UNIVERSITY OF CAMBRIDGE

MAT0

MATHEMATICAL TRIPOS Part IA

Friday 5 June 2026 2:00pm to 5:00pm

PAPER 2

Before you begin read these instructions carefully

The examination paper is divided into two sections. Each question in Section II carries twice the number of marks of each question in Section I. Section II questions also carry an alpha or beta quality mark and Section I questions carry a beta quality mark.

Candidates may obtain credit from attempts on **all four** questions from Section I and **at most five** questions from Section II. Of the Section II questions, no more than three may be on the same course.

Write on **one side** of the paper only and begin each answer on a separate sheet.

Write legibly; otherwise you place yourself at a grave disadvantage.

At the end of the examination:

Separate your answers to each question.

Complete a gold cover sheet **for each question** that you have attempted, and place it at the front of your answer to that question.

Complete a green main cover sheet listing **all the questions** that you have attempted.

Every cover sheet must also show your Blind Grade Number and desk number.

Tie up your answers and cover sheets into **a single bundle**, with the main cover sheet on the top, and then the cover sheet and answer for each question, in the numerical order of the questions.

STATIONERY REQUIREMENTS
Gold cover sheets
Green main cover sheet
Treasury tag

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

SECTION I

1C Differential Equations

Consider the integral

$$I(x) = \int_0^1 y^2 \, e^{xy} \, dy \qquad \text{for } x \in \mathbb{R}.$$

By differentiating under the integral sign show that

$$x^2 \frac{d^2 I(x)}{dx^2} - 12 I(x) = e^x (x-4).$$

2C Differential Equations

(a) Find two solutions of

$$x^2 y'' + 4xy' + (2 + x^2) y = 0, \qquad \text{for } x > 0,$$

and prove explicitly by computing their *Wronskian* $W(x)$ that they are independent. [*Hint: You may find the substitution* $y(x) = z(x)/x^2$ *helpful.*]

(b) Find a particular integral for

$$x^2 y'' + 4xy' + (2 + x^2) y = x^2, \qquad \text{for } x > 0.$$

3F Probability

Let X and Y have the joint density function f given by

$$f(x,y) = \frac{\sqrt{3}}{4\pi} e^{-(x^2 - xy + y^2)/2}, \quad x, y \in \mathbb{R}.$$

(i) Find the marginal density of X. Find the marginal density of Y. Are X and Y independent?

(ii) Find the conditional density of Y given X. What is the expectation of Y given X?

4F Probability

Suppose we have two decks of n cards, numbered $1, 2, \ldots, n$. The two decks are shuffled so that the order of cards in each deck is uniformly distributed and independent of the order in the other deck. We say a match occurs at position i if the ith cards from each deck have the same number. Let S_n denote the total number of matches.

(i) Find $\mathbb{E}S_n$. Find $\mathrm{var}(S_n)$.

(ii) Show that for any non-negative random variable X with finite variance,

$$\mathbb{P}(X > 0) \geqslant \frac{(\mathbb{E}X)^2}{\mathbb{E}(X^2)} .$$

(iii) Show that

$$\mathbb{P}(S_n > 0) \geqslant \frac{1}{2} .$$

[You may use standard inequalities from lectures if you state them clearly.]

SECTION II

5C Differential Equations

Consider the second order differential equation

$$y'' + (\alpha - x^2)y = 0,$$

where α is a real positive constant parameter.

(i) Find the value of the parameter α such that

$$y_0(x) = e^{-x^2/2}$$

is a solution of the equation.

(ii) Substitute

$$y(x) = e^{-x^2/2} f(x),$$

and find a second order differential equation for f.

(iii) Find a recurrence relation between the coefficients of the power series solutions for $f(x)$

$$f(x) = \sum_{n=0}^{\infty} a_n x^n.$$

(iv) Impose the initial condition $y'(0) = 0$. For what values of α is there a solution with $f(x)$ a polynomial of degree $2N$?

(v) Hence, or otherwise, find the solution of

$$y'' + (9 - x^2)y = 2e^{-x^2/2},$$

satisfying $y'(0) = 0$ and $y(0) = 1$.

6C Differential Equations

Consider the function

$$U(x,y) = x^3 - 3x + xy^2 + y^2.$$

(i) Find the stationary points of $U(x,y)$ and identify the local minimum. Sketch the contours $U(x,y) = \text{constant}$.

(ii) A particle with position $(x(t), y(t))$ at time t moves with velocity

$$(\dot{x}(t), \dot{y}(t)) = -\nabla U(x,y).$$

Show that $U(x(t), y(t))$ is non-increasing function of t along any solution trajectory. If $x = 2$ and $y = 1$ at $t = 0$, what happens to the solution as $t \to \infty$? [Justify your answer with reference to your sketched contours.]

(iii) Find the particle's trajectory $y(x)$ in the vicinity of the local minimum found in part (i).

(iv) Consider the modified system where a periodic external force is applied along the y direction

$$\frac{dx}{dt} = -\frac{\partial U(x,y)}{\partial x}$$
$$\frac{dy}{dt} = -\frac{\partial U(x,y)}{\partial y} + \cos(\omega t),$$

with ω a real constant. Show that $U(x(t), y(t))$ is no longer necessarily non-increasing. Find how the solutions $(x(t), y(t))$ change in the vicinity of the minimum due to the periodic forcing term.

7C Differential Equations

Let $u(x,t)$ be a real-valued function satisfying the following equation for x and $t \in \mathbb{R}$:

$$\frac{\partial^2 u}{\partial t^2} + \gamma \frac{\partial u}{\partial t} - c^2 \frac{\partial^2 u}{\partial x^2} + \omega^2 u = f(x,t), \qquad (*)$$

where γ, ω and c are real positive parameters and $f(x,t)$ is a real function.

(a) For $f(x,t) = f(t)$ consider spatially uniform solutions of $(*)$ of the form $u(x,t) = y(t)$.

(i) Show that y satisfies

$$\ddot{y} + \gamma \dot{y} + \omega^2 y = f(t).$$

(ii) Assume $f(t) = 0$. Write down the spatially uniform solutions $y(t)$ and identify the underdamped, overdamped and critically damped solutions according to the values of γ and ω. State the long-time behaviour of the solutions $y(t)$.

(iii) Consider the case of light damping in which $\gamma = \sqrt{3}\omega$ and let

$$f(t) = I\delta\left(t - \frac{\pi}{2\omega}\right), \qquad I > 0,$$

where δ is the Dirac δ-function. Find a particular solution of the equation for $t \in [0, \pi/\omega]$ such that $y = 0$ at $t = 0$ and $t = \pi/\omega$.

(b) For $f(x,t) = 0$ consider solutions of $(*)$ of the form $u(x,t) = \phi(\xi)$, where $\xi = x - X(t)$ with $X(t)$ a smooth real function of t, such that $\phi(\xi) \to 0$ and $\phi'(\xi) \to 0$ as $|\xi| \to \infty$.

Derive the equation that such solutions satisfy. Deduce that either $\phi \equiv 0$ (trivial solution), or $\ddot{X} + \gamma \dot{X} = 0$ for all t. In the second case, solve for $X(t)$ and describe the long-time behaviour of the solution as $t \to \infty$.

8C Differential Equations

(i) A bacterial culture is inside a test tube at room temperature θ_0. The tube is placed in a hot incubator and kept there for a time T, then removed and placed on a bench at room temperature for the same time T.

The culture temperature $\theta(t)$ changes at a rate proportional to the temperature difference between the culture and its surroundings, with proportionality constant $\alpha > 0$. The surroundings are at temperature $\theta_1 > \theta_0$ (inside the incubator) during warming and θ_0 (room temperature) during cooling.

Write down the differential equations for $\theta(t)$ during the warming and cooling phases, and find the unique solutions using the initial conditions.

(ii) The bacteria have an enzyme and the concentration of the enzyme $N(t)$ decreases during the warming and cooling process at a rate proportional to its current concentration and a temperature-dependent denaturation rate

$$\beta(\theta(t)) = \beta_{\max} \frac{\theta(t) - \theta_0}{\theta_1 - \theta_0},$$

with $\beta_{\max}$ a positive constant.

Find an implicit equation for the time T required so that the concentration of enzyme is reduced by a factor of 100 by the end of the entire warming and cooling cycle.

(iii) Show that for small values of α such that $\alpha T \ll 1$, the required time is proportional to $1/\sqrt{\alpha}$, and find the constant of proportionality in terms of $\beta_{\max}$.

9F Probability

A *graph* on a set V is a set of some unordered pairs of (distinct) elements of V: we call these the *edges* of the graph and the elements of V are called the *vertices*. A *random graph* with n vertices $v_1, v_2, \ldots, v_n$ is drawn by adding an edge with probability p between v_i and v_j for all $i \neq j$ independently.

A triangle is a set of three distinct vertices v_i, v_j, v_k that are all connected to each other. Let T be the number of triangles in this random graph.

(i) Find the expectation $\mathbb{E}(T)$.

(ii) Let $p = 1/n^{\alpha}$. Show that if $\alpha > 1$, then $\mathbb{P}(T = 0) \to 1$ as $n \to \infty$.

(iii) Show that if $p \in (0, 1)$ is such that $\text{var}(T)/\mathbb{E}(T)^2 \to 0$ as $n \to \infty$, then $\mathbb{P}(T = 0) \to 0$ as $n \to \infty$.

(iv) Now let $p = 1/n^{\alpha}$ with $0 < \alpha < 1$. Show that $\mathbb{P}(T = 0) \to 0$ as $n \to \infty$.

[You may use standard inequalities from lectures if you state them clearly.]

10F Probability

Consider a biased coin whose probability of head H is p and tail T is $1-p$ for some $p \in (0,1)$. Successive tosses of the coin are independent.

(i) What is the expected number of tosses to get a head H?

(ii) What is the expected number of tosses to get two consecutive heads HH?

(iii) What is the expected number of tosses to get the sequence HT?

(iv) What is the expected number of tosses to get n consecutive heads?

(v) Suppose the coin is tossed repeatedly until the total number of heads is n. Let $p_k(n)$ denote the probability that k tosses are required. Find $p_k(n)$.

For any real numbers $-\infty < a < b < \infty$, find integers $k_a(n), k_b(n)$ such that

$$\sum_{k=k_a(n)}^{k_b(n)} p_k(n) \to \frac{1}{\sqrt{2\pi}} \int_a^b e^{-x^2/2} dx\,, \quad \text{as } n \to \infty.$$

[You may quote standard results from lectures. You may use that the variance of a Geometric(p) random variable is $(1-p)/p^2$.]

11F Probability

(a) Let $(S_n : n \geqslant 0)$ be a simple symmetric random walk on $\mathbb{Z}$ with $S_0 = 0$, that is defined as

$$\mathbb{P}(S_n - S_{n-1} = 1) = \mathbb{P}(S_n - S_{n-1} = -1) = \frac{1}{2} \text{ for all } n \geqslant 1\,.$$

For any $a \in \mathbb{Z}, a \geqslant 1$, find the probability that (S_n) hits a before hitting -1. Let T denote the time at which (S_n) first hits -1 or a, i.e.,

$$T = \min\{n \geqslant 0 : S_n \in \{-1, a\}\}\,.$$

Find $\mathbb{E}(T)$.

(b) Now, for $n \geqslant 2$, let $\mathbb{Z}_n$ be the circle graph consisting of the vertices $0, 1, \ldots, n-1$ and edges between vertices $k, k+1$ for $k = 0, 1, \ldots, n-2$ and an edge between $n-1$ and 0. Let $(Y_n : n \geqslant 0)$ be a simple symmetric random walk on $\mathbb{Z}_n$ with $Y_0 = 0$.

(i) Let T_k denote the first time the random walk (Y_n) has hit k distinct vertices, for $k = 1, 2, \ldots, n$. Clearly $T_1 = 0$. Find $\mathbb{E}(T_{k+1} - T_k)$ for $k = 1, 2, \ldots, n-1$. Find $\mathbb{E}(T_n)$, where T_n is the first time the random walk has hit all the vertices.

(ii) Let Z denote the last new vertex to be hit by the random walk (Y_n). Find the distribution of Z.

[You may use the following facts: Conditional on $Y_{T_k} = i$, $(Y_{T_k+n} : n \geqslant 0)$ is again distributed as a simple symmetric random walk starting at i. Similarly, if τ_i is the first time (Y_n) hits i, $(Y_{\tau_i+n} : n \geqslant 0)$ is distributed as a simple symmetric random walk starting at i.]

12F Probability

In a branching process, the probability that an individual has exactly k offspring is given by p_k, for $k = 0, 1, 2, \ldots$, and all individuals reproduce independently. Let X_n denote the size of the nth generation. Assume that $X_0 = 1$ and $p_0 > 0, p_2 > 0$ and let $\mu := \mathbb{E}X_1$. Let $F_n(s)$ be the generating function of X_n. Thus

$$F_1(s) = \mathbb{E}(s^{X_1}) = \sum_{k=0}^{\infty} p_k s^k, \quad |s| \leqslant 1\,.$$

(i) Show that $F_{n+1}(s) = F_n(F_1(s))$.

(ii) Compute $\mathbb{E}X_n$.

(iii) Show that if $\mu \leqslant 1$, the branching process eventually becomes extinct with probability 1.

(iv) Now let $p_k = 2^{-k-1}$ for $k = 0, 1, 2 \ldots$. Show that the branching process becomes extinct with probability 1.

Find $F_n(s)$. Compute the probability that the nth generation is empty.

END OF PAPER