UNIVERSITY OF CAMBRIDGE

MAT2

MATHEMATICAL TRIPOS Part II

Friday 12 June 2026 9:00am to 12:00pm

PAPER 4

Before you begin read these instructions carefully.

The examination paper is divided into two sections. Each question in Section II carries twice the number of marks of each question in Section I. Section II questions also carry an alpha or beta quality mark and Section I questions carry a beta quality mark.

Candidates may obtain credit from attempts on **at most six questions** from Section I and from any number of questions from Section II.

Write on **one side** of the paper only and begin each answer on a separate sheet.

Write legibly; otherwise you place yourself at a grave disadvantage.

At the end of the examination:

Separate your answers to each question.

Complete a gold cover sheet **for each question** that you have attempted, and place it at the front of your answer to that question.

Complete a green main cover sheet listing **all the questions** that you have attempted.

Every cover sheet must also show your Blind Grade Number and desk number.

Tie up your answers and cover sheets into **a single bundle**, with the main cover sheet on the top, and then the cover sheet and answer for each question, in the numerical order of the questions.

STATIONERY REQUIREMENTS
Gold cover sheets
Green main cover sheet
Script paper
Rough paper

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

SECTION I

1G Number Theory

Define what we mean by a *Carmichael number*.

Show that if N is a Carmichael number and p is an odd prime such that $p \mid N$, then $p-1 \mid N-1$.

Show that if N is a Carmichael number, then N is square-free.

Deduce that if N is a Carmichael number and p is an odd prime such that $p \mid N$, then $p \leqslant \sqrt{N}$.

2I Topics in Analysis

(i) Let $t_1, t_2, \ldots, t_n \in [-1,1]$ be any set of n distinct numbers. Show that there exist unique numbers $A_1, A_2, \ldots, A_n$ such that

$$\int_{-1}^{1} p(t)dt = \sum_{j=1}^{n} A_j p(t_j)$$

holds for every polynomial p of degree $n-1$ or less.

(ii) For $n = 1, 2, \ldots$, let p_n be the Legendre polynomial of degree n. Suppose that $t_1, t_2, \ldots, t_n \in [-1,1]$ are the roots of p_n, and $A_1, A_2, \ldots, A_n$ are the numbers corresponding to p_n as in part (i). Prove that the integration formula in part (i) is now valid for any polynomial of degree $2n-1$ or less.

[You may assume without proof that p_n has n distinct roots in $[-1,1]$.]

(iii) Let $t_1, t_2, \ldots, t_n$ and $A_1, A_2, \ldots, A_n$ be as in part (ii). Show that $A_j > 0$ for each $j = 1, 2, \ldots, n$ and that $\sum_{j=1}^{n} A_j = 2$. Show further that if p is any polynomial of degree $2n-1$ or less, then

$$\left|\sum_{j=1}^{n} A_j p(t_j) - \int_{-1}^{1} f(t)dt\right| \leqslant 4 \sup_{t\in[-1,1]} |f(t) - p(t)|.$$

holds for all continuous functions f.

3H Coding & Cryptography

Describe the Rabin public encryption scheme. [You should describe both encryption and decryption.]

Suppose Alice chooses to use the Rabin scheme with $N = 2773$. To be sure I am in contact with Alice, I send her the encrypted message $r \equiv 25 \bmod 2773$. On receiving r, Alice uses her knowledge of N and returns $2355 \bmod 2773$. Show that I can now decrypt all messages sent to Alice.

4I Automata and Formal Languages

If $f, g\colon \mathbb{N} \to \{0,1\}$, define $f+g\colon \mathbb{N} \to \{0,1\}$ to be their pointwise sum modulo 2.

(a) Show that if f and g are computable, then so is $f+g$.

For a function $f\colon \mathbb{N} \to \{0,1\}$, let $X_f = \{n \in \mathbb{N} : f(n) = 1\}$.

(b) If X_f and X_g are both recursively enumerable, must X_{f+g} be recursively enumerable? Justify your claim.

Let Σ be any finite alphabet and $\beta\colon \Sigma^* \to \mathbb{N}$ be any bijection. We say that a function $f\colon \mathbb{N} \to \{0,1\}$ is *β-context free* if $\beta^{-1}(X_f)$ is a context free language.

(c) If f and g are both β-context free, must $f+g$ be β-context free? Justify your claim.

[*You may use results and examples from the lectures without proof, provided that you state them clearly.*]

5K Statistical Modelling

Let $\{f(y;\theta) \mid \theta \in \Theta \subseteq \mathbb{R}\}$ be an exponential family with natural parameter θ. Define the corresponding generalised linear model for the data $(X_i, Y_i) \in \mathbb{R}^p \times \mathbb{R}$, $i = 1, \ldots, n$ with parameter vector $\beta = (\beta_1, \ldots, \beta_p) \in \mathbb{R}^p$.

Find the log-likelihood function and the normal equations that the maximum likelihood estimator $\hat{\beta} \in \mathbb{R}^p$ must satisfy. What is the geometric interpretation of the normal equations?

Describe a two-sided hypothesis test for $H_0 : \beta_1 = 0$ versus $H_1 : \beta_1 \neq 0$.

6C Mathematical Biology

A delay model for a population N_t is given by

$$N_{t+1} = N_t \exp[r(1 - N_{t-1}/K)]\,,$$

where t is discrete time and $r, K > 0$.

Show that there is a stable, non-zero fixed point for $0 < r < 1$.

Show that at $r = 1$ the steady state bifurcates to a periodic solution of period 6.

7D Further Complex Methods

State the identity theorem for analytic functions.

Define what is meant by the *analytic continuation* of a function.

Consider

$$F(z) = \int_{-\infty}^{\infty} \frac{e^{-5it}}{t^2 - tz + ti - iz} dt,$$

defined for $\operatorname{Im} z < 0$. Why is $F(z)$ analytic for $\operatorname{Im} z < 0$? Construct an analytic continuation of $F(z)$ to $\operatorname{Im} z > 0$ and show that

$$\widetilde{F}(z) = \int_{-\infty}^{\infty} \frac{e^{-5it}}{t^2 - tz + ti - iz} dt, \quad \operatorname{Im} z \neq 0$$

cannot be such an analytic continuation.

Let

$$D = \{z \in \mathbb{C} : \operatorname{Re} z > 0\}$$

and define a sequence $z_n = 1/n$.

Suppose G is analytic on D and satisfies $G(z_n) = 0$ for all $n \in \mathbb{N}$.

Does this information determine G uniquely on D? Justify your answer.

8A Classical Dynamics

Consider Hamiltonian dynamics with a time-independent Hamiltonian $H(q^a, p_b)$ on phase space $\mathbb{R}^{2n}$.

(a) Prove that energy is conserved in this system. Show that the Hamiltonian vector field V_H associated to H has vanishing divergence.

(b) Let $\Phi^t : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ denote Hamiltonian flow, so that a system whose initial positions and momenta lie in some region U will be found in the region $\Phi^t(U)$ after time t. Show that the regions $\Phi^t(U)$ have the same phase space volume for all t. [You may assume without proof that the Jacobian matrix $J(t)$ of the coordinate transformation $\Phi^t : (q^a(0), p_b(0)) \mapsto (q^a(t), p_b(t))$ obeys $J^{-1}\partial_t J = \operatorname{div}(V_H) 1_{2n\times 2n}$.]

(c) A classical system is governed by the Hamiltonian $H = \frac{1}{2}(p \cdot p + \cosh(q \cdot q) - 1)$. The system's initial conditions are such that its energy E lies in the range $0 \leqslant E \leqslant 1/2$. Suppose the initial region U is small enough that the system is definitely outside U after some time T. Show that there must exist integers $n > m$ such that

$$\Phi^{nT}(U) \cap \Phi^{mT}(U) \neq \emptyset\,.$$

Explain why this implies that there are initial conditions arbitrarily close to $(q^a(0), p_b(0))$ where the system returns arbitrarily close to its starting point.

9D Cosmology

A fermion species X of mass m_X is in thermal equilibrium in the early universe with number density given by

$$n_X = \frac{4\pi}{h^3}\int_0^\infty \frac{p^2\,dp}{\exp[\mathcal{E}(p)/(kT)]+1},$$

where $\mathcal{E}(p) = \sqrt{p^2c^2 + m_X^2c^4}$. Assume the species remains in equilibrium until it becomes non-relativistic.

(a) In the limit $kT \ll m_X\,c^2$, expand $\mathcal{E}(p)$ to leading order, showing that the integral reduces to a Gaussian and obtain the number density,

$$n_X = \left(\frac{2\pi\, m_X\, kT}{h^2}\right)^{3/2} \exp\bigl[-m_X\, c^2/(kT)\bigr]\,.$$

[*Hint: You may assume* $\int_0^\infty e^{-\lambda x^2}dx = \frac{1}{2}\sqrt{\pi/\lambda}$ *for* $\lambda > 0$.]

(b) The fermion X decouples at a temperature $kT_{\rm d} = m_X\, c^2/\alpha$ with $\alpha \gg 1$. Using the photon number density at the same temperature, $n_\gamma = 16\pi\,\zeta(3)(kT/hc)^3$, show that the fermion-to-photon ratio at $T = T_{\rm d}$ is

$$r(\alpha) \equiv \frac{n_X}{n_\gamma} = \frac{\sqrt{2\pi}}{8\,\zeta(3)}\,\alpha^{3/2}\,e^{-\alpha}\,.$$

(c) Assume no significant entropy production after decoupling so that r remains constant. Consider whether the fermion X could be a viable cold dark matter candidate by comparing its mass density ρ_X to the baryon mass density $\rho_B = m_p\, n_B$, where m_p is the proton mass. You may assume that the baryon-to-photon ratio is $\eta = n_B/n_\gamma \simeq 6\times10^{-10}$ and that the measured density parameters today for baryons and dark matter are $\Omega_B \simeq 0.05$ and $\Omega_{DM} \simeq 0.25$ respectively. Hence, or otherwise, determine that for fixed α the required mass for this to be a dark matter fermion is

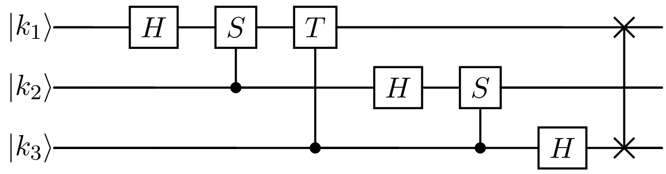
$$m_X \simeq \frac{\mu}{r(\alpha)}\, m_p\,,$$

where you should specify the numerical constant μ.

10E Quantum Information and Computation

(a) Let QFT_N denote the quantum Fourier transform over $\mathbb{Z}_N = \{0, 1, \ldots, N-1\}$. How does QFT_N act on a state $|k\rangle$, where $k \in \{0, \ldots, N-1\}$?

(b) Consider the following 3-qubit circuit



where

$$H = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}, \qquad S = \begin{bmatrix} 1 & 0 \\ 0 & e^{i\pi/2} \end{bmatrix}, \qquad T = \begin{bmatrix} 1 & 0 \\ 0 & e^{i\pi/4} \end{bmatrix},$$

and $\times\!\!-\!\!\!-\!\!\times$ in the final step of the circuit denotes the SWAP gate, defined through $\mathrm{SWAP}\,|a\rangle\,|b\rangle = |b\rangle\,|a\rangle$ for all single-qubit states $|a\rangle$ and $|b\rangle$. Evaluate the actions of the circuit on the following states (i) $|000\rangle$ and (ii) $|011\rangle$, and show that these are identical to the actions of QFT_8 on the states $\{|k\rangle\}$, where $|k\rangle = |k_1 k_2 k_3\rangle$.

(c) Let $|\phi\rangle = \frac{1}{\sqrt{A}}\sum_{j=0}^{A-1}|x_0 + jr\rangle$ be a 3-qubit state, with $x_0 \in \mathbb{Z}_8$ and $A = 8/r$ an integer for some $r \in \mathbb{Z}_8$. Show that the amplitudes of $\mathrm{QFT}_8\,|\phi\rangle$ are concentrated on multiples of A. If one measures this state in the basis $\{|0\rangle, |1\rangle, \ldots, |7\rangle\}$, do the measurement outcomes and their probabilities depend on x_0? Justify your answer.

SECTION II

11G Number Theory

Define what it means for a binary quadratic form to be *positive definite*, and what it means for two binary quadratic forms $f = (a, b, c)$ and $g = (a', b', c')$ to be *equivalent*. Define what it means for a positive definite binary quadratic form to be *reduced*.

Show that every positive definite binary quadratic form is equivalent to a reduced form. Show further that if $f = (a, b, c)$ is a reduced positive definite binary quadratic form of discriminant d, then $|b| \leqslant a \leqslant \sqrt{|d|/3}$ and $b \equiv d \mod 2$.

Find congruence conditions for a prime p to be represented by the form $x^2 + 7y^2$. [You should state carefully any further results from lectures that you require.]

12I Topics in Analysis

(a) State Liouville's theorem on approximation of algebraic numbers by rationals.

(b) Consider the continued fraction associated with a positive irrational number α,

$$\alpha = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \dots}}}$$

where a_n, $n = 1, 2, \ldots$, are positive integers and a_0 is a non-negative integer. Assuming that the nth convergent p_n/q_n is determined from

$$\begin{pmatrix} p_n & q_n \end{pmatrix} = \begin{pmatrix} a_n & 1 \end{pmatrix} \begin{pmatrix} a_{n-1} & 1 \\ 1 & 0 \end{pmatrix} \cdots \begin{pmatrix} a_0 & 1 \\ 1 & 0 \end{pmatrix},$$

show that $q_n p_{n-1} - q_{n-1} p_n = (-1)^n$ for all $n \in \{1, 2, 3, \ldots\}$. Show further that

$$\left| \alpha - \frac{p_n}{q_n} \right| \leqslant \frac{1}{q_n q_{n+1}}.$$

Deduce that for each irrational α there are infinitely many fractions k/m (with k, m co-prime integers) such that

$$\text{(i)} \quad \left| \alpha - \frac{k}{m} \right| < \frac{1}{m^2};$$

$$\text{(ii)} \quad \left| \alpha - \frac{k}{m} \right| < \frac{1}{2m^2}.$$

(c) Suppose that an infinite continued fraction is constructed according to the following rule: if p_n/q_n is the n-th convergent, then $a_{n+1} = q_n^n$. Let α be the real number associated with this continued fraction. Prove that α is transcendental.

13K Statistical Modelling

Let $(X_1, Y_1, Z_1), \ldots, (X_n, Y_n, Z_n) \in \mathbb{R}^3$ be n independent and identically distributed samples generated by the R code below. Suppose the R object `X` stores $X_1, \ldots, X_n$, object `Y` stores $Y_1, \ldots, Y_n$, and object `Z` stores $Z_1, \ldots, Z_n$.

(a) The `R` function `cov(X,Y)` returns the sample covariance between `X` and `Y` defined as

$$S_{XY} = \frac{1}{n-1} \sum_{i=1}^{n} (X_i - \bar{X})(Y_i - \bar{Y}),$$

where $\bar{X}$ is the sample mean of $X_1, \ldots, X_n$ and $\bar{Y}$ is similarly defined. Show that this is an unbiased estimator of $\mathsf{Cov}(X_1, Y_1)$.

(b) Consider the following R code and output.

```
> n <- 1000
> Z <- rnorm(n)
> U <- rnorm(n)
> X <- Z + U + rnorm(n)
> Y <- X + U + rnorm(n)
> coef(lm(Y ~ X))
(Intercept)           X
-0.01579827  1.36753704
> cov(X, Y) / cov(X, X)
[1] 1.367537
```

Why is the coefficient of `X` in the linear model identical to the ratio of covariances?

How do you expect the coefficient of `X` in this linear model to behave when n increases? Find the limiting value of this coefficient when $n \to \infty$ in the example above.

(c) Define the *two-stage least squares* estimator of the causal effect of X on Y using Z as the instrumental variable, and show that in the setting above, it can be calculated by `cov(Z, Y) / cov(Z, X)`. What do you expect the two-stage least squares estimator to converge to when $n \to \infty$ in this example? Justify your answer.

14C Mathematical Biology

In a forest the number of trees of height s, at time t, is denoted by $n(s,t)$. Trees grow at a rate $g(s) > 0$. The production rate of seedlings per tree is $b(s)$, and the death rate per tree is $\mu(s)$. The evolution of the forest is described by the equation

$$\frac{\partial n}{\partial t} + \frac{\partial (g\, n)}{\partial s} = -\mu\, n\ .$$

(a) Explain why this equation is subject to the boundary condition

$$g(0)n(0,t) = \int_0^\infty ds\ b(s)\, n(s,t)\,.$$

(b) A separable solution takes the form $n(s,t) = \widetilde{n}(s)e^{rt}$ with r constant.

(i) Derive an expression for $\widetilde{n}(s)$ and show how the boundary condition yields an integral expression for r.

(ii) Suppose that $g(s) = g$ and $\mu(s) = \mu$, both constant, while $b(s) = B$ for $s > s_0$ and zero otherwise, for constants B and s_0. What is the mean number of seedlings per tree? What value of B ensures that the tree population is independent of time?

(iii) Suppose now that g and μ are constant but that trees only produce seedlings when they are a particular height $s^\star$, so that $b(s) = B\,\delta(s - s^\star)$, where δ is the Dirac delta function. What is the value of r?

15A Classical Dynamics

A table tennis bat has principal moments of inertia $I_1 < I_2 < I_3$ about the principal axes in the body frame.

(a) Write down Euler's equations describing torque-free rotation of the bat with angular velocity $\boldsymbol{\omega}$. Show that uniform rotation around any of the principal axes is a solution of the equations of motion.

(b) Show that the kinetic energy $T(\boldsymbol{\omega})$ and magnitude of the angular momentum $\mathbf{L}$ are constants of the motion. Show that steady rotations around the principal axes correspond to points where the surfaces $T(\boldsymbol{\omega}) = \text{const}$ and $\mathbf{L}^2 = \text{const}$ are tangent in $\boldsymbol{\omega}$-space.

(c) Let $\boldsymbol{\omega} = (\omega_1, \omega_2, \omega_3)$ in the principal body frame. Eliminate ω_2 to obtain an equation relating ω_1 and ω_3. Sketch the curves described by this equation in the (ω_1, ω_3) plane and relate your sketches to the three-dimensional geometric picture in part (b). Briefly explain what these curves imply for the motion of the bat if it is perturbed slightly from the steady rotations of part (b).

(d) George throws the table tennis bat so that it has initial angular momentum $\boldsymbol{\omega} = (\epsilon u(t), \Omega, \epsilon v(t))$, with $\epsilon \ll 1$. Determine $u(t)$ and $v(t)$. Is your result consistent with your expectation from part (c)?

16J Logic & Set Theory

Work in ZFC in this question.

(a) Define the following:

(i) a *transitive set*,

(ii) the *transitive closure* $\mathrm{tcl}(x)$ *of a set* x,

(iii) the levels $\mathbf{V}_\alpha$ of the *von Neumann hierarchy*, and

(iv) the *Mirimanoff rank* ϱ.

(b) Prove that for every set x and $\gamma < \varrho(x)$, there is some $y \in x$ such that $\gamma \leqslant \varrho(y)$.

(c) For any set x, show that $\varrho(x) = \varrho(\mathrm{tcl}(x))$.

(d) By induction or otherwise, prove that for every transitive set x and $\gamma < \varrho(x)$, there is some $y \in x$ such that $\gamma = \varrho(y)$.

We call a set x *hereditarily countable* if $\mathrm{tcl}(x)$ is countable and write $\mathbf{HC}$ for the collection of hereditarily countable sets.

(e) Show that $\mathbf{HC} \subseteq \mathbf{V}_{\omega_1}$.

(f) Show that $|\mathbf{HC}| = 2^{\aleph_0}$.

17J Graph Theory

In this question all graphs have at least 3 vertices.

(a) Let G be a graph of order n with minimum degree δ. What does it mean to say that G is *Hamiltonian*? Show that if $\delta \geqslant \frac{n}{2}$, then G is Hamiltonian. For each $n \geqslant 3$, give examples to show that this result does not remain true if we weaken the condition to $\delta \geqslant \frac{n}{2} - 1$ (for n even) or $\delta \geqslant \frac{n-1}{2}$ (for n odd).

(b) Let G be a Hamiltonian graph. Show that G has the property that for all subsets V_0 of $V(G)$, the number of connected components after removing V_0 is at most $|V_0|$.

Is it true that every graph with this property must be Hamiltonian? Give a proof or a counterexample.

(c) Let G be a graph and let $\chi(G)$ be its chromatic number. A *Hamilton path* is a path that visits each vertex of G exactly once. Suppose that G has the property that for every pair of vertices there exists a Hamilton path between the two vertices. Show that $\chi(G) \geqslant 3$.

18F Galois Theory

(a) State Artin's theorem on the invariants of a field L under the action of a finite subgroup of $\mathrm{Aut}(L)$. Deduce that any finite group G arises as the Galois group of some extension of fields.

(b) Define the *elementary symmetric polynomials* $s_1, \ldots, s_n \in \mathbb{Z}[X_1, \ldots, X_n]$. Let $K = \mathbb{C}(s_1, \ldots, s_n) \subset L = \mathbb{C}(X_1, \ldots, X_n)$. Compute the degree $[L : K]$ using Artin's theorem. Show that there exists an element $f \in L$ such that $f^2 \in K$ but $f \notin K$.

(c) Let $M = \mathbb{C}(X, Y)$ and let σ, $\tau \in \mathrm{Aut}(M/\mathbb{C})$ be the automorphisms given by

$$\sigma(X) = \zeta_n X, \qquad \sigma(Y) = \zeta_n^{-1} Y,$$

$$\tau(X) = Y, \qquad \tau(Y) = X,$$

where ζ_n is a primitive nth root of unity. Show that σ and τ generate a group G isomorphic to the dihedral group D_{2n}, and that $M^G = \mathbb{C}(X^n + Y^n, XY)$.

19F Representation Theory

(a) Let H, K be subgroups of a finite group G, V a complex representation of H, and W a complex representation of K.

Suppose $|H|$ and $|K|$ are coprime. Compute the inner product $\langle \operatorname{Ind}_H^G V, \operatorname{Ind}_K^G W \rangle_G$.

(b) Let G be a finite group of odd order. Let $\chi_1, \ldots, \chi_r$ be the characters of the irreducible complex representations of G.

Show that the map $g \mapsto g^2$ is a bijection on G. Let χ be the character of an irreducible complex representation of G, and write $\widetilde{\chi}$ for the function $g \mapsto \chi(g^2)$. Show that

(i) $\langle \widetilde{\chi}, \widetilde{\chi} \rangle = 1$,

(ii) $\widetilde{\chi} = \sum_{i=1}^{r} n_i \chi_i$, for some $n_i \in \mathbb{Z}$.

Deduce that $\widetilde{\chi}_1, \ldots, \widetilde{\chi}_r$ are some permutation of $\chi_1, \ldots, \chi_r$.

20G Number Fields

(a) Show that there are no integers x, y, z, t (not all zero) with

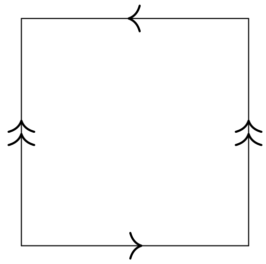
$$x^3 + 7y^3 + 49z^3 - 21xyz = 3t^3.$$

Deduce that if $K = \mathbb{Q}(\sqrt[3]{7})$ then there does not exist $\theta \in K$ with $N_{K/\mathbb{Q}}(\theta) = 3$.

(b) State Dedekind's criterion. Assuming that $\mathcal{O}_K = \mathbb{Z}[\sqrt[3]{7}]$, factor 2, 3, 5 and 7 as products of prime ideals in $\mathcal{O}_K$, and compute the class group of K.

21H Algebraic Topology

Let K be the Klein bottle obtained by identifying the sides of the unit square as shown in the figure, and let $x_0 \in K$ be the image of the corners of the square.



Give a cell complex description for K, and use it to deduce a presentation for $\pi_1(K, x_0)$.

State the Galois correspondence for based covering spaces. Describe based covering spaces $p : (\widetilde{K}, \widetilde{x}_0) \to (K, x_0)$ such that the corresponding subgroup of $\pi_1(K, x_0)$ is isomorphic to (i) $\mathbb{Z}^2$; and (ii) $\mathbb{Z}$.

Given a based covering map $p : (\widetilde{K}, \widetilde{x}_0) \to (K, x_0)$, show that if $\widetilde{K}$ is compact, then $p_* \pi_1(\widetilde{K}, \widetilde{x}_0)$ must have finite index as a subgroup of $\pi_1(K, x_0)$. Is the converse true?

22I Linear Analysis

(a) Define the *dual space* X^* of a real normed space X. Define the *dual operator* T^* of a bounded linear map $T \colon X \to Y$ between normed spaces X and Y, and show that T^* is a bounded linear map.

(b) Let X and Y be normed spaces. Show that if $X \sim Y$, *i.e.* X and Y are isomorphic, then $X^* \sim Y^*$. Show also, that if $X \cong Y$, *i.e.* X and Y are isometrically isomorphic, then $X^* \cong Y^*$.

(c) Let X be a Banach space such that $X^* \cong \ell_2$. Show that $X \cong \ell_2$. [You may use the fact that the canonical embedding $x \mapsto \hat{x} : X \to X^{**}$ is isometric, where $\hat{x}(f) = f(x)$.]

(d) Let K be an infinite compact Hausdorff space. Fix pairwise distinct elements $k_1, k_2, \ldots$ in K and, for fixed $y = (y_n) \in \ell_1$, consider the map $\varphi_y : C(K) \to \mathbb{R}$ given by

$$\varphi_y(f) = \sum_n f(k_n) y_n \,.$$

Using this map, or otherwise, show that ℓ_1 is isometrically isomorphic to a subspace of $C(K)^*$.

(e) Is it true that $C[0,1]^* \cong \ell_1$? [*Hint: You may wish to consider the following elements of* $C[0,1]^*$*: for fixed* $x \in [0,1]$*, the map* $\delta_x(f) = f(x)$*.*]

23H Analysis of Functions

(a) State and prove the Banach–Alaoglu theorem for separable Banach spaces. [You do not need to prove metrizability of the topology that appears in the theorem.]

(b) Let $\Phi \subset C_c^\infty(\mathbb{R})$ be a fixed set of functions and let $p \in (1, \infty)$. Write

$$H = \left\{ f \in L^p(\mathbb{R}) : \int \varphi(x) f(x) dx = 0 \text{ for all } \varphi \in \Phi \right\},$$

and let $g_1, g_2, g_3 \in L^p(\mathbb{R})$ be fixed functions. Prove that there is $f_0 \in H$ such that

$$\|f_0 - g_1\|_p + \|f_0 - g_2\|_p + \|f_0 - g_3\|_p = \inf_{f \in H} \left(\|f - g_1\|_p + \|f - g_2\|_p + \|f - g_3\|_p \right),$$

where $\|h\|_p$ denotes the L^p-norm of the function h. [In part (b), you may use without proof any results from the course provided they are stated clearly.]

24F Algebraic Geometry

Define the *degree* of a divisor D on a (smooth irreducible projective) curve C. What does it mean for a divisor to be *principal*? Define the *divisor class group* $\mathrm{Cl}(C)$ and the subgroup $\mathrm{Cl}^0(C)$.

Compute the canonical class of $\mathbb{P}^1$. Compute the canonical class of a hyperelliptic curve C.

Prove that a curve C has $\mathrm{Cl}^0(C)$ trivial if and only if C is isomorphic to $\mathbb{P}^1$.

Prove that if C has genus $g \geqslant 1$ and $p_0 \in C$ then the map $\phi : C \to \mathrm{Cl}^0(C)$ given by $p \mapsto [p - p_0]$ is injective. Using this, or otherwise, show that for a curve of genus $g \geqslant 1$, which is not hyperelliptic, the divisor $p + q$ is linearly equivalent to the divisor $r + s$ if and only if $\{p, q\} = \{r, s\}$ as sets.

25I Differential Geometry

(a) Suppose $X \subset \mathbb{R}^N$ is a k-dimensional manifold. Let us define

$$TX = \bigcup_{p \in X} \{p\} \times T_p X,$$

i.e. the elements of TX are pairs (p, v) where $p \in X$ and $v \in T_p X$. This may be considered as a subset $TX \subset \mathbb{R}^N \times \mathbb{R}^N$.

Show that TX is a $2k$-dimensional manifold.

[*You may wish to consider the following maps:* $(q, w) \mapsto (\phi(q), d\phi|_q(w))$ *where* ϕ *is a local parametrisation of* X.]

(b) For $f : X \to Y$ a smooth map of manifolds, define what it means for $y \in Y$ to be a *regular value* of f. State, without proof, the preimage theorem.

(c) Now let $X \subset \mathbb{R}^3$ be a surface and consider the subset

$$SX = \{(p, v) \in TX : I_p(v, v) = 1\}$$

where I_p denotes the first fundamental form of X at p, and TX is defined in part (a). Show that SX is a submanifold of TX of dimension 3.

26L Probability and Measure

(a) What does it mean to say that a sequence of real-valued random variables $(X_n : n \geqslant 1)$ *converges in law*?

(b) State a characterization of convergence in law in terms of the sequence of characteristic functions $(\Phi_{X_n} : n \geqslant 1)$.

Let $(X_n : n \geqslant 1)$ be a sequence of independent random variables such that

$$\mathbb{P}(X_n = 1) = \mathbb{P}(X_n = -1) = 1/2, \quad n \geqslant 1.$$

Set

$$S_n = \sum_{j=1}^{n} \frac{X_j}{2^j}, \quad T_n = \sum_{j=1}^{n} \frac{X_j + 1}{3^j}.$$

(c) Show that $S_n \to S$ almost surely for some random variable S and determine the law of S.

[You may use the following identity

$$\sin\left(\frac{\xi}{2^n}\right) \prod_{j=1}^{n} \cos\left(\frac{\xi}{2^j}\right) = \frac{\sin \xi}{2^n}. \quad]$$

(d) Show that $T_n \to T$ almost surely for some random variable T and that T is neither a discrete random variable nor does it have a density with respect to Lebesgue measure.

[*Hint: You may find it helpful to work out the distributions of* T_1, T_2, T_3 *explicitly.*]

27K Applied Probability

(a) Consider an $M/M/\infty$ queue with arrival rate $\lambda > 0$ and service rate $\mu > 0$. Let $(X_t)_{t \geqslant 0}$ be the Markov chain representing the number of busy servers at time t. Suppose $X_0 = 0$.

(i) Find the distribution of X_t.

(ii) Find the invariant distribution of $(X_t)_{t \geqslant 0}$.

(iii) Using your expression for the distribution of X_t from part (i), show by direct computation that

$$\lim_{t \to \infty} \mathbb{P}(X_t = k) = \pi_k, \qquad \text{for all } k = 0, 1, 2, \ldots$$

where $\pi = (\pi_0, \pi_1, \ldots)$ is the invariant distribution derived in part (ii).

(b) Let $\lambda : \mathbb{R}^d \to [0, \infty)$ be a non-negative function such that $\int_A \lambda(x)\,dx < \infty$ for every bounded subset $A \subset \mathbb{R}^d$. State what it means for a point process Π on $\mathbb{R}^d$ to be a *Poisson process with intensity function* λ.

(c) Assume that viruses are distributed in $\mathbb{R}^2$ according to a homogeneous spatial Poisson process Π with a constant intensity $\lambda > 0$. A virus can infect any point within a disc of random radius X centred at its location, where X is distributed as Uniform$[0, R]$, for some fixed $R > 0$. The infection distances of the viruses are independent of each other and of Π. Find the distribution of the number of viruses infecting the origin $(0, 0)$.

28L Principles of Statistics

Let $\{f(x;\theta) : \theta \in \mathbb{R}\}$ be a regular statistical model and let X have density $f(.;\theta)$. Let $\delta = \delta(X)$ be an estimator for θ with finite variance and having a differentiable bias function

$$b(\theta) = \mathbb{E}_\theta[\delta] - \theta.$$

(a) Define the *Fisher information* $I(\theta)$ for the model and state the Cramér–Rao lower bound for the variance of δ. [You are not expected to set out regularity conditions for its validity.]

(b) State what is meant by the *quadratic risk* $R(\delta, \theta)$.

(c) Suppose that quadratic risk satisfies $R(\delta,\theta) \leqslant \gamma I(\theta)^{-1}$ for some constant $0 < \gamma < 1$. Use the Cramér–Rao lower bound to show that the derivative of the bias must satisfy

$$b'(\theta) \leqslant -\left(1 - \sqrt{\gamma - I(\theta)b(\theta)^2}\right).$$

Consider from now on the model $X \sim N(\theta, 1)$. The maximum likelihood estimator $\delta_*(X) = X$ is known to have a constant risk 1. We consider a "super-efficient" estimator $\delta(X)$ that achieves a risk $R(\delta,\theta) \leqslant \gamma < 1$ for all θ in some interval $[\theta_1, \theta_2]$.

(d) Using the result from part (c), show that the change in bias over this interval satisfies

$$b(\theta_2) - b(\theta_1) \leqslant -(1 - \sqrt{\gamma})(\theta_2 - \theta_1).$$

Deduce that the length of the interval $L = \theta_2 - \theta_1$ satisfies

$$L \leqslant \frac{2\sqrt{\gamma}}{1 - \sqrt{\gamma}}.$$

(e) Consider, for the same model, the linear shrinkage estimator $\delta_c(X) = cX$ for a fixed constant $0 \leqslant c < 1$. Fix $\gamma \in (c^2, 1)$ and set

$$I_\gamma = \{\theta : R(\delta_c, \theta) \leqslant \gamma\}.$$

Show that I_γ is an interval of length of L_γ, where

$$L_\gamma = \frac{2\sqrt{\gamma - c^2}}{1 - c}.$$

Verify that this length satisfies the bound derived in part (d).

29L Stochastic Financial Models

Let $W = (W_t)_{t\geqslant 0}$ be a Brownian motion and let $X_t = W_t + ct$ for some constant c.

(a) Use the reflection principle and the Cameron–Martin theorem to show, for all $x \geqslant 0$ and $T \geqslant 0$,

$$\mathbb{P}\left(\max_{0\leqslant t\leqslant T} X_t \geqslant x\right) = \mathbb{P}(X_T \geqslant x) + e^{2cx}\mathbb{P}(-X_T \geqslant x).$$

Consider a market with a stock with time-t price $S_t = S_0 e^{\mu t + \sigma W_t}$ and interest rate r, where S_0, σ, r are positive constants and μ a real constant. Let T be a fixed maturity date, and let $\pi(Y)$ denote the time-0 Black–Scholes price of a European contingent claim with time-T payout Y.

(b) Compute $\pi(\log S_T)$.

(c) A trader aims to replicate (i.e. to delta-hedge) the payout $\log S_T$. At time 0, how many shares of the stock should the trader hold?

(d) Show that

$$\pi\left(\max_{0\leqslant t\leqslant T} S_t + \min_{0\leqslant t\leqslant T} S_t\right) = \pi\left(S_0 + \frac{\sigma^2}{2}\int_0^T S_t dt + S_T\right).$$

[*Hint: You may find the following identities useful. For $Y \geqslant 0$ and $\theta \geqslant 0$,*

$$\mathbb{E}(e^{\theta Y}) = 1 + \int_0^\infty \theta e^{\theta x}\mathbb{P}(Y \geqslant x)dx, \quad \mathbb{E}(e^{-\theta Y}) = \int_0^\infty \theta e^{-\theta x}\mathbb{P}(Y \leqslant x)dx. \quad]$$

30K Mathematics of Machine Learning

Suppose $(x_1, y_1), \ldots, (x_n, y_n) \in \mathbb{R}^p \times \{-1, 1\}$ are i.i.d. input–output pairs. Describe the procedure of *gradient boosting* using a base regression procedure $\hat{h}$ and using the logistic loss, computing any derivatives involved.

Consider now the base regression procedure given by the empirical risk minimiser with squared error loss over the hypothesis class

$$\mathcal{H} := \{x_0 \mapsto \beta \operatorname{sgn}(x_{0j} - \alpha) : j = 1, \ldots, p,\ \alpha, \beta \in \mathbb{R}\}.$$

Explain how an empirical risk minimiser may be computed in $O(np)$ operations. [You may assume for simplicity that for each j, the n numbers $x_{1j}, x_{2j}, \ldots, x_{nj}$ are all distinct, and moreover, that we have already computed the permutations π_j such that $x_{\pi_j(1)j} < x_{\pi_j(2)j} < \cdots < x_{\pi_j(n)j}$, for each j.]

31D Asymptotic Methods

Consider the ordinary differential equation

$$y'' + \lambda^2 q(x) y = 0, \qquad (\dagger)$$

defined on the interval $[a,b]$, where $q(x) > 0$ is analytic and λ is real.

(a) Derive the asymptotic WKBJ approximation to y on $[a,b]$, in the form:

$$y(x) \sim q^{-1/4}\left(Ae^{i\lambda \int_a^x \sqrt{q(t)}dt} + Be^{-i\lambda \int_a^x \sqrt{q(t)}dt}\right),$$

for $\lambda \to \infty$, where A and B are complex constants.

(b) Impose the boundary conditions $y(a) = y(b) = 0$ and regard λ as an eigenvalue of the problem, taking discrete values λ_n.

(i) Show that $\lambda_n \sim E\,n$, as $n \to \infty$, where E is a constant you must determine. Give the form of the associated eigenfunction y_n.

(ii) Show, by direct substitution, that the asymptotic eigenfunctions in part (i) obey the orthogonality condition:

$$\int_a^b y_n(x) y_m(x) q(x) dx = 0,$$

for $n \neq m$.

(iii) Let $q(x) = x^{-2}$ and set $a = 1$ and $b = e$. Solve $(\dagger)$ exactly and obtain exact expressions for the eigenvalues λ_n. Compare these with their WKBJ approximations. Show that the magnitude of any discrepancy is consistent with the order of your asymptotic approximation.

32B Dynamical Systems

(a) Let $F : \mathbb{R} \to \mathbb{R}$ be a continuous one-dimensional map of the real line to itself. Define what it means for

(i) F to have a *horseshoe*,

(ii) F to be *chaotic* according to Glendinning's definition.

Prove that if F has a 3-cycle, $x_1 \to x_2 \to x_3$ with $x_1 < x_2 < x_3$, then F is chaotic.

(b) Now consider the map F defined by

$$x_{n+1} = F(x_n; \mu) := x_n(px_n^2 + qx_n - 1 - \mu),$$

where $p > 0$ and $q > 0$ are constants and μ is a parameter.

(i) One fixed point of F is at $x = 0$. Determine its linear stability. Find the other fixed points of F. Hence show that there are bifurcations at $\mu = -2$, $\mu = 0$, and a third value $\mu = \mu(q, p)$ to be determined. Determine the stability of the non-zero fixed points, explaining your reasoning, and sketch the (μ, x) bifurcation diagram showing the locus and stability of all the fixed points. State the type of each bifurcation present.

(ii) Now set $q = 0$ and consider the bifurcation at $\mu = 0$ of $x = 0$. For $0 < \mu \ll 1$, by considering x_{n+2} as a function of x_n, assuming that $x_n = O(\mu^{1/2})$ and retaining terms up to $O(\mu^{3/2})$ only, show that there is a stable period-2 orbit with

$$|x_n| = \sqrt{\frac{\mu}{p}}$$

to leading order in μ.

33A Principles of Quantum Mechanics

Consider a particle of mass m and spin $s = 1/2$ moving in a spherically symmetric potential $V(|\mathbf{X}|)$ with Hamiltonian

$$H_0 = \frac{\mathbf{P}^2}{2m} + V(|\mathbf{X}|)\,.$$

(a) Let $\mathbf{J}$, $\mathbf{L}$, and $\mathbf{S}$ be respectively the total angular momentum, the orbital angular momentum and the spin operators.

(i) State all the commutation relations satisfied by $\mathbf{L}$, $\mathbf{S}$ and $\mathbf{J}$. Briefly explain why we can choose a basis $\{|n, \ell, s, m, \sigma\rangle\}$ for the Hilbert space, labelled by the quantum numbers of the operators $(H_0, L^2, S^2, L_z, S_z)$ respectively.

(ii) Show that the energy E_n of $|n, \ell, s, m, \sigma\rangle$ is independent of m and σ. Hence compute the degeneracy of the energy eigenstates for a given fixed ℓ and $s = 1/2$.

(iii) Briefly explain why we may also choose a basis $\{|n, \ell, s, j, j_z\rangle\}$ for the Hilbert space, labelled by the quantum numbers of the operators $(H_0, L^2, S^2, J^2, J_z)$ respectively.

(iv) For fixed $\ell \geqslant 1$ and $s = 1/2$, list the allowed values of j. You may use that addition of spin and orbital angular momentum works in the same way as addition of angular momentum of two subsystems. Hence, in the basis $\{|n, \ell, s, j, j_z\rangle\}$ compute the degeneracy of the energy eigenstates for a given fixed ℓ and $s = 1/2$ and compare it to your previous result. [Note that in this question the eigenvalue of $J_z/\hbar$ is called j_z to distinguish it from that of $L_z/\hbar$, which is called m.]

(b) Now the same particle evolves according to the new Hamiltonian

$$H = H_0 + H_{\text{SO}}, \qquad H_{\text{SO}} = \lambda\, \mathbf{L} \cdot \mathbf{S},$$

where $\lambda > 0$ is a real constant and H_{SO} is the spin-orbit interaction.

(i) By recalling the relation among $\mathbf{J}$, $\mathbf{L}$, and $\mathbf{S}$ or otherwise, show that $\mathbf{L} \cdot \mathbf{S}$ can be written in terms of J^2, L^2 and S^2. Use this fact to briefly explain why one of the two bases considered in part (a) will be more convenient for studying the spectrum of the new Hamiltonian.

(ii) In the basis $\{|n, \ell, s, j, j_z\rangle\}$ compute the energy eigenvalues of H and their degeneracy for all states with fixed n, for $\ell \geqslant 1$ and $s = 1/2$. Discuss how the spin-orbit coupling lifts part of the degeneracy encountered in part (a). [The quantum number n still labels the eigenvalues E_n of H_0.]

(iii) The eigenvalues of $\lambda\, \mathbf{L} \cdot \mathbf{S}$ are the spin-orbit energy shifts. Show that the degeneracy-weighted average of these eigenvalues vanishes.

34E Applications of Quantum Mechanics

Consider a spinless particle of charge q and mass m in three dimensions under the influence of a uniform magnetic field, where the vector potential $\mathbf{A}$ satisfies

$$\mathbf{A} = \frac{B}{2}(-y, x, 0) \ ,$$

with B constant, and electrostatic potential $\phi = 0$.

(a) Write down the Hamiltonian and the Schrödinger equation for this particle.

(b) Consider the operators

$$\boldsymbol{\rho} = \frac{1}{qB}\hat{\mathbf{z}} \times (\mathbf{p} - q\mathbf{A}) \ , \qquad \mathbf{R} = \mathbf{r} - \boldsymbol{\rho} \ ,$$

where $\mathbf{r}$ and $\mathbf{p}$ are the usual position and momentum operators, and $\hat{\mathbf{z}}$ is the unit vector along the z-axis. Show that the only non-zero commutators involving components of $\boldsymbol{\rho}$ and $\mathbf{R}$ are $[\rho_x, \rho_y]$ and $[R_x, R_y]$.

(c) Given the following definitions

$$a = \frac{1}{r_B}\frac{1}{\sqrt{2}}(\rho_x + i\rho_y) \ , \qquad b = \frac{1}{r_B}\frac{1}{\sqrt{2}}(R_x - iR_y) \ ,$$

where $r_B^2 = \hbar/(qB)$, evaluate $[a, a^\dagger]$ and $[b, b^\dagger]$.

(d) When the particle is confined to move in the xy-plane, show that the Hamiltonian can be written as

$$H = \hbar\omega\left(a^\dagger a + \frac{1}{2}\right) \ ,$$

where ω is a parameter you should identify.

(e) Given that there exists a unique state $|\psi\rangle$ satisfying

$$a|\psi\rangle = b|\psi\rangle = 0 \ ,$$

what conclusions can be drawn about the allowed energies of the Hamiltonian and their degeneracies? [*Hint: Consider the expectation value of* $R_x^2 + R_y^2$ *in an eigenstate of the system and how it relates to a typical radius of motion.*]

35E Statistical Physics

(a) For any gas with internal energy E, temperature T, volume V and pressure p, use the Maxwell relation derived from varying the free energy $F(V,T)$ to show that

$$\left(\frac{\partial E}{\partial V}\right)_T = T^2 \left[\frac{\partial}{\partial T}\left(\frac{p}{T}\right)\right]_V .$$

For the remaining parts of this problem, consider a classical but non-ideal monatomic gas, obeying the van der Waals equation of state

$$p = \frac{NkT}{V - bN} - a\frac{N^2}{V^2} ,$$

where a and b are constants.

(b) Calculate the internal energy E of the non-ideal gas. Assume that the heat capacity of the gas is $C_V = \frac{3}{2}kN$ in the dilute gas limit, $V/N \to \infty$ (with N constant), and show that this formula for C_V continues to be valid even at finite V/N.

(c) Consider a Carnot engine, whose cylinder is filled with the same non-ideal gas as in the previous part.

(i) Under isothermal expansion of the volume from V_1 to V_2 at some temperature T, how much heat ΔQ is obtained from the thermal bath?

(ii) Under adiabatic expansion (or contraction) of the volume V, how does the temperature T vary with the volume V?

(iii) Suppose a complete, reversible Carnot cycle is performed, involving a hot bath with temperature T_H and a cold bath with temperature T_C. Give an expression for the efficiency η of the engine, and discuss whether it can depend on the parameters a and b. Justify your answer with reference to physical principles.

36D Electrodynamics

(a) By considering the force per unit volume, $\mathbf{f} = \rho\mathbf{E} + \mathbf{J} \times \mathbf{B}$, on a charge density ρ and current density $\mathbf{J}$ due to an electric field $\mathbf{E}$ and magnetic field $\mathbf{B}$, show from Maxwell's equations that

$$\frac{\partial g_i}{\partial t} + \frac{\partial \sigma_{ij}}{\partial x_j} = -f_i \,,$$

where $\mathbf{g} = \epsilon_0 \mathbf{E} \times \mathbf{B}$ and

$$\sigma_{ij} = -\epsilon_0 \left(E_i E_j - \frac{1}{2}|\mathbf{E}|^2 \delta_{ij} \right) - \frac{1}{\mu_0} \left(B_i B_j - \frac{1}{2}|\mathbf{B}|^2 \delta_{ij} \right) .$$

Give a physical interpretation of $\mathbf{g}$ and σ_{ij}.

(b) A homogeneous, linear dielectric with relative permittivity ϵ_r fills the region $z > 0$, while the region $z < 0$ is vacuum. State the junction conditions for $\mathbf{E}$ and the electric displacement $\mathbf{D}$ at the plane $z = 0$ assuming there is no free surface charge density or current.

A point charge q is held at rest at position $(0, 0, -d)$. Let $R_1 = \sqrt{x^2 + y^2 + (z + d)^2}$ be the distance of the point (x, y, z) from the charge q, and $R_2 = \sqrt{x^2 + y^2 + (z - d)^2}$ be the distance from the point $(0, 0, d)$. Using the method-of-images ansatz $\mathbf{E} = -\boldsymbol{\nabla}\Phi$, where

$$\Phi = \begin{cases} \dfrac{1}{4\pi\epsilon_0} \left(\dfrac{q}{R_1} + \dfrac{q'}{R_2} \right) & \text{for } z < 0 \,, \\[2ex] \dfrac{q''}{4\pi\epsilon_0 \epsilon_r R_1} & \text{for } z > 0 \,, \end{cases}$$

determine q'/q and q''/q in terms of ϵ_r.

Find the components of $\mathbf{E}$ in the plane $z = 0^-$ just outside the dielectric.

(c) For the set-up in part (b), integrate the relevant components of σ_{ij} from part (a) over the plane $z = 0^-$ to determine the force on the charge q due to the dielectric. Comment on your result.

[*You may assume the integrals*

$$2\pi \int_0^\infty \frac{r\,dr}{(r^2 + d^2)^3} = \frac{\pi}{2d^4} \quad \textit{and} \quad 2\pi \int_0^\infty \frac{r^3\,dr}{(r^2 + d^2)^3} = \frac{\pi}{2d^2} \,.$$

]

37A General Relativity

(a) The metric for de Sitter spacetime can be written

$$ds^2 = -\left(1 - r^2/b^2\right) dt^2 + \left(1 - r^2/b^2\right)^{-1} dr^2 + r^2\left(d\theta^2 + \sin^2\theta\, d\phi^2\right),$$

using static, spherically symmetric coordinates (t, r, θ, ϕ), where b is a positive constant and units with $c = 1$ are chosen. Defining

$$u = t - b \tanh^{-1}(r/b) \quad \text{for} \quad 0 \leqslant r < b\,,$$

find the components of the vector field $U_\mu = \partial_\mu u$ in the (t, r, θ, ϕ) coordinate system and verify that U_μ is null.

Find the form of the metric in coordinates (u, r, θ, ϕ) and hence show that the metric is non-singular on the surface $r = b$.

(b) Show that null geodesics which are radial ($d\theta = d\phi = 0$) must obey either

$$\frac{du}{dr} = 0 \qquad \text{or} \qquad \frac{du}{dr} = -\frac{2}{1 - r^2/b^2}\,.$$

Determine which of these geodesics are outgoing, $dr/dt > 0$, and which are ingoing, $dr/dt < 0$, for $0 \leqslant r < b$. Sketch members of both families in the r-t_* plane, where $t_* = r + u$, for $0 \leqslant r < b$.

Is it possible for a light signal travelling radially inwards in the region $r < b$ to have been sent from a point in the region $r > b$?

(c) Consider an observer travelling radially outward from $r = 0$ on a timelike trajectory $r(\tau)$, where τ is proper time. By finding a suitable trajectory explicitly, show that it is possible for the observer to reach the surface $r = b$ after finite proper time has elapsed.

38C Fluid Dynamics

Starting from the Navier–Stokes equations for incompressible viscous flow with conservative forces, obtain the vorticity equation (in standard notation)

$$\frac{\partial \boldsymbol{\omega}}{\partial t} + (\mathbf{u} \cdot \boldsymbol{\nabla})\boldsymbol{\omega} = (\boldsymbol{\omega} \cdot \boldsymbol{\nabla})\mathbf{u} + \nu \nabla^2 \boldsymbol{\omega}.$$

Consider a vorticity field $\boldsymbol{\omega} = \big(0, 0, \omega(r,t)\big)$ in cylindrical polar coordinates (r, θ, z), where ω is a smooth function and is exponentially small as $r \to \infty$. If the corresponding velocity field is $\big(0, v(r,t), 0\big)$ and v is finite as $r \to 0$, find $v(r,t)$ in terms of $\omega(r,t)$.

Suppose now that the velocity field is $\big(-\alpha r,\, v(r,t),\, 2\alpha z\big)$, where α is a positive constant. Show that the vorticity is still $\big(0, 0, \omega(r,t)\big)$. Find the scalar partial differential equation satisfied by $\omega(r,t)$ and deduce that the total circulation,

$$\Gamma = 2\pi \int_0^\infty r\omega(r,t)\, dr\,,$$

is constant in time.

Obtain the steady solution $\omega_s(r)$ for a given value of Γ.

[*Hint: In cylindrical polar coordinates (r, θ, z), the Laplacian of a function $f = f(r)$ is given by*

$$\nabla^2 f(r) = \frac{1}{r}\frac{\partial}{\partial r}\left(r \frac{\partial f}{\partial r}\right)\,. \,]$$

39C Waves

The Rankine-Hugoniot relations describing the jump conditions across a steady shock in a perfect fluid with constant specific heats are

$$[\rho \mathbf{u} \cdot \mathbf{n}] = 0 \, ,$$
$$[\rho \mathbf{u}(\mathbf{u} \cdot \mathbf{n}) + p\mathbf{n}] = 0 \, ,$$
$$[\rho(\mathbf{u} \cdot \mathbf{n})(h + \frac{1}{2}\mathbf{u}^2)] = 0 \, ,$$

where ρ, p and $\mathbf{u}$ are the fluid density, pressure and velocity, $\mathbf{n}$ is the direction normal to the shock, $h = c^2/(\gamma - 1)$ is the enthalpy, γ is the adiabatic index, c is the local sound speed, and $[X]$ represents the jump in the quantity X across the shock.

(a) Briefly explain the physical meaning of each of these jump conditions.

(b) In the case of a normal shock, show that

$$M_2^2 = \frac{2 + (\gamma - 1)M_1^2}{2\gamma M_1^2 - (\gamma - 1)} \, ,$$

where M_1 is the Mach number upstream of the shock, and M_2 is the Mach number downstream.

(c) Find the equation relating the temperatures upstream and downstream of a normal shock, given that

$$\frac{\rho_2}{\rho_1} = \frac{M_1^2(\gamma + 1)}{2 + (\gamma - 1)M_1^2} \qquad (\dagger)$$

in such a shock, where ρ_1 and ρ_2 are upstream and downstream densities, respectively.

(d) Now suppose that the shock is oblique, inclined at angles θ_1 and θ_2 to the upstream and downstream flows respectively (so that in the case of a normal shock $\theta_1 = \theta_2 = \pi/2$). Show how equation (†) is modified in an oblique shock, and derive a single equation for $\tan\theta_2$ in terms of θ_1 and M_1. Determine the maximum possible flow deflection when $M_1 \to \infty$.

[*Recall that:* $\tan(A - B) = \dfrac{\tan A - \tan B}{1 + \tan A \tan B}$.]

40B Numerical Analysis

A nonnegative matrix is a matrix whose entries are nonnegative ($\geqslant 0$). Let A, M, N be three real $n \times n$ matrices satisfying $A = M - N$. The pair of matrices M, N is a *regular splitting* of A, if M is nonsingular and M^{-1} and N are nonnegative.

With a regular splitting, we associate the iteration

$$x_{k+1} = M^{-1}Nx_k + M^{-1}b, \qquad x_0, b \in \mathbb{R}^n, \quad k = 0, 1, 2, \ldots . \tag{$\dagger$}$$

The questions below address the conditions for such an iteration to converge for all x_0.

Each result must be proved directly rather than quoting from lectures, except that you may assume the existence of Jordan block structure.

(a) Show that the sequence A^k, $k = 0, 1, \ldots$, converges to zero if and only if $\rho(A) < 1$, where $\rho(A)$ denotes the spectral radius of A.

(b) Show that the series $\sum_{k=0}^{\infty} A^k$ converges if and only if $\rho(A) < 1$, and that, under this condition, $I - A$ is nonsingular and the limit of the series is equal to $(I - A)^{-1}$.

(c) Let $B \in \mathbb{R}^{n \times n}$ be a nonnegative matrix. Show that if $\rho(B) < 1$, then $(I - B)^{-1}$ is nonnegative.

(d) Let M, N be a regular splitting of a matrix A. Then, show that $\rho(M^{-1}N) < 1$ if and only if A^{-1} exists and is non-negative.

Deduce that if $Ax^* = b$ then ($\dagger$) converges to x^* for all x_0 if and only if A^{-1} exists and is non-negative.

[You may use the fact that if G is a nonnegative matrix, there is a nonnegative eigenvector x with eigenvalue $\rho(G)$.]

END OF PAPER