UNIVERSITY OF CAMBRIDGE

MAT1

MATHEMATICAL TRIPOS Part IB

Tuesday 9 June 2026 9:00am to 12:00pm

PAPER 1

Before you begin read these instructions carefully

The examination paper is divided into two sections. Each question in Section II carries twice the number of marks of each question in Section I. Section II questions also carry an alpha or beta quality mark and Section I questions carry a beta quality mark.

Candidates may obtain credit from attempts on **at most four** questions from Section I and **at most six** questions from Section II.

If you attempt any of the joint Complex Analysis and Complex Methods questions, you may submit an answer to at most one of the two sub-questions.

Write on **one side** of the paper only and begin each answer on a separate sheet.

Write legibly; otherwise you place yourself at a grave disadvantage.

At the end of the examination:

Separate your answers to each question.

Complete a gold cover sheet **for each question** that you have attempted, and place it at the front of your answer to that question.

Complete a green main cover sheet listing **all the questions** that you have attempted.

Every cover sheet must also show your Blind Grade Number and desk number.

Tie up your answers and cover sheets into **a single bundle**, with the main cover sheet on the top, and then the cover sheet and answer for each question, in the numerical order of the questions.

STATIONERY REQUIREMENTS
Gold cover sheets
Green main cover sheet
Treasury tag

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

SECTION I

1G Linear Algebra

Let $V = P_2(\mathbb{R})$ be the vector space of polynomials with real coefficients of degree at most 2.

(a) Show that

$$\langle p, q \rangle = \int_0^1 p(t)q(t)\,\mathrm{d}t$$

defines an inner product on V.

(b) Let W be the subspace of V spanned by the polynomials $\{1, t\}$. Find an orthonormal basis for W.

2F Topological Spaces

Let $(X, \mathcal{T})$ be a topological space. What is a *neighbourhood* of a point $x \in X$? What does it mean for a sequence in X to *converge* to some $x \in X$?

Recall that $(X, \mathcal{T})$ is metrisable if $\mathcal{T}$ is induced by a metric d on X. Given metrics d and d' on X, show that the following are equivalent:

(1) d and d' induce the same topology on X;

(2) the convergent sequences with respect to d and d' are the same, i.e. for every sequence (x_n) in X, $\exists x \in X$ such that $x_n \to x$ with respect to d $\iff$ $\exists y \in X$ such that $x_n \to y$ with respect to d'.

It follows that the convergent sequences uniquely determine the topology of a metrisable space. Does the same hold for a general topological space? Briefly justify your answer.

3 Complex Analysis OR Complex Methods

This is the joint question for Complex Analysis/Complex Methods. Attempt only ONE of the sub-questions. On your answer sheet, specify the question number as either "3.1G" or "3.2C".

(3.1G) Complex Analysis

(a) Let f be holomorphic on the punctured disc $\{z \in \mathbb{C} \colon 0 < |z - z_0| < r\}$ for some $z_0 \in \mathbb{C}$ and $r > 0$. Define the *residue* of f at z_0. State and prove a formula for calculating the residue at a pole of order 2 as a derivative of an appropriate function.

(b) Consider the function

$$f(z) = \frac{1}{z^2(z+1)(z+2)}.$$

Find the Laurent series expansion of $f(z)$ about 0 in the annulus

$$A = \{z \in \mathbb{C} \colon 1 < |z| < 2\}.$$

(3.2C) Complex Methods

For each of the following functions of a complex variable, state the nature of the point $z = 0$, classifying any singularities. Where it is meaningful, state the residue of the function at $z = 0$.

$$\sin(z)\,, \qquad \frac{1}{\sin(z)}\,, \qquad \frac{z}{\sin(z)}\,, \qquad \sin\left(\frac{1}{z}\right)\,, \qquad z\sin\left(\frac{1}{z}\right)\,, \qquad \frac{1}{\sin\left(\frac{1}{z}\right)}\,.$$

4B Variational Principles

State Fermat's principle.

Consider a light ray travelling between two points in the horizontal-vertical $x - y$ plane in a medium such that the speed of light is $c(x)$. Starting from Fermat's principle, derive Snell's law, namely that

$$\frac{\sin\theta}{c(x)}$$

is a constant, where θ is the angle the ray makes with the horizontal at any point.

[You may state then use a suitable Euler-Lagrange equation without proof.]

If $c(x) = 1/\sqrt{3-x}$ and a light ray travels through the origin at an angle $\pi/4$ to the horizontal, what is its angle to the horizontal at $x = 1$?

5C Numerical Analysis

A ν-stage explicit Runge–Kutta method for solving the ordinary differential equation (ODE) $d\boldsymbol{y}/dt = \boldsymbol{f}(t, \boldsymbol{y})$ has the form

$$\begin{aligned}
\boldsymbol{k}_1 &= f(t_n, \boldsymbol{y}_n)\,, \\
\boldsymbol{k}_2 &= f(t_n + c_2 h, \boldsymbol{y}_n + h c_2 \boldsymbol{k}_1)\,, \\
\boldsymbol{k}_3 &= f(t_n + c_3 h, \boldsymbol{y}_n + h(a_{3,1}\boldsymbol{k}_1 + a_{3,2}\boldsymbol{k}_2))\,, \qquad a_{3,1} + a_{3,2} = c_3\,, \\
&\vdots \\
\boldsymbol{k}_\nu &= f\left(t_n + c_\nu h, \boldsymbol{y}_n + h\sum_{j=1}^{\nu-1} a_{\nu,j}\boldsymbol{k}_j\right), \qquad \sum_{j=1}^{\nu-1} a_{\nu,j} = c_\nu\,, \\
\boldsymbol{y}_{n+1} &= \boldsymbol{y}_n + h\sum_{l=1}^{\nu} b_l \boldsymbol{k}_l\,,
\end{aligned}$$

where $h > 0$ is the step size.

(i) Apply the method to the ODE $dy/dt = \lambda y$, where λ is a complex constant, to show that

$$y_{n+1} = r(z) y_n\,,$$

where $z = h\lambda$ and $r(z)$ is a polynomial of degree ν (which you do not need to determine).

(ii) Show that, for the method to be of order ν, $r(z)$ must have the form of the truncated exponential series

$$r(z) = \sum_{l=0}^{\nu} \frac{z^l}{l!}\,.$$

(iii) For the methods considered in part (ii) in the cases $\nu = 2$ and $\nu = 3$, determine which part (if any) of the imaginary axis belongs to the linear stability domain.

6H Statistics

Let $X_1, \ldots, X_n \overset{\text{i.i.d.}}{\sim} U[\theta, 1]$ where $\theta < 1$.

(a) What does it mean for a statistic T to be *sufficient* for θ? Find such a sufficient statistic T.

(b) State the Rao–Blackwell theorem.

(c) Find, with justification, an unbiased estimator $\hat{\theta}$ of θ with mean squared error

$$\mathbb{E}(\hat{\theta} - \theta)^2 = \frac{(1-\theta)^2}{n(n+2)}.$$

[*Hint: First consider an unbiased estimator of* θ *based on just a single observation. The variance of the minimum of* n *independent* $U[0,1]$ *random variables is*

$$\frac{n}{(n+1)^2(n+2)}. \quad]$$

7H Optimisation

Solve the following optimisation problem using the simplex algorithm:

$$\begin{array}{rl} \text{maximise} & x_1 + x_2 + x_3 \\ \text{subject to} & (x_1 - x_2 + x_3)^2 \leqslant 4, \\ & 2x_1 + 3x_2 + x_3 \leqslant 4, \qquad x_1, x_2, x_3 \geqslant 0. \end{array}$$

Suppose the constraint $2x_1 + 3x_2 + x_3 \leqslant 4$ is replaced by $2x_1 + 3x_2 + x_3 \leqslant 4 + \epsilon$. Give an expression for the optimal solution that is valid for all sufficiently small non-zero ϵ.

SECTION II

8E Linear Algebra

(a) Describe (without proof) what it means to put an $n \times n$ complex matrix into *Jordan normal form*. Explain (without proof) the sense in which the Jordan normal form is unique.

(b) Let V_2 be the set of complex polynomials in z, w of degree at most 2 in each variable. You may assume this is a 9-dimensional vector space over $\mathbb{C}$, with basis $\{z^i w^j : 0 \leqslant i, j \leqslant 2\}$.

(i) For $f \in V_2$, consider the map $D : V_2 \to V_2$ given by

$$D(f)(z,w) = \frac{\partial f}{\partial z} + \frac{\partial f}{\partial w}.$$

Explain briefly why 0 is the only eigenvalue of D. Find the corresponding eigenspace.

(ii) Determine the Jordan normal form of D.

(c) Let A be an $n \times n$ complex matrix and let $g(X)$ be a polynomial with complex coefficients.

(i) By considering the Jordan normal form of A, or otherwise, show that if the eigenvalues of A are $\lambda_1, \ldots, \lambda_n$ then the eigenvalues of $g(A)$ are $g(\lambda_1), \ldots, g(\lambda_n)$.

(ii) Let $B = \begin{pmatrix} a & d & c & b \\ b & a & d & c \\ c & b & a & d \\ d & c & b & a \end{pmatrix}$ be a complex matrix. Express B in the form $g(A)$ for a certain polynomial g that you should determine, and with $A = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$. Find the eigenvalues of B.

[*Hint: Compute powers of A.*]

9E Groups, Rings and Modules

Let R be a non-zero commutative ring with a multiplicative identity.

(i) Recall that an element $r \in R$ is nilpotent if $r^n = 0$ for some $n > 0$. Let N be the set of nilpotent elements of R.

Show that N is an ideal of R. What are the nilpotent elements in R/N?

(ii) Show that a polynomial $f(X) = a_0 + a_1 X + \cdots + a_n X^n \in R[X]$ is nilpotent if and only if $a_0, a_1, \ldots, a_n \in N$.

(iii) Show that if $u \in R$ is a unit and $b \in N$ then their sum $u + b$ is a unit. Deduce that if a polynomial $f(X) = a_0 + a_1 X + \cdots + a_n X^n \in R[X]$ is such that a_0 is a unit in R and $a_1, \ldots, a_n \in N$, then $f(X)$ is a unit.

(iv) Show that if $f(X) \in R[X]$ is a zero divisor then there is some non-zero $r \in R$ such that $rf(X) = 0$. [*Hint: Suppose not, and choose a non-zero* $g(X) \in R[X]$ *of minimal degree such that* $f(X)g(X) = 0$.]

Consider a polynomial $f(X) = a_0 + a_1 X + \cdots + a_n X^n \in \mathbb{Z}[X]$ and let $m \geqslant 2$. Write $f(X) \bmod m$ when we consider its coefficients modulo m. Deduce that if $f(X) \bmod m$ is a zero divisor in $(\mathbb{Z}/m\mathbb{Z})[X]$ then $\gcd(a_0, \ldots, a_n, m) > 1$.

10G Analysis II

(a) Let (f_n) be a sequence of real-valued functions defined on a set $E \subset \mathbb{R}$. State the definition of *uniform convergence* of (f_n) to a function f on E.

(b) State and prove the Weierstrass M-test for the uniform convergence of a series of functions on E.

(c) For $x > 1$, define

$$\zeta(x) = \sum_{n=1}^{\infty} \frac{1}{n^x}.$$

Show that the series defining $\zeta(x)$ converges for $x > 1$ and that ζ is continuous on $(1, \infty)$.

(d) Consider the series defined by

$$S(x) = \sum_{n=1}^{\infty} \frac{\cos(n^3 x)}{n^4}, \qquad T(x) = \sum_{n=1}^{\infty} \frac{\sin(n^3 x)}{n}$$

for $x \in \mathbb{R}$.

(i) Evaluate

$$\int_0^{\pi} S(x)\,\mathrm{d}x$$

by integrating the series $S(x)$ termwise. Justify why this operation is valid.

(ii) Does the series $T(x)$ converge uniformly on $(0, \pi)$? Justify your answer.

11 Complex Analysis OR Complex Methods

This is the joint question for Complex Analysis/Complex Methods. Attempt only ONE of the sub-questions. On your answer sheet, specify the question number as either "11.1G" or "11.2C".

(11.1G) Complex Analysis

(a) Let $D = \{z \in \mathbb{C}\colon |z| < 1\}$ denote the open unit disc of the complex plane. Let $f\colon D \to D$ be a holomorphic function such that $f(0) = 0$. Show that $|f(z)| \leqslant |z|$ for all $z \in D$. Deduce that if such a function f also satisfies $|f(z_0)| = |z_0|$ for some $z_0 \in D \setminus \{0\}$, then there exists $a \in \mathbb{C}$ such that $f(z) = az$ for all $z \in D$.

(b) Let Ω be the following domain defined by the intersection of two discs:

$$\Omega = \{z \in \mathbb{C} : |z| < 1\} \cap \{z \in \mathbb{C} : |z - 1| < 1\}.$$

Find, with justification, a conformal bijection f mapping Ω onto the upper half-plane $\mathbb{H} = \{w \in \mathbb{C}\colon \mathrm{Im}(w) > 0\}$.

(11.2C) Complex Methods

Let $z = x + iy$ and $\zeta = \xi + i\eta$ be two complex variables. Consider the map $z \mapsto f(z)$ given by the function

$$f(z) = \zeta = \frac{z + \sqrt{z^2 - c^2}}{c},$$

where c is a positive real constant and the branch cut is taken along the real axis from $-c$ to $+c$.

(i) Show that the map is conformal away from the branch cut.

(ii) Verify that

$$\frac{1}{\zeta} = \frac{z - \sqrt{z^2 - c^2}}{c}$$

and use this property to express z in terms of ζ.

(iii) For any $R > 1$, show that the circle $|\zeta| = R$ is the image under f of an ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \qquad (*)$$

and express the semi-axes $a, b > 0$ of the ellipse in terms of c and R. Show further that the exterior of the ellipse maps to the region $|\zeta| > R$.

[QUESTION CONTINUES ON THE NEXT PAGE]

(iv) Let $w = u + iv$ be a complex variable. Explain why the function

$$g(\zeta) = w = \frac{\zeta - R}{\zeta + R}$$

corresponds to a conformal map of the region $|\zeta| > R$ to the right half-plane $u > 0$. [Properties of Möbius maps may be quoted without proof.]

(v) Suppose we wish to find a solution of Laplace's equation in the exterior of the ellipse $(*)$ that equals $+1$ on the upper half of the ellipse $(y > 0)$, equals -1 on the lower half of the ellipse $(y < 0)$ and tends to zero as $\sqrt{x^2 + y^2} \to \infty$. By considering the function $\log w$ (with the standard branch cut) and quoting any relevant properties of analytic functions, express this solution in terms of a function of z.

12D Methods

(i) Define the two functions $f(x)$ and $F(x)$ on the domain $x \in [0, 1)$ as:

$$f(x) \;=\; 1 - 2x\,, \qquad F(x) \;=\; x(1 - x)\,.$$

Now consider the odd extension of these functions to the domain $[-1, 1)$, labelled respectively as $\bar{f}(x)$, $\bar{F}(x)$. Determine the coefficients for the Fourier series for both of these odd functions on $[-1, 1)$. Compare the rates of convergence of the series and offer a brief explanation.

(ii) Consider the eigenfunction problem

$$\mathcal{L}\,y \;\equiv\; -\,y'' - 2y' \;=\; \lambda y\,,$$

with boundary conditions $y(0) = 0$ and $y(1) = 0$. Demonstrate that the eigenvalues are given by $\lambda = n^2\pi^2 + 1$ for $n = 1, 2, 3, \ldots$ and determine the associated eigenfunctions $y_n(x)$. Put $\mathcal{L}\,y = \lambda y$ in Sturm-Liouville form and obtain the orthogonality relation of these eigenfunctions.

Now consider the inhomogeneous problem,

$$\mathcal{L}\,y \;\equiv\; -\,y'' - 2y' \;=\; e^{-x} f(x)\;, \qquad y(0) = y(1) = 0\,,$$

where the function $f(x) = 1 - 2x$ is the same as in part (i). Assuming the set of eigenfunctions $\{y_n(x)\}$ is complete, seek a series solution of the form $y(x) = \sum_n a_n\, y_n(x)$, using an appropriate expansion for the source term. Explicitly evaluate the series coefficients a_n for this solution.

13B Quantum Mechanics

(i) Why does the Schrödinger equation not apply to massless particles?

(ii) A one-dimensional infinite potential well has a vanishing potential for $-a < x < a$ and an infinite potential elsewhere. Show that the stationary states ψ_n of a quantum mechanical particle of mass $m > 0$ can be written

$$\psi_n(x) = A \sin\left(\omega(x+a)\right),$$

where you should determine each of ω, A and the energies E_n of the stationary states explicitly in terms of m, a, $\hbar$ and $n = 1, 2, \ldots$ Show explicitly that the ψ_n are orthonormal. You may assume that they are also complete.

(iii) Consider the preparation of a particular state ψ in this potential well:

$$\psi(x) = \begin{cases} \sqrt{\frac{2}{a}} \sin\left(\frac{2\pi x}{a}\right), & 0 < x < a\,, \\ 0\,, & -a < x \leqslant 0\,. \end{cases}$$

Many such states are prepared, each in a similar infinite one-dimensional potential well, in different experiments. Calculate the average E_{av} of the energies measured immediately after their preparation.

(iv) By calculating E_{av} in terms of the energies of the Hamiltonian eigenstates, show that

$$\pi^2 = B \sum_{n=0}^{\infty} \frac{(2n+1)^2}{[(2n+1)^2 - 16]^2}$$

for some constant B, which you should find.

[*Hint: You may find the following identity helpful,*

$$\sin\theta_1 \sin\theta_2 = \frac{\cos(\theta_1 - \theta_2) - \cos(\theta_1 + \theta_2)}{2}\,. \qquad]$$

14D Electromagnetism

(a) A steady current I flows along a curve $\mathcal{C}$ formed by a perfectly conducting wire. Assume that the vector potential $\mathbf{A}$ at a point $\mathbf{x}$ away from the wire is given by

$$\mathbf{A}(\mathbf{x}) \;=\; \frac{\mu_0 I}{4\pi} \int_{\mathcal{C}} \frac{d\mathbf{x}'}{|\mathbf{x}-\mathbf{x}'|}\,.$$

Show that $\mathbf{A}(\mathbf{x})$ satisfies the Coulomb gauge condition $\nabla \cdot \mathbf{A} = 0$ with appropriate boundary conditions for $\mathcal{C}$. Deduce the Biot-Savart law for the magnetic field,

$$\mathbf{B}(\mathbf{x}) \;=\; \frac{\mu_0 I}{4\pi} \int_{\mathcal{C}} \frac{d\mathbf{x}' \times (\mathbf{x}-\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|^3}\,.$$

(b) An infinite wire carrying a current I runs along the z-axis from $z = -\infty$ to the origin $(0,\,0,\,0)$, where it turns at right angles and continues along the negative y-axis to $y = -\infty$. Consider the semi-infinite segment of wire along the negative z-axis only and use the Biot-Savart law to show that its contribution to the magnetic field is given in cylindrical polar coordinates (r, ϕ, z) (with $r^2 \equiv x^2 + y^2$) by

$$\mathbf{B}(\mathbf{x}) \;=\; \frac{\mu_0}{4\pi}\frac{I}{r}\left(1 - \frac{z}{\sqrt{r^2+z^2}}\right)\hat{\boldsymbol{e}}_\phi\,.$$

By considering Cartesian coordinates, obtain the complete solution for $\mathbf{B}$ from the wire which also includes the semi-infinite segment along the negative y-axis. Briefly discuss the limiting values of the magnetic field along the line $x = 0$, $y = 1$ in the asymptotic limits $z \to \pm\infty$.

15A Fluid Dynamics

An inviscid, incompressible fluid flow is governed by the Euler equations

$$\rho \left[\frac{\partial \boldsymbol{u}}{\partial t} + (\boldsymbol{u} \cdot \nabla)\boldsymbol{u} \right] = -\nabla p \,, \qquad \nabla \cdot \boldsymbol{u} = 0 \,,$$

where the density ρ is a constant.

(a) Derive the evolution equation satisfied by the vorticity $\boldsymbol{\omega} = \nabla \times \boldsymbol{u}$ and express it in a form in which the incompressibility of the flow is explicit.

(b) Consider the velocity field $\boldsymbol{u}$ whose Cartesian components are

$$\boldsymbol{u} = (-\frac{1}{2}\alpha(t)x, -\frac{1}{2}\alpha(t)y, \alpha(t)z),$$

where $\alpha(t)$ is a given function of time.

(i) Show that the flow is incompressible.

(ii) Suppose that a spatially uniform but temporally varying rotation is added to the flow so that

$$\boldsymbol{\omega}(t) = (0, 0, \omega(t)).$$

Using the vorticity equation derived in part (a), obtain an equation for the rate of change of $\omega(t)$.

(iii) Solve the obtained equation for $\omega(t)$ in terms of $\alpha(t)$ and the initial value $\omega(0) = \omega_0$. Hence describe how the vorticity magnitude changes when $\alpha(t) > 0$ and explain your result in terms of vortex stretching.

16C Numerical Analysis

(i) Let $p \in \mathbb{P}_n$ interpolate the function $f \in C^{n+1}[-1,1]$ at the $n+1$ distinct points $x_0, \cdots, x_n \in [-1,1]$. Prove that the interpolation error for any $x \in [-1,1]$ can be expressed as

$$f(x) - p(x) = \frac{f^{(n+1)}(\xi)}{(n+1)!}\,\Pi(x)\,,$$

where

$$\Pi(x) = \prod_{i=0}^{n}(x - x_i)$$

and $\xi \in [-1,1]$ is some number that depends on x. [You may assume Rolle's theorem.]

(ii) Let $\theta = \arccos x$ and let $T_{n+1}(x) = \cos((n+1)\theta)$ be the Chebyshev polynomial of degree $n+1$. In the case of the Chebyshev points,

$$x_i = \cos\theta_i\,, \qquad \theta_i = \left(i + \frac{1}{2}\right)\frac{\pi}{n+1}\,, \qquad 0 \leqslant i \leqslant n\,,$$

which are the zeros of $T_{n+1}(x)$, show that

$$|\Pi(x)| \leqslant \left(\frac{1}{2}\right)^n$$

for all $x \in [-1,1]$.

[*Hint: The identity* $\cos((n+1)\theta) + \cos((n-1)\theta) = 2\cos\theta\cos(n\theta)$ *can be rewritten as a recurrence relation for the (non-monic) polynomial* $T_n(x)$*. Hence, or otherwise, deduce the coefficient of* x^n *in* $T_{n+1}(x)$*.*]

(iii) For the function

$$f(x) = \cos(kx)\,,$$

where k is a positive constant, show that $f(x) - p(x) \to 0$ as $n \to \infty$ for all $x \in [-1,1]$ in the case of the Chebyshev points.

(iv) In the case of equally spaced points,

$$x_i = -1 + \left(i + \frac{1}{2}\right)\frac{2}{n+1}\,, \qquad 0 \leqslant i \leqslant n\,,$$

show that

$$\Pi(1) > \left(\frac{2}{e}\right)^n.$$

[*Hint: Use a graphical or similar argument to show that*

$$\log\left(\frac{3}{2}\right) + \log\left(\frac{5}{2}\right) + \cdots + \log\left(\frac{2n+1}{2}\right) > \int_1^{n+1} \log x\,dx\,.]$$

17H Statistics

Consider the normal linear model $Y = X\beta + \varepsilon$ where X is a known matrix of predictors, $\beta \in \mathbb{R}^p$ is an unknown vector of parameters, and $\varepsilon \sim N_n(0, \sigma^2 I)$ is a vector of normal errors with $\sigma > 0$. Let $X = (X_0, X_1)$, where $X_0 \in \mathbb{R}^{n \times p_0}$ and $X_1 \in \mathbb{R}^{n \times p_1}$ with $p_0 + p_1 = p$, and correspondingly write $\beta^\top = (\beta_0^\top, \beta_1^\top)$ where $\beta_0 \in \mathbb{R}^{p_0}$ and $\beta_1 \in \mathbb{R}^{p_1}$. We wish to test the null hypothesis H_0 that $\beta_1 = 0$ against the alternative that $\beta_1 \neq 0$. We assume throughout that X_0 has full column rank.

(i) Write down the maximum likelihood estimator $(\hat{\beta}_0, \hat{\sigma}^2)$ of (β_0, σ^2) in the model given by the null hypothesis H_0.

(ii) Assuming that X has full column rank, write down the F-test statistic for testing H_0.

(iii) Let $R = Y - X_0\hat{\beta}_0$ be the residuals from fitting the model given by H_0. Show that under the null, the distribution of R/σ is the same as that of $(I - P_0)Z$ where $Z \sim N_n(0, I)$ and $P_0 \in \mathbb{R}^{n \times n}$ is a matrix you should specify.

(iv) Let $X_{1j} \in \mathbb{R}^n$ be the jth column of the matrix X_1 and suppose $\|X_{1j}\| = \sqrt{n}$ for all j. It is known that if $\zeta \sim N(0,1)$, then $\mathbb{P}(|\zeta| > t) \leqslant e^{-t^2/2}$ for $t \geqslant 0$. Show that for each j,

$$\mathbb{P}\left(\frac{1}{\sqrt{n}}\left|X_{1j}^\top (I - P_0)Z\right| > t\right) \leqslant e^{-t^2/2}.$$

(v) Let $X_{1j} \in \mathbb{R}^n$ be the jth column of the matrix X_1. We do not assume that X has full column rank, but suppose that σ is known. Consider the test statistic

$$T = \frac{1}{\sigma\sqrt{n}} \max_{j=1,\dots,p_1} |X_{1j}^\top R|.$$

Let $\alpha \in (0,1)$. Show that the test that rejects H_0 when $T \geqslant \sqrt{2\log(p_1/\alpha)}$ has size at most α.

[*Hint: Given events $A_1, \dots, A_{p_1}$, a union bound yields* $\mathbb{P}(\cup_{j=1}^{p_1} A_j) \leqslant \sum_{j=1}^{p_1} \mathbb{P}(A_j)$.]

(vi) Now consider the case where $\sigma > 0$ is unknown (and X does not necessarily have full column rank). Find, with justification, a test statistic for the null H_0 against the alternative $\beta_1 \neq 0$ that under the null has the same distribution as

$$\max_{j=1,\dots,p_1} \frac{|X_{1j}^\top (I - P_0)Z|}{\|(I - P_0)Z\|}.$$

18H Markov Chains

(a) Let $(X_n)_{n\geqslant 0}$ be a Markov chain with transition matrix P. What is a *stopping time* of $(X_n)_{n\geqslant 0}$? What is the *strong Markov property*?

(b) The chatbot for dogs, ChatGPET 1.0, produces sequences consisting of three sounds: a bark, a growl and a whine. Given a sound X_0 of this form initiated by a dog, at time t for $t = 1, 2, \ldots$, the chatbot produces a sound X_t such that $X_0, X_1, \ldots$ is a Markov chain with transition matrix

$$P = \begin{pmatrix} 0 & 1/2 & 1/2 \\ 2/3 & 0 & 1/3 \\ 2/3 & 1/3 & 0 \end{pmatrix}.$$

Here rows 1, 2 and 3 correspond to bark, growl and whine, respectively. Suppose $X_0 =$ bark. Determine the expected time for ChatGPET 1.0 to produce:

(i) a growl;

(ii) a growl immediately followed by a whine.

(c) It was found that dogs find too many barks in the conversation over-stimulating, but this can be mitigated by a number of whines. The newest version, ChatGPET 2.0, therefore includes the following additional safety feature: writing N_b for the number of barks produced by the chatbot, and N_w for the number of whines, when $N_b - N_w \geqslant m$ for some $m \in \mathbb{N}$, the conversation is terminated. By appropriately augmenting the state space of the Markov chain in part (b), find the expected length of the sequence produced by ChatGPET 2.0.

[*Hint: It may help to set β, γ, ω to be the expected times for $N_b - N_w$ to increase by 1 when starting with a bark, growl and whine respectively.*]

END OF PAPER