UNIVERSITY OF CAMBRIDGE

MAT0

MATHEMATICAL TRIPOS Part IA

Wednesday 10 June 2026 2:00pm to 5:00pm

PAPER 4

Before you begin read these instructions carefully

The examination paper is divided into two sections. Each question in Section II carries twice the number of marks of each question in Section I. Section II questions also carry an alpha or beta quality mark and Section I questions carry a beta quality mark.

Candidates may obtain credit from attempts on **all four** questions from Section I and **at most five** questions from Section II. Of the Section II questions, no more than three may be on the same course.

Write on **one side** of the paper only and begin each answer on a separate sheet.

Write legibly; otherwise you place yourself at a grave disadvantage.

At the end of the examination:

Separate your answers to each question.

Complete a gold cover sheet **for each question** that you have attempted, and place it at the front of your answer to that question.

Complete a green main cover sheet listing **all the questions** that you have attempted.

Every cover sheet must also show your Blind Grade Number and desk number.

Tie up your answers and cover sheets into **a single bundle**, with the main cover sheet on the top, and then the cover sheet and answer for each question, in the numerical order of the questions.

STATIONERY REQUIREMENTS
Gold cover sheets
Green main cover sheet
Treasury tag

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

SECTION I

1F Numbers and Sets

Let $u_n = (3+\sqrt{5})^n + (3-\sqrt{5})^n, \quad \text{for } n = 0,1,2,\ldots.$

(i) Show that u_n is an integer for all $n = 0,1,2,\ldots$.

(ii) Show that u_n is divisible by 2^n for all $n = 0,1,2,\ldots$.

[*Hint: Find integers* $a, b \in \mathbb{Z}$ *such that* $u_n = au_{n-1} + bu_{n-2}$ *for* $n \geqslant 2$.]

2D Numbers and Sets

(a) Find all solutions of

$$5z \equiv 2 \bmod 17 \ \text{ and } \ 3z \equiv 5 \bmod 19.$$

(b) Suppose Alice sends Bob a message encrypted using RSA with $(N, e) = (33, 7)$. How should Bob decode the received message 29 mod 33? Explain your answer and explain why RSA is believed to be secure.

3B Dynamics and Relativity

(i) Determine the mass, time, and length dimension of Newton's constant G.

(ii) Use dimensional analysis to derive Kepler's third law.

(iii) General relativity is a theory of gravity that involves two fundamental constants: G and the speed of light c. Use dimensional analysis to relate the radius R of a black hole to its mass M.

(iv) The theory of elementary particles involves Planck's constant $\hbar$ and the speed of light c. Use dimensional analysis to relate the linear size (i.e. length) λ of a particle to its mass m.

(v) Combining these results, what is the threshold mass above which an elementary particle become a black hole?

[Throughout this question, you may ignore overall numerical constants.]

4B Dynamics and Relativity

Define the *proper time*. Explain why it is appropriate to define the 4-velocity in terms of proper time.

In an inertial frame, Bob traces out a path in Minkowski space given by

$$X^\mu = (ct, R\sin(\omega t), R\cos(\omega t), 0)\ .$$

What requirements on ω and R ensure this is a time-like trajectory?

Bob starts his journey at $t = 0$. How old is he, in terms of the parameters R, ω, and c, when he next returns to his starting point in this frame?

Compute the 4-velocity U^μ. What is the Lorentz invariant $U \cdot U$?

SECTION II

5F Numbers and Sets

(i) Find which of the following numbers is irrational with proper justification.

(a) $\log 2$ (here log is logarithm to the base e)
(b) $\log_2 3$
(c) $ae + be^{-1}$ for a, b non-zero integers
(d) the real root of the equation $x^3 + 4x - 3$

(ii) Let $n \geqslant 2$ be an integer. Let

$$H_n := 1 + \frac{1}{2} + \cdots + \frac{1}{n}.$$

Show that H_n cannot be an integer.

[You may use the fact that e is transcendental.]

6D Numbers and Sets

Let $\mathbb{Z}_n = \{0, 1, \ldots, n-1\}$ with arithmetic modulo n.

(i) Suppose $n = p$ is a prime. Show that all non-zero elements of $\mathbb{Z}_p$ are units.

(ii) Show that if $n = p$ is an odd prime and $a \neq 0$ is a square modulo n, then a has exactly 2 square roots. Does this hold if n is composite?

(iii) Prove that

$$(n-1)! \equiv -1 \bmod n$$

if and only if n is a prime.

(iv) Evaluate $16! \bmod 19$.

(v) Suppose $n = p$ is a prime. Show that if $a \neq 0$ is a square modulo p, then $a^{\frac{p-1}{2}} \equiv 1 \bmod p$ and if a is not a square, then $a^{\frac{p-1}{2}} \equiv -1 \bmod p$.

[*Fermat's Little Theorem can be used as long as it is clearly stated.*]

7E Numbers and Sets

For each of the statements (a)–(e) below, determine whether it is true or false, justifying appropriately your answer with a proof or counterexample. You may refer without proof to standard results in the course.

(a) Let $f : A \to B$ be injective. There exists a $C \subset B$ and an injective $g : C \to A$ such that $g \circ f = \mathrm{id}$.

(b) Suppose $f : A \to B$ is surjective and $g : B \to C$ is injective. Then $g \circ f$ is injective.

(c) Suppose $f : A \to B$ is injective and $g : B \to C$ is surjective. Then $g \circ f$ is surjective.

(d) Suppose $f : A \to B$ is not injective and $g : B \to C$ is a map. Then $g \circ f$ is not injective.

To introduce the final statement (e) to be given below, let A and B be non-empty finite sets with $|A| = |B| + 1 \geqslant 4$, and define $F_{A,B} = \{f : A \to B : f \text{ surjective}\}$ and $G_{B,A} = \{g : B \to A : g \text{ injective}\}$. [Note that these sets are non-empty.] Define a map $\Psi : F_{A,B} \to G_{B,A}$ taking $f \mapsto g$ where g is defined as follows: For each $b \in B$, pick $a \in A$ such that $f(a) = b$ and set $g(b) = a$. [Note that Ψ indeed defines a map from the sets indicated.]

(e) The map Ψ is injective.

Does the validity of (e) change if we assume $|A| = |B| + 1 = 2$? What can you say in the case $|A| = |B| + 1 = 3$?

8E Numbers and Sets

(a) For given $n \geqslant 1$, let $\mathcal{P}_n$ denote the set of all polynomials in n variables $x_1, \ldots, x_n$ with rational coefficients, not all vanishing. (For instance, an example of a member of the set $\mathcal{P}_2$ is the polynomial $p(x_1, x_2) = x_1^3 x_2 + \frac{1}{2} x_2^2$, while the 0 polynomial is excluded from the set.) Show that the set $\mathcal{P}_n$ is countable for all $n \geqslant 1$.

(b) Now define the set

$$X_n := \{(x_1, x_2, \ldots, x_n) \in \mathbb{R}^n : \exists p \in \mathcal{P}_n : p(x_1, x_2, \ldots, x_n) = 0\} .$$

Is X_n countable? Does your answer depend on n?

(c) Let S be a set with the following properties:

(i) $\mathbb{R} \supset S \supset \mathbb{Q}$

(ii) If $x \in S$, then $\sqrt{|x|} \in S$

(iii) If $x \in S$ and $0 \neq y \in S$, then $xy \in S$, $x + y \in S$, $-y \in S$ and $y^{-1} \in S$

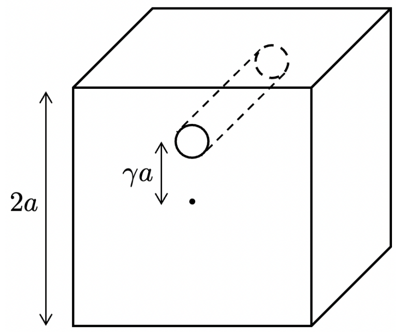
(iv) If $\tilde{S} \subset \mathbb{R}$ satisfies (i), (ii) and (iii), then $\tilde{S} \supset S$

Show that the set S is uniquely defined and is countable.

[Hint: *Exhibit S as a union $S = \cup_{n \geqslant 0} S_n$ of countable sets S_n satisfying (iii) and the property that if $x \in S_n$ then $\sqrt{|x|} \in S_{n+1}$. To construct S_n, again try and exhibit these as countable unions $S_n = \cup_{m \geqslant 0} S_{n,m}$.*]

9B Dynamics and Relativity

A cube of uniform density has sides of length $2a$. A cylindrical hole of radius r is drilled through the cube, parallel to one edge, equidistant from two other edges, and a distance γa from the centre, as shown in the figure. Here $\gamma < 1$ and $r < (1-\gamma)a$.



The cube oscillates about a frictionless, cylindrical horizontal rod of radius r that passes through the hole. What is the period of small oscillations?

[You may quote the parallel axis theorem without proof.]

10B Dynamics and Relativity

A particle of mass m moves in three dimensions in a potential $V(\mathbf{x})$.

(a) Write down the equation of motion of the particle. Show that there is a conserved energy E.

(b) What is meant by a *central potential*? Show that the angular momentum $\mathbf{L} = m\mathbf{x}\times\dot{\mathbf{x}}$ is conserved when the potential is central. Explain why conservation of angular momentum implies that the motion is restricted to a plane.

(c) Suppose now that the particle has electric charge q and, in addition to the central potential, moves in a magnetic field $\mathbf{B}(\mathbf{x})$. Write down the equation of motion of the particle and show that the energy E is still conserved.

The magnetic field is given by

$$\mathbf{B}(\mathbf{x}) = \frac{\beta}{r^2}\hat{\mathbf{r}}$$

with β constant, $r = |\mathbf{x}|$ and $\hat{\mathbf{r}} = \mathbf{x}/r$. Show that

$$\mathbf{J} = \mathbf{L} - \kappa\hat{\mathbf{r}}$$

is conserved for some constant κ that you should determine. Hence show that the motion of the particle is restricted to lie on a cone. Find an expression for the opening angle α of the cone.

(d) Show that the problem considered in part (c) reduces to motion in the radial direction along the cone with the effective potential given by

$$V_{\text{eff}}(r) = \frac{J^2\sin^2\alpha}{2mr^2} + V(r)\ .$$

11B Dynamics and Relativity

(a) A non-inertial frame S' rotates with angular velocity $\boldsymbol{\omega}$ with respect to an inertial frame S. Show that, from the perspective of S', the equation of motion for a particle of mass m, subjected to a force $\mathbf{F}$, takes the form

$$m\ddot{\mathbf{x}} = \mathbf{F} - 2m\boldsymbol{\omega} \times \dot{\mathbf{x}} - m\dot{\boldsymbol{\omega}} \times \mathbf{x} - m\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{x}) \ .$$

(b) A horizontal square hoop ABCD is made of fine smooth wire and has side length $2a$. When stationary, the coordinates of the corners are $A = (0,0,0)$, $B = (2a,0,0)$, $C = (2a,2a,0)$ and $D = (0,2a,0)$. The hoop rotates about a vertical axis through A with time-dependent angular velocity

$$\boldsymbol{\omega}(t) = -\frac{\alpha}{t}\hat{\mathbf{z}}$$

where $\alpha > 0$. A small bead can slide along the side BC. Let s be the distance of the bead from the vertex B. At time $t = t_0 > 0$, the bead is at rest at the point $s = s_0$.

For what values of α is there a choice of s_0 such that the bead remains stationary on the wire? The bead is now nudged gently so it moves away from s_0. What is the most general form of the solution $s(t)$? Where does the bead end up?

12B Dynamics and Relativity

(a) A pion with rest mass m_π decays to an electron of rest mass m_e and a neutrino which we approximate as massless. Find an expression for the speed of the electron in the pion rest frame.

(b) A positron and electron have the same rest mass m_e. A moving positron with energy E annihilates with a stationary electron, producing two photons. What is the minimum E for which the two photons may move at right angles to each other.

(c) A Higgs boson, with rest mass m, moves in the x-direction. It decays into two photons with energies E_1 and E_2. Show that the angle θ between the photons is given by

$$\sin\frac{\theta}{2} = \frac{mc^2}{2\sqrt{E_1 E_2}} \ .$$

Find an expression for the speed of the Higgs boson in terms of θ if, in the rest frame of the Higgs boson, the photons are emitted in opposite directions along the y-axis.

END OF PAPER