta)$.

(i) By using the method of separation of variables $u(r,\theta,t)=R(r)\Theta(\theta)T(t)$, show that the radial equation (Bessel's equation) can be written in Sturm-Liouville form as

$$-\frac{d}{dr}\left[r\frac{dR}{dr}\right] + \frac{m^2}{r}R = \lambda r R\,, \qquad (*)$$

where m (an integer) and λ are separation constants.

(ii) After making the substitution $z=\sqrt{\lambda}r$ in $(*)$ to eliminate λ, use the Frobenius method $R = z^p \sum_{k=0}^{\infty} a_k\, z^k$ to find a recurrence relation between the a_k and the a_{k-2} coefficients of the power series solution which is non-singular at the origin ($r=0$).

(iii) You may assume the following expression for the resulting power series expansion of the Bessel function $J_m(z)$ [*you are not required to derive this*]:

$$J_m(z) = \left(\frac{z}{2}\right)^m \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\,(n+m)!}\left(\frac{z}{2}\right)^{2n}, \qquad (m \geqslant 0).$$

Deduce the following recurrence relations

$$\frac{d}{dz}(z^m J_m(z)) = z^m J_{m-1}(z) \qquad \text{and} \qquad \frac{d}{dz}(z^{-m} J_m(z)) = -z^{-m} J_{m+1}(z)\,.$$

(iv) The $J_m(z)$ are oscillatory functions with zeros at positions $z = j_{m1}, j_{m2}, j_{m3}, \ldots$ with $0 < j_{m1} < j_{m2} < \ldots$. Assume that $J_m(j_{mn}r)$ are the radial eigenfunctions for the wave equation with Dirichlet boundary conditions on the unit disc, with normalisation given by $\mathcal{N}_{mn} = \int_0^1 [J_m(j_{mn}r)]^2\, r dr = \frac{1}{2}[J_m'(j_{mn})]^2$.

Write down a general expansion for a purely radial solution for the membrane wave equation specified above on the unit disc. Give expressions for the specific solution satisfying the initial conditions $\phi(r,\theta)=1-r$ and $\psi(r,\theta)=0$ and integrate to find the eigenfunction coefficients explicitly.

15B Quantum Mechanics

Muonic hydrogen is like ordinary hydrogen, where the electron of mass m_e has been replaced by a muon, of identical charge and of mass $m_\mu > m_e$ but $m_\mu \ll m_p$, where m_p is the mass of a proton.

(i) Write down the time-independent Schrödinger equation for the wavefunction of muonic hydrogen.

(ii) Explain the origin of each term in this equation, briefly.

(iii) What assumptions must you make for the equation that you have written in part (i) to hold?

(iv) The first excited state of muonic hydrogen where the muon has zero orbital angular momentum can be written

$$\psi(r) = A\left(1 - \frac{r\alpha}{2}\right)\exp(-\alpha r/2),$$

where A is a normalisation constant. Assuming that this form of ψ satisfies your answer to part (i), find E in terms of α and then determine α in terms of the other constants.

(v) Verify that ψ satisfies the equation you have given in answer to part (i).

(vi) Sketch $\psi(r)/A$ as a function of r. On the same graph, sketch and label the radial wavefunction divided by its normalising factor for the first excited state of ordinary hydrogen where the electron has zero orbital angular momentum.

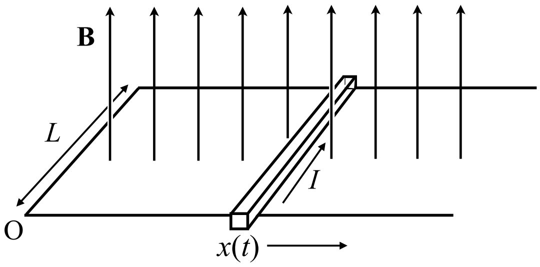
[*Hint: The time-independent Schrödinger equation for the wavefunction of the hydrogen atom is*

$$-\frac{\hbar^2}{2m_e r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\psi}{\partial r}\right) + \frac{1}{2m_e r^2}L^2\psi - \frac{e^2}{4\pi\epsilon_0 r}\psi = E\psi\,. \quad]$$

16D Electromagnetism

(a) Consider a perfect conductor made up of a closed, but movable, circuit $\mathcal{C}$ in the presence of a magnetic field $\mathbf{B}$. State Faraday's Law of Induction, defining all quantities.

(b) Suppose that a rectangular circuit $\mathcal{C}$ lying in the xy-plane (see diagram) is made up of two parallel metal rails placed a distance L apart and connected at one end by a straight wire, with all three elements perfectly conducting. Lying perpendicular to these rails is a movable conducting bar (mass m) with resistance R that can travel without friction in the x-direction. A uniform time-varying magnetic field $\mathbf{B}$ points along the z-axis as shown. [*You may neglect any magnetic fields due to current flow.*]



(i) Find the current $I(t)$ that passes through the bar and hence find the Lorentz force acting on the bar. For a decaying magnetic field, falling as $B_z = B_0/(1+t)$, $(t \geqslant 0)$, show that the acceleration of the bar at $x(t)$ satisfies the differential equation:

$$\ddot{x} + \frac{\alpha}{(1+t)^2}\,\dot{x} - \frac{\alpha}{(1+t)^3}\,x = 0\,, \qquad (*)$$

where you should specify α in terms of B_0, L, R and m.

(ii) For simplicity, set $\alpha = 1$ in $(*)$. Solve for the motion of the bar, assuming it starts initially at position $x(0) = x_0$ and at rest $\dot{x}(0) = 0$.

[*Hint: Set* $u = 1 + t$. *Note that* $x_1(u) = u$ *is a solution, so seek a second in the form* $x_2(u) = u\,w(u)$.]

(iii) Describe the motion of the bar and calculate its asymptotic velocity for $t \gg 1$. What is the total mechanical work done on the bar? Given the orientation of the magnetic field $\mathbf{B}$, specify the direction of the induced current I around the circuit. Briefly explain why this and the subsequent motion are consistent with Lenz's Law.

17C Numerical Analysis

Let $f \in C^5[-1,1]$. The two-point Gauss–Legendre quadrature rule is

$$\int_{-1}^{1} f(x)\,dx \approx f(-s) + f(s)\,, \qquad \text{where} \quad s = \frac{1}{\sqrt{3}}\,.$$

The linear functional

$$L(f) = \int_{-1}^{1} f(x)\,dx - f(-s) - f(s)$$

quantifies the error in the quadrature rule.

(i) Work out $L(p)$ explicitly when $p \in \mathbb{P}_5$, i.e. $p(x) = \sum_{j=0}^{5} c_j x^j$. What is the highest degree n of polynomial for which the quadrature rule is exact?

(ii) State the Peano kernel theorem. Evaluate the Peano kernel for the error in the two-point Gauss–Legendre quadrature rule. Assuming that the Peano kernel is non-negative, deduce that

$$|L(f)| \leqslant \frac{||f^{(n+1)}||_\infty}{N}\,,$$

where $||f||_\infty = \max_{x\in[-1,1]} |f(x)|$ and N is an integer which you should determine.

(iii) Show that the upper bound in part (ii) is sharp in the case of a quartic polynomial.

18H Markov Chains

Let P be a transition matrix on state space I. What does it mean for a distribution π to be an *invariant distribution*? Consider an irreducible Markov chain on a finite state space with invariant distribution π. State the relationships between π and

(i) the expected return time to a state i, and

(ii) the expected time spent in a state i between visits to a state j.

A burglar is trying to guess a PIN given by a pair of digits in $\{0,\ldots,7\}^2$. Rather than trying each possibility systematically, the burglar, in a panic, instead tries a random sequence of PINs. At each step he chooses one of the 2 components with probability $1/2$, and uniformly at random changes the digit to one of the 7 other possibilities. Suppose the burglar initially tries $(6,0)$.

(a) What is the expected number of steps before the burglar tries $(6,0)$ again?

(b) What is the expected number of times that both digits will be odd before it is $(6,0)$ again?

(c) What is the expected number of steps that the burglar would take to reach $(6,7)$?

(d) After the first burglar has left, a second burglar arrives who modifies the first burglar's strategy as follows: after selecting a component, rather than choosing from the remaining digits uniformly at random, he picks each of 6 and 7 (if it is not the current digit) twice as often as the remaining options. Thus, for instance, the transition from $(6,0)$ to $(7,0)$ has probability $\frac{1}{2}\times\frac{2}{8}$, and that from $(0,0)$ to $(7,0)$ has probability $\frac{1}{2}\times\frac{2}{9}$. The second burglar initially tries $(0,0)$. What is the expected number of times he tries $(6,7)$ before he tries $(0,0)$ again?

END OF PAPER