UNIVERSITY OF CAMBRIDGE

MAT0

MATHEMATICAL TRIPOS Part IA

Monday 8 June 2026 9:00am to 12:00pm

PAPER 3

Before you begin read these instructions carefully

The examination paper is divided into two sections. Each question in Section II carries twice the number of marks of each question in Section I. Section II questions also carry an alpha or beta quality mark and Section I questions carry a beta quality mark.

Candidates may obtain credit from attempts on **all four** questions from Section I and **at most five** questions from Section II. Of the Section II questions, no more than three may be on the same course.

Write on **one side** of the paper only and begin each answer on a separate sheet.

Write legibly; otherwise you place yourself at a grave disadvantage.

At the end of the examination:

Separate your answers to each question.

Complete a gold cover sheet **for each question** that you have attempted, and place it at the front of your answer to that question.

Complete a green main cover sheet listing **all the questions** that you have attempted.

Every cover sheet must also show your Blind Grade Number and desk number.

Tie up your answers and cover sheets into **a single bundle**, with the main cover sheet on the top, and then the cover sheet and answer for each question, in the numerical order of the questions.

STATIONERY REQUIREMENTS
Gold cover sheets
Green main cover sheet
Treasury tag

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

SECTION I

1D Groups

In the following groups determine the number of elements of order 4. Justify your answers.

(a) $C_2 \times C_4$ (where C_n denotes the cyclic group of order n).

(b) S_4 (where S_n denotes the symmetric group of degree n).

(c) S_6.

(d) A_6 (where A_n denotes the alternating group of degree n).

2D Groups

Define the centre, $Z(G)$, of a group G and show that it is an abelian normal subgroup of G.

Are the following true or false? Justify your answers.

(a) If K is a normal subgroup of order 2, then $K \leqslant Z(G)$.

(b) If $G/Z(G)$ is cyclic, then G is abelian.

(c) There exists a nontrivial group with trivial centre.

3A Vector Calculus

Consider a curve $\mathbf{x}(t)$ in $\mathbb{R}^3$. Define the *arc length* s, *tangent vector* $\mathbf{t}$, *principal normal vector* $\mathbf{n}$ and *curvature* κ.

Show that $\mathbf{t}$ is perpendicular to $\mathbf{n}$.

Compute $\mathbf{t}$, $\mathbf{n}$ and κ for the curve

$$\mathbf{x}(t) = \left(e^t \cos t,\, e^t \sin t,\, \sqrt{2}e^t\right) .$$

4A Vector Calculus

State the *divergence theorem* for a smooth vector field $\mathbf{F}(\mathbf{x})$ in $\mathbb{R}^3$ and a volume V enclosed by a piecewise smooth closed surface $S = \partial V$.

By explicitly calculating the volume and surface integrals, verify the divergence theorem for the case where

$$\mathbf{F}(\mathbf{x}) = \mathbf{x}\, e^{-\sqrt{x^2+y^2+z^2}} ,$$

and V is a sphere of radius a centred on the origin.

SECTION II

5D Groups

Let G be a finite group and $g \in G$. Define the *order* of g. Show that if $g^n = e$, then the order of g divides n.

State and prove Lagrange's theorem.

Deduce that the order of g divides the order of G.

Let N be a normal subgroup of G. Explain how G/N inherits the structure of a group.

Suppose N has index m in G and the order of N is n. Further, suppose m and n are coprime. Show the following:

(a) $N = \{a \in G : a^n = e\}$.

(b) $N = \{b^m : b \in G\}$.

6D Groups

(i) Define $\mathrm{GL}_2(\mathbb{C})$ and $\mathrm{SL}_2(\mathbb{C})$. Show that $\mathrm{SL}_2(\mathbb{C})$ is a normal subgroup of $\mathrm{GL}_2(\mathbb{C})$ and identify the quotient group $\mathrm{GL}_2(\mathbb{C})/\mathrm{SL}_2(\mathbb{C})$.

(ii) Define the Möbius group $\mathcal{M}$. Explain how $\mathcal{M}$ can be identified as a quotient group of $\mathrm{GL}_2(\mathbb{C})$.

(iii) Show that $\mathcal{M}$ acts transitively on $\mathbb{C} \cup \{\infty\}$. Show that any non-identity element of $\mathcal{M}$ has one or two fixed points. Identify the stabiliser of ∞ and denote it by S.

(iv) Show that every element of $\mathcal{M}$ is conjugate to an element of S.

7D Groups

Let G be a finite group. Define what it means to say G acts on a set X. Explain how, if $|X| = n$, this yields a homomorphism from G to S_n, where S_n denotes the symmetric group of degree n.

Let H be a subgroup of G of index n. Suppose $|G|$ does not divide $n!$. Show that H contains a nontrivial normal subgroup of G.

Let L be a subgroup of G of index p, where p is the smallest prime divisor of $|G|$. Show that L is a normal subgroup of G.

If M is a subgroup of G of index p, where p is prime, does it follow that M is normal?

8D Groups

Let S_n denote the symmetric group of degree $n \geqslant 2$. Explain how a permutation $\sigma \in S_n$ can be written as a product of disjoint cycles.

Show that S_n can be generated by transpositions.

Define A_n, the alternating group of degree n.

For any even m with $2 \leqslant m \leqslant n$, show that A_n can be generated by m-cycles. [*Hint: Write* $(a, b)(a, c)$ *as a product of two* m*-cycles.*]

For any even m with $2 \leqslant m \leqslant n$, show that S_n can be generated by m-cycles.

Can S_n be generated by m-cycles if m is odd? Justify your answer.

9A Vector Calculus

(a) Using suffix notation, prove the vector identity

$$\boldsymbol{\nabla} \times (\mathbf{F} \times \mathbf{G}) = (\boldsymbol{\nabla} \cdot \mathbf{G})\mathbf{F} - (\boldsymbol{\nabla} \cdot \mathbf{F})\mathbf{G} + (\mathbf{G} \cdot \boldsymbol{\nabla})\mathbf{F} - (\mathbf{F} \cdot \boldsymbol{\nabla})\mathbf{G}\,.$$

[You may use without proof any standard results involving the antisymmetric tensor ϵ_{ijk}.]

(b) Define what is meant by an *irrotational* vector field. Prove that if $\mathbf{E} = \boldsymbol{\nabla} f$, where f is a smooth scalar field, then $\mathbf{E}$ is irrotational.

(c) Consider cylindrical polar coordinates (ρ, ϕ, z). For each of the following vector fields $\mathbf{E}$, explicitly compute

$$I = \oint_C \mathbf{E} \cdot d\mathbf{x}\,,$$

where C is a circle of radius a about the z-axis at $z = 0$ traversed in the direction of increasing ϕ. For each case, find a scalar field f such that $\mathbf{E} = \boldsymbol{\nabla} f$.

(i) $\mathbf{E} = \frac{1}{\rho}\,\mathbf{e}_\phi$ for $\rho \neq 0$.

(ii) $\mathbf{E} = \left[(-2\rho^2 + 1)\sin\phi\,\mathbf{e}_\rho + \cos\phi\,\mathbf{e}_\phi - 2z\rho\sin\phi\,\mathbf{e}_z\right] e^{-\rho^2 - z^2}$.

[*Hint: In cylindrical polar coordinates* $\boldsymbol{\nabla} f = \frac{\partial f}{\partial \rho}\mathbf{e}_\rho + \frac{1}{\rho}\frac{\partial f}{\partial \phi}\mathbf{e}_\phi + \frac{\partial f}{\partial z}\mathbf{e}_z$.]

In the case of part (ii), how could we have inferred the value of I without explicit computation? What about the value of I in part (i)?

10A Vector Calculus

(a) State *Stokes' theorem* for a smooth vector field $\mathbf{F}$ in $\mathbb{R}^3$ and a piecewise smooth open surface S with a piecewise smooth boundary curve $C = \partial S$. [Carefully specify any directions/orientations.]

(b) Consider $\mathbf{F} = (-y^3, x^3, -x^3 + y^3)$. Calculate $\boldsymbol{\nabla} \times \mathbf{F}$ and explicitly verify that $\boldsymbol{\nabla} \cdot (\boldsymbol{\nabla} \times \mathbf{F}) = 0$.

(c) Consider an open surface S defined in two pieces: $x^2 + y^2 = a^2$ for $0 \leqslant z \leqslant h$ and the disc $x^2 + y^2 \leqslant a^2$ for $z = h$, where a and h are positive constants. Sketch S and illustrate the direction of $d\mathbf{S}$ for each piece of the surface.

(d) Verify Stokes' theorem for $\mathbf{F}$ in part (b) with S in part (c) by explicitly calculating the surface and line integrals.

(e) Now consider an open surface S' defined by $x^2 + y^2 + 3z^2 + 5z^4 = a^2$ for $z \geqslant 0$, where a is a positive constant. Calculate

$$\int_{S'} (\boldsymbol{\nabla} \times \mathbf{F}) \cdot d\mathbf{S}\,,$$

for $\mathbf{F}$ given in part (b).

11A Vector Calculus

(a) Consider a scalar field $\psi(\mathbf{x})$ that satisfies

$$\nabla^2\psi - m^2\psi = \rho \text{ in } V, \quad \partial\psi/\partial\mathbf{n} = f \text{ on } S,$$

where V is a region of $\mathbb{R}^3$ with boundary S a closed surface, ρ and f are given functions on V and S respectively, and m is a real constant. Show that ψ is unique for $m \neq 0$. Does the conclusion change if $m = 0$?

(b) The electric field $\mathbf{E}(\mathbf{x})$ satisfies $\boldsymbol{\nabla} \cdot \mathbf{E} = \rho/\epsilon_0$ and $\boldsymbol{\nabla} \times \mathbf{E} = \mathbf{0}$, where $\rho(\mathbf{x})$ is the charge density and ϵ_0 is a constant (the permittivity of free space).

(i) Briefly explain how an electric potential $\phi(\mathbf{x})$ can be defined and show this satisfies Poisson's equation.

(ii) Consider a sphere of radius b with charge density

$$\rho(\mathbf{x}) = \begin{cases} 0 & \text{for } r < a \\ \rho_0/r & \text{for } a \leqslant r \leqslant b\,, \end{cases}$$

where $b > a > 0$ and ρ_0 are constants, and $r = |\mathbf{x}|$. Find the electric potential ϕ in the three regions $r < a$, $a \leqslant r \leqslant b$ and $r > b$, assuming that $\phi = \phi(r)$ is spherically symmetric, ϕ and ϕ' are continuous everywhere, and $\phi \to 0$ as $r \to \infty$.

(iii) Compute the electric field outside a sphere of radius $c < b$ with uniform charge density (i.e. ρ constant) and total charge Q. For $r > b$ show that this electric field is equivalent to the electric field outside the sphere in part (ii) if that sphere also has total charge Q.

[*Hint:* $\nabla^2\phi(r) = \frac{1}{r}\frac{d^2}{dr^2}(r\phi)$.]

12A Vector Calculus

In this question consider real Cartesian tensors in $\mathbb{R}^3$.

(a) Define a *tensor* T of rank k, explaining carefully the meaning of the notation in your definition.

(b) If A is a tensor of rank p and B is a tensor of rank q, define what is meant by the contraction of one index of A with one index of B. What is the rank of the resulting tensor? For the particular case $p = 3$ and $q = 3$, show that this contraction of A and B does result in a tensor.

(c) Define an *isotropic* tensor and write down the most general isotropic tensors of rank 0, 1, 2 and 3.

(d) For each of the following integrals, briefly indicate why the integral is an isotropic tensor, state the rank of the tensor and compute the integral,

(i) $\int_S x_i e^{-r^2} dS$,

(ii) $\int_S x_i x_j e^{-r^2} dS$,

(iii) $\int_S x_i x_j x_k e^{-r^2} dS$,

where S is the surface of a sphere of radius a centred on the origin.

(e) Consider the tensor

$$T_{ij}(\mathbf{u}, \mathbf{v}) = \int_S (x_i - u_i)(x_j - v_j) e^{-r^2} dS \,,$$

where $\mathbf{u}$ and $\mathbf{v}$ are fixed vectors. Show that it has the form $T_{ij} = \alpha\delta_{ij} + \beta u_i v_j$, where α and β are scalars. Determine α and β.

END OF PAPER