UNIVERSITY OF CAMBRIDGE

MAT0

MATHEMATICAL TRIPOS Part IA

Thursday 4 June 2026 9:00am to 12:00pm

PAPER 1

Before you begin read these instructions carefully

The examination paper is divided into two sections. Each question in Section II carries twice the number of marks of each question in Section I. Section II questions also carry an alpha or beta quality mark and Section I questions carry a beta quality mark.

Candidates may obtain credit from attempts on **all four** questions from Section I and **at most five** questions from Section II. Of the Section II questions, no more than three may be on the same course.

Write on **one side** of the paper only and begin each answer on a separate sheet.

Write legibly; otherwise you place yourself at a grave disadvantage.

At the end of the examination:

Separate your answers to each question.

Complete a gold cover sheet **for each question** that you have attempted, and place it at the front of your answer to that question.

Complete a green main cover sheet listing **all the questions** that you have attempted.

Every cover sheet must also show your Blind Grade Number and desk number.

Tie up your answers and cover sheets into **a single bundle**, with the main cover sheet on the top, and then the cover sheet and answer for each question, in the numerical order of the questions.

STATIONERY REQUIREMENTS
Gold cover sheets
Green main cover sheet
Treasury tag

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

SECTION I

1A Vectors and Matrices

(a) For $z \in \mathbb{C}$ define the principal value of $\log z$.

(b) Find the principal value of $\log(1+i)$ and all other possible values.

(c) Find all solutions $z \in \mathbb{C}$ of $(z-i)^5 = 2$ and sketch them on an Argand diagram.

2C Vectors and Matrices

The matrix

$$A = \begin{pmatrix} 1 & -1 & 0 \\ 2 & 2 & 1 \end{pmatrix}$$

represents a linear map $\Phi : \mathbb{R}^3 \to \mathbb{R}^2$ with respect to the bases

$$B = \left\{ \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \right\} \qquad C = \left\{ \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\}$$

Find the matrix A' that represents Φ with respect to the bases

$$B' = \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\} \qquad C' = \left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \end{pmatrix} \right\}$$

3E Analysis I

(a) Define the *radius of convergence* R for both real and complex power series.

(b) Consider the complex power series $\sum_{n=0}^{\infty} a_n z^n$ defined by coefficients a_n given by $a_{2n} = (-1)^n,\ a_{2n+1} = 0$. Show that

(i) $\frac{1}{(z-i)(z+i)} = \sum_{n=0}^{\infty} a_n z^n$ for $|z| < 1$,

(ii) $\sup_{|z|<1} |\sum_{n=0}^{\infty} a_n z^n| = \infty$.

What is the radius of convergence R of this series?

(c) Suppose $\sum_{n=0}^{\infty} b_n x^n$ is a *real* power series with radius of convergence $R = 1$. Must $\sup_{x \in (-1,1)} |\sum_{n=0}^{\infty} b_n x^n| = \infty$?

4E Analysis I

(a) State the Bolzano–Weierstrass theorem for sequences in $\mathbb{R}$.

(b) Let $a_n \in \mathbb{C}$ be a sequence. State what it means for the sequence to *converge* to some $a \in \mathbb{C}$.

(c) Let $a_n \in \mathbb{C}$ be a sequence. Show that either there is a subsequence a_{n_k} which converges to some $a \in \mathbb{C}$, or else a subsequence such that $|a_{n_k}| \geqslant k^2$. [You may use the Bolzano–Weierstrass theorem for sequences in $\mathbb{R}$ from (a) without proof.] Are these two alternatives exclusive?

SECTION II

5A Vectors and Matrices

(a) Define the *scalar product* of two vectors in $\mathbb{R}^n$ and the *norm* of a vector in $\mathbb{R}^n$.

(b) State and prove the *Cauchy–Schwarz inequality* for vectors in $\mathbb{R}^n$.

(c) Consider a plane P in $\mathbb{R}^3$ containing points $\mathbf{a} = (1,2,0)$, $\mathbf{b} = (1,1,0)$ and $\mathbf{c} = (0,1,1)$. Write an equation for P in the form $\mathbf{r} \cdot \hat{\mathbf{n}} = \beta$ where $\hat{\mathbf{n}}$ is a unit vector and $\beta \in \mathbb{R}$. [You should determine $\hat{\mathbf{n}}$ and β.]

(d) Find the shortest distance between the plane P and the point $\mathbf{d} = (2,0,3)$.

(e) Consider a sphere with equation $|\mathbf{r} - \mathbf{f}| = 2\sqrt{2}$, where $\mathbf{f} = (g,0,0)$ and g is a real constant. Find a condition on g for the sphere to intersect with the plane P. Describe this intersection geometrically, specifying how it depends on the value of g.

6C Vectors and Matrices

(a) Let A be a 3×3 matrix satisfying

$$A^3 - 4A^2 + 5A - 2I = 0,$$

where I denotes the identity matrix.

(i) Show that A is invertible.

(ii) Express A^{-1} as a polynomial in A.

(iii) Show that every power A^k ($k \geqslant 3$) can be written as a linear combination of I, A and A^2.

(b) Let A and B be similar $n \times n$ matrices so that $B = S^{-1}AS$ for some invertible matrix S.

(i) Show that $\ker B = S^{-1}(\ker A)$ and deduce that $\dim(\ker A) = \dim(\ker B)$.

(ii) State the rank-nullity theorem and use it to prove that $\operatorname{rank}(A) = \operatorname{rank}(B)$

(iii) Show that

$$\operatorname{rank}(A^k) = \operatorname{rank}(B^k) \quad \text{for all } k \geqslant 1.$$

(c) State the Cayley–Hamilton theorem for an $n \times n$ matrix B.
Let B satisfy $B^k = 0$ for some $k > n$. Use the Cayley–Hamilton theorem to obtain an expression for B^n in terms of $I, B, \ldots, B^{n-1}$ and deduce that $B^n = 0$.

7B Vectors and Matrices

Let H be an $n \times n$ Hermitian matrix with distinct eigenvalues μ_i, $i = 1, \ldots, n$.

(a) Show that $\mu_i \in \mathbb{R}$ and that the eigenvectors are orthogonal. Explain how to construct a unitary matrix U such that $H = U^\dagger \Lambda U$ where Λ is diagonal.

(b) Define the characteristic polynomial $\chi_H(\lambda)$ of H and give an expression for it in terms of the eigenvalues μ_i.

(c) The complex matrix M has components $M_{ij} = \delta_{ij} - x_i y_j^*$ where $x_i, y_i \in \mathbb{C}$ are components of normalised vectors $\mathbf{x}^* \cdot \mathbf{x} = \mathbf{y}^* \cdot \mathbf{y} = 1$. By considering the eigenvalues of M, or otherwise, show that

$$\det M = 1 - \mathbf{y}^* \cdot \mathbf{x} \ .$$

(d) The real matrix P has components $P_{ij} = \delta_{ij} - u_i u_j$ where $u_i \in \mathbb{R}$ are components of a normalised vector $\mathbf{u}$ with $\mathbf{u} \cdot \mathbf{u} = 1$. Show that, for $\lambda \neq \mu_i$,

$$\chi_{HP}(\lambda) = \chi_H(\lambda)\, f(\lambda) \quad \text{with} \quad f(\lambda) = 1 - \mathbf{u} \cdot (H - \lambda I)^{-1} H \mathbf{u} \ ,$$

where I denotes the identity matrix.

(e) Show that $\mathbf{u}$ is an eigenvector of HP with eigenvalue zero.

Denote the remaining eigenvalues of HP as σ_i, with $i = 1, \ldots, n-1$, ordered such that $\sigma_i < \sigma_{i+1}$. Take $H = \Lambda = \mathrm{diag}(\mu_1, \ldots, \mu_n)$ the diagonal matrix with eigenvalues ordered such that $\mu_i < \mu_{i+1}$.

Show that the eigenvalues σ_i of HP interlace the eigenvalues of H, such that

$$\mu_1 \leqslant \sigma_1 \leqslant \mu_2 \leqslant \sigma_2 \leqslant \ldots \leqslant \sigma_{n-1} \leqslant \mu_n \ .$$

8B Vectors and Matrices

A collection of three 2×2 matrices σ_a, with $a = 1, 2, 3$, are defined by

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \ , \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \ , \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \ .$$

(a) Which of the σ_a are Hermitian? Show that

$$\sigma_a \sigma_b = \delta_{ab} I + i \varepsilon_{abc} \sigma_c \ ,$$

where I denotes the identity matrix.

(b) The exponential of a matrix M is defined by the Taylor expansion

$$e^M = \sum_{k=0}^{\infty} \frac{M^k}{k!} \ .$$

Let $n_a \in \mathbb{R}$, with $a = 1, 2, 3$, be the components of a unit three-dimensional vector $\mathbf{n}$, i.e. $n_a n_a = 1$. Introduce the 2×2 matrix $\mathbf{n} \cdot \boldsymbol{\sigma} = n_1 \sigma_1 + n_2 \sigma_2 + n_3 \sigma_3$.

Compute $R = e^{i\theta\, \mathbf{n} \cdot \boldsymbol{\sigma}}$ for $\theta \in \mathbb{R}$. Show that R is unitary.

(c) Let $x_a \in \mathbb{R}$ be the components of a general three-dimensional vector $\mathbf{x}$, and $A = \mathbf{x} \cdot \boldsymbol{\sigma}$. Compute the determinant and trace of A.

Let

$$A' = R A R^{\dagger} \ .$$

Show that A' is Hermitian, and that $\det A' = \det A$ and $\operatorname{Tr} A' = \operatorname{Tr} A$. Hence argue that $A' = \mathbf{x}' \cdot \boldsymbol{\sigma}$ for some $\mathbf{x}'$ with $\mathbf{x}' \cdot \mathbf{x}' = \mathbf{x} \cdot \mathbf{x}$.

(d) Compute the components x'_a of $\mathbf{x}'$ in terms of x_a, n_a and θ.

[*Note: Index summation convention is used throughout this question.*]

9E Analysis I

(a) We say that a function $f : \mathbb{R} \to \mathbb{R}$ is *real analytic* if for all $x_0 \in \mathbb{R}$, there exists an $\epsilon > 0$, and a coefficient sequence $a_n \in \mathbb{R}$, $n = 0, 1, \ldots$ such that for all $x \in (-\epsilon, \epsilon)$, the function $f(x_0 + x)$ may be represented as the convergent series

$$f(x_0 + x) = \sum_{n=0}^{\infty} a_n x^n. \qquad (*)$$

Show that if $f(x) = P(x) : \mathbb{R} \to \mathbb{R}$ is a polynomial, then $f(x)$ is real analytic.

(b) State Taylor's theorem. [You may use any of the forms for the remainder term from the course.]

(c) Show that if $f : \mathbb{R} \to \mathbb{R}$ is infinitely differentiable and if for every $-\infty < a < b < \infty$, there exists a $C(a, b) > 0$ such that $|f^{(n)}(x)| \leqslant C^{n+1} n!$ for all $n \geqslant 0$ and all $x \in [a, b]$, then it follows that f is real analytic. [Hint: *Apply Taylor's theorem in sufficiently small intervals* $(x_0 - \epsilon, x_0 + \epsilon)$ *to show that* $f(x_0 + x)$ *may be written as* $(*)$ *with* $a_n = \frac{1}{n!} f^{(n)}(x_0)$.]

(d) Give an example of a real analytic function $f : \mathbb{R} \to \mathbb{R}$ which is not a polynomial. [Justify your answer.]

10E Analysis I

(a) State and prove Rolle's theorem for real valued functions $f : [a, b] \to \mathbb{R}$. [Be sure to state carefully the precise differentiability assumptions necessary.]

(b) Infer from the above that if $f : [a, b] \to \mathbb{R}$ is continuous and $f'(x)$ is defined and strictly positive for all $x \in (a, b)$, then $f(b) > f(a)$.

(c) Let $f : \mathbb{R} \to \mathbb{R}$ be a twice continuously differentiable function such that $f(0) = 0$, $f'(0) = 1$, $f''(0) = 0$, and such that $f'''(x)$ exists for all $x > 0$ and is strictly negative. Show that there must exist a unique $x_0 > 0$ such that $f(x_0) = 0$ and $f(x) > 0$ for all $x \in (0, x_0)$.

(d) Give the standard definitions of $\sin x$ and $\cos x$ in terms of power series. [In what follows, you may assume without proof that these series converge for all $x \in \mathbb{R}$ and are differentiable, that $\sin' x = \cos x$, $\cos' x = -\sin x$ and that the trigonometric identities $\cos^2 x + \sin^2 x = 1$, $\sin(x + y) = \sin x \cos y + \sin y \cos x$ hold.] Show that there must exist an $x_0 \in \mathbb{R}$, $x_0 > 0$ such that $\sin(x_0) = 0$, $\sin(x) > 0$ for all $0 < x < x_0$.

[Note that the number x_0 is usually denoted as π.]

(e) Show that $\sin(x + 2x_1) = \sin x$ for all $x \in \mathbb{R}$ if and only if $x_1 = nx_0$ for some $n \in \mathbb{Z}$, with x_0 $(= \pi)$ the unique number whose existence was shown in (d).

11E Analysis I

(a) State the Fundamental Theorem of Calculus (both parts, concerning derivatives of integrals and integrals of derivatives).

(b) Let $f : [0, a) \to \mathbb{R}$ be continuous and satisfy

$$|f(x)| \leqslant |f(0)| + \int_0^x g(\tilde{x})|f(\tilde{x})|d\tilde{x}$$

for some nonnegative continuous function $g : [0, a) \to \mathbb{R}$. Show that

$$|f(x)| \leqslant |f(0)| \exp \int_0^x g(\tilde{x})d\tilde{x}.$$

If $f(0) = 0$, deduce that $f(x) = 0$ for all $0 \leqslant x < a$. [You may use standard properties of the exponential function $\exp : \mathbb{R} \to \mathbb{R}_+$ proven in the course, and the fact that it admits a continuously differentiable inverse, denoted $\log : \mathbb{R}_+ \to \mathbb{R}$, with $(\log x)' = 1/x$.]

(c) Let f_1, f_2 be two continuously differentiable solutions to the equation

$$f'(x) = (1 + \sqrt{x})f^2(x)$$

on some interval $[0, a)$, with $f_1(0) = f_2(0)$. Deduce that $f_1 = f_2$ identically on $[0, a)$. [Hint: *Consider the differential equation satisfied by the difference.*]

12E Analysis I

For each of the statements below, determine whether it is true or false, justifying appropriately your answer with a proof or counterexample. You may refer to standard results in the course.

(a) The function $f : [0,1] \to \mathbb{R}$ defined by $f(x) = 0$ if $x \in \mathbb{Q} \cap [0,1]$, $f(x) = 1$ if $x \notin \mathbb{Q} \cap [0,1]$ is Riemann integrable.

(b) Suppose $f : [a,b] \to \mathbb{R}$ is Riemann integrable. Then f is continuous at all but finitely many points of $[a,b]$.

(c) Suppose $f_i : [a,b] \to \mathbb{R}$, $i = 1, \ldots,$ is a sequence of Riemann integrable functions such that $\lim_{i\to\infty} f_i(x)$ exists for all $x \in [a,b]$. Define $f(x) = \lim_{i\to\infty} f_i(x)$. Then $f(x)$ is Riemann integrable.

(d) If $f : [a,b] \to \mathbb{R}$ is Riemann integrable, $f \geqslant 0$ and $\int_a^b f(x)dx = 0$, then $f = 0$ identically.

(e) Let $F : [a,b] \to \mathbb{R}$ be differentiable at all but countably many points and suppose that F' extends to a Riemann integrable function $f : [a,b] \to \mathbb{R}$, i.e. there exists a Riemann integrable $f : [a,b] \to \mathbb{R}$ such that, *for all x such that $F'(x)$ is defined*, we have $f(x) = F'(x)$. Then

$$\int_a^b f(x)dx = F(b) - F(a).$$

(f) If $f : (a,b) \to \mathbb{R}$ is continuous and there exists a constant $0 < C < \infty$ such that $\int_{\tilde{a}}^{\tilde{b}} |f|^p dx \leqslant C^p$ for all $a < \tilde{a} < \tilde{b} < b$ and all integers $p \geqslant 1$, then $\sup_{(a,b)} |f(x)| \leqslant C$.

END OF PAPER