UNIVERSITY OF CAMBRIDGE

MAT2

MATHEMATICAL TRIPOS Part II

Thursday 11 June 2026 9:00am to 12:00pm

PAPER 3

Before you begin read these instructions carefully.

The examination paper is divided into two sections. Each question in Section II carries twice the number of marks of each question in Section I. Section II questions also carry an alpha or beta quality mark and Section I questions carry a beta quality mark.

Candidates may obtain credit from attempts on **at most six questions** from Section I and from any number of questions from Section II.

Write on **one side** of the paper only and begin each answer on a separate sheet.

Write legibly; otherwise you place yourself at a grave disadvantage.

At the end of the examination:

Separate your answers to each question.

Complete a gold cover sheet **for each question** that you have attempted, and place it at the front of your answer to that question.

Complete a green main cover sheet listing **all the questions** that you have attempted.

Every cover sheet must also show your Blind Grade Number and desk number.

Tie up your answers and cover sheets into **a single bundle**, with the main cover sheet on the top, and then the cover sheet and answer for each question, in the numerical order of the questions.

STATIONERY REQUIREMENTS
Gold cover sheets
Green main cover sheet
Script paper
Rough paper

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

SECTION I

1G Number Theory

Define the *Jacobi symbol* $\left(\frac{a}{n}\right)$ for $a \in \mathbb{Z}$ and $n \geqslant 1$ an odd integer. State and prove the reciprocity law for the Jacobi symbol. [You may assume the reciprocity law for the Legendre symbol.]

Compute $\left(\frac{5}{959}\right)$. Determine whether the congruence $x^2 \equiv 5 \mod 959$ has a solution.

2I Topics in Analysis

Define a *dense subset* of a metric space.

State a version of the Baire category theorem.

Let A_j be a sequence of subsets of $[0,1]$ such that for each $N \geqslant 1$, $\cup_{j=N}^{\infty} A_j$ is open and dense in $[0,1]$. Prove that the set S of points $x \in [0,1]$ such that $x \in A_j$ for infinitely many j is dense. Must S be open? Must it be true that $\cap_{j=1}^{\infty} A_j \neq \emptyset$? Justify your answers.

3H Coding & Cryptography

Let $\mathbb{F}_2$ denote the field of order 2. What does it mean to say $C \subseteq \mathbb{F}_2^n$ is a *cyclic code*?

Show that there is a bijection between the cyclic codes of length n and factors of $X^n - 1$ over the field $\mathbb{F}_2$.

How many cyclic codes are there of length 7?

4I Automata and Formal Languages

(a) Define the notion of a *deterministic finite state automaton* $D = (Q, \Sigma, \delta, q_0, F)$, its *extended transition function* $\widehat{\delta}$, and its *accepted language* $\mathcal{L}(D)$. What is a *regular language*?

(b) State the pumping lemma for regular languages.

(c) Let $\Sigma = \{a, b\}$. Determine whether the following languages are regular. Justify your answer.

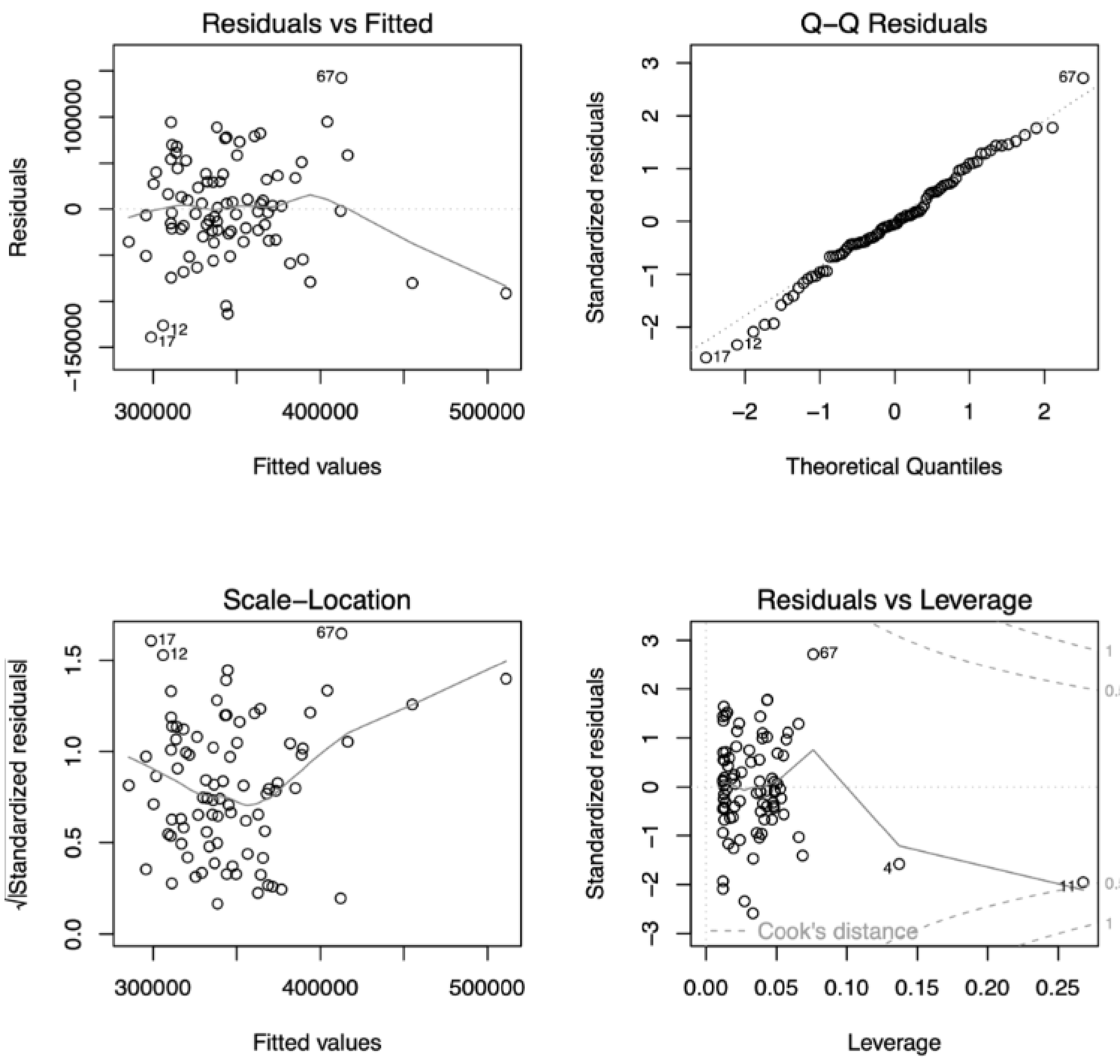
(i) The set of words containing no occurrences of 100 successive b's.

(ii) The set $\{w_i : i \geqslant 1\}$ where w_i is recursively defined as follows: $w_1 = ab$ and $w_{i+1} = w_i w_i$.

5K Statistical Modelling

Consider the dataset `HousePrices` which includes the house sale prices along with various factors that might be relevant for predicting the sale price. Consider the `R` code and output below.

```
> head(HousePrices, 3)
  Bedrooms Bathrooms Living.area Lot.size Year.built Property.tax Sale.price
1        4         1        1380     6000       1948         8360     350000
2        4         2        1761     7400       1951         5754     360000
3        4         2        1564     6000       1948         8982     350000
> par(mfrow = c(2,2))
> plot(lm(Sale.price ~ Bedrooms + Living.area, data = HousePrices))
```



Describe your observations in each plot above. Does the model appear to be adequate in light of these plots? If not, what might you do to improve the model?

6C Mathematical Biology

Bacteria have a population density $n(x,t)$ obeying the reaction-diffusion equation

$$\frac{\partial n}{\partial t} = \frac{\partial^2 n}{\partial x^2} - n(n-A)(n-B)\,,$$

with $0 < A < B$.

(a) Find the stable, homogeneous fixed points.

(b) Suppose $n(x,t) \to 0$ as $x \to -\infty$ and $n(x,t) \to C$ as $x \to +\infty$, with C a non-zero stable homogeneous fixed point.

Show that the system supports a solution of the form $n(x,t) = f(x - ct)$, for c a constant, and f some suitable function you need not determine.

What further condition on A and B ensures that the population does not die out in the future?

7D Further Complex Methods

Consider the Fuchsian ordinary differential equation

$$w'' + p(z)w' + q(z)w = 0$$

with three regular singular points.

Provide a definition of *regular singular points* and explain what the *Papperitz symbol*

$$P\left\{\begin{matrix} a & b & c & \\ \alpha & \beta & \gamma & ;\, z \\ \alpha' & \beta' & \gamma' & \end{matrix}\right\}$$

represents.

For

$$z(1-z)w'' + (2-3z)w' - w = 0\,, \qquad (\dagger)$$

show that there are two regular singular points a and b in $\mathbb{C}$. Where is the third regular singular point c located?

Find α, α', β and β' by considering corresponding indicial equations.

Write $(\dagger)$ as a hypergeometric equation, hence determine γ and γ' and deduce the Papperitz symbol.

8A Classical Dynamics

What is the *Hamiltonian vector field* associated to a smooth function f on phase space? What does it mean for a coordinate transformation on phase space to be *canonical*?

Let D_f and D_g be Hamiltonian vector fields associated to smooth functions f and g on phase space. Show that the commutator $[D_f, D_g] = D_{\{f,g\}}$.

Suppose phase space has generalised coordinates $z^A = (q^a, p_b)$. Show that the coordinate transformations

$$z^A \mapsto Z^A(z, s) = \exp\left(-sD_f\right) z^A$$

define a 1-parameter family of canonical transformations. [*Hint: You may find it helpful to differentiate with respect to s.*]

9D Cosmology

(a) The equations governing a period of inflation in a homogenous and isotropic universe are

$$H^2 = \left(\frac{\dot{a}}{a}\right)^2 = \frac{1}{3M_{\rm Pl}^2}\left(\frac{1}{2}\dot{\phi}^2 + V(\phi)\right), \qquad \ddot{\phi} + 3H\dot{\phi} + V_{,\phi} = 0\,,$$

where a is the scale factor, ϕ the inflaton scalar, and $M_{\rm Pl}$ the Planck mass. For the exponential potential, $V(\phi) = V_* e^{-\phi/M_{\rm Pl}}$ there is an exact scaling solution of the form

$$a(t) = a_0\, t^2, \qquad H(t) = \frac{2}{t}\,, \qquad \phi(t) \;=\; \phi_* + 2M_{\rm Pl}\ln(t/t_*)\,.$$

Verify that $a(t)$ and $\phi(t)$ simultaneously satisfy the Friedmann and scalar field equations, provided the initial parameters satisfy a consistency condition:

$$\phi_* = M_{\rm Pl}\ln\left(\frac{V_* t_*^2}{10M_{\rm Pl}^2}\right)\,.$$

Does this power law expansion fulfill the criterion for an inflationary universe? The number of e-folds between time t and t_* is defined by $\mathcal{N} \equiv \ln\left(\frac{a(t)}{a(t_*)}\right)$. How far must the field ϕ traverse from ϕ_* in order for $\mathcal{N} \approx 60$ e-folds?

(b) Briefly explain the horizon problem in a decelerating FLRW universe. Describe how an inflationary phase resolves this problem.

Consider a radiation-dominated universe with scale factor $a(t) \propto t^{1/2}$, evolving from $t \sim 10^{-35}$ s to the present day $t_0 \sim 10^{17}$ s. By extrapolating the present horizon scale backward in time, show that its physical size at $t \sim 10^{-35}$ s was of order one metre.

Assume that for $t < 10^{-35}$ s the universe underwent a de Sitter-like inflationary phase with constant Hubble parameter H, which then reheated and transitioned into a radiation-dominated era at $t \sim 10^{-35}$ s. Approximately estimate the number $\mathcal{N}$ of inflationary e-folds required for the entire observable universe to have originated within a single Hubble volume at the beginning of inflation.

10E Quantum Information and Computation

Let $f : \{0,1\}^n \to \{0,1\}$ be a Boolean function. We say that f is *k-balanced* if $f(x) = 1$ for exactly k inputs and $f(x) = 0$ for $2^n - k$ inputs. Thus f is constant if $k = 0$ or $k = 2^n$, and f is perfectly balanced if $k = 2^{n-1}$. In the following, assume that we are given a quantum oracle U_f for the function f.

Starting with initial state $|0\rangle^{\otimes n}|-\rangle$ and applying the standard Deutsch–Jozsa algorithm, express the probability of obtaining the n-bit string $000\ldots0$ in the final measurement, in terms of k and n. Explain qualitatively how this probability behaves in the following cases

(i) f is k-balanced with $k \notin \{0, 2^{n-1}, 2^n\}$,

(ii) f is constant,

(iii) f is perfectly balanced.

Discuss why the Deutsch–Jozsa algorithm no longer perfectly distinguishes between the two cases, constant or k-balanced for $k \notin \{0, 2^{n-1}, 2^n\}$. Given $k \notin \{0, 2^{n-1}, 2^n\}$ and $\varepsilon \in (0,1)$, state an algorithm that distinguishes between the two cases in such a way that the following two conditions hold.

- If f is constant, the algorithm always correctly identifies f as constant.
- If f is k-balanced, the algorithm correctly identifies f as k-balanced with probability at least $1 - \varepsilon$.

For this algorithm, state how the number of uses of the oracle U_f depends on ε and k.

SECTION II

11G Number Theory

(a) Define the *convergents* $(p_n/q_n)_{n\geqslant 0}$ of the continued fraction expansion $[a_0, a_1, a_2, ...]$.

Show that for all $n \geqslant 1$, $p_n q_{n-1} - p_{n-1} q_n = (-1)^{n+1}$. Show further that if $n \geqslant 1$, $\beta > 0$ and $\alpha = [a_0, a_1, \ldots, a_n, \beta]$, then

$$\alpha = \frac{p_n \beta + p_{n-1}}{q_n \beta + q_{n-1}}$$

and α lies strictly between p_n/q_n and p_{n-1}/q_{n-1}.

(b) Given $\theta \in \mathbb{R} \setminus \mathbb{Q}$, let $[a_0, a_1, a_2, ...]$ be the continued fraction expansion of θ with convergents $(p_n/q_n)_{n\geqslant 0}$. Let $b_n = q_n^2 |\theta - p_n/q_n|$.

By considering $b_n/q_n^2 + b_{n+1}/q_{n+1}^2$, show that for every $n \geqslant 1$,

$$\frac{q_{n+1}}{q_n} = \frac{1 \pm \sqrt{1 - 4b_n b_{n+1}}}{2b_n}.$$

Deduce that

$$2c_n a_{n+1} \leqslant 2b_n a_{n+1} \leqslant \sqrt{1 - 4b_n b_{n+1}} + \sqrt{1 - 4b_{n-1} b_n} \leqslant 2\sqrt{1 - 4c_n^2},$$

where $c_n = \min\{b_{n-1}, b_n, b_{n+1}\}$.

(c) Deduce from part (b) that if $\theta \in \mathbb{R} \setminus \mathbb{Q}$, then there are infinitely many rationals p/q such that

$$\left|\theta - \frac{p}{q}\right| \leqslant \frac{1}{\sqrt{5}q^2}.$$

12J Automata and Formal Languages

(a) Define the notion of a *register machine*, describe its *computation* on a given input, and how a given register machine is *encoded by a natural number*.

(b) For $m, k \in \mathbb{N} = \{0, 1, 2, 3, \ldots\}$, define the *partial function* $f_{m,k}$ and the set W_m.

(c) State the recursion theorem.

(d) Let $\mathbf{Inf} = \{m \in \mathbb{N} : \mathrm{W}_m \text{ is infinite}\}$ and $\mathbf{Fin} = \{m \in \mathbb{N} : \mathrm{W}_m \text{ is finite}\}$. Give explicit numbers i and j such that $i \in \mathbf{Inf}$ and $j \in \mathbf{Fin}$.

(e) Using the numbers i and j from part (d), show that the functions

$$g_0(n) = \begin{cases} j & \text{if } n \in \mathbf{Inf}, \\ i & \text{if } n \in \mathbf{Fin} \end{cases} \qquad \text{and} \qquad g_1(n) = \begin{cases} i & \text{if } n \in \mathbf{Inf}, \\ j & \text{if } n \in \mathbf{Fin} \end{cases}$$

are not total recursive. Is there a total recursive function h such that $n \in \mathbf{Inf}$ if and only if $h(n) \in \mathbf{Fin}$? Justify your answer.

[*You may use theorems from the course without proof, provided that you state them clearly.*]

13E Mathematical Biology

At any time $t \in \mathbb{R}$, a certain species has a population size $n \in \mathbb{Z}^+$. The probability rate for the species to evolve from size n to size $n+r$, with $r \in \mathbb{Z}$, is given by $W(n,r)$.

(a) Write down the master equation that governs the probability distribution $P(n,t)$.

(b) In what circumstance can the discrete population size n be approximated by a continuous variable $x \in \mathbb{R}^+$? By making appropriate assumptions, derive the Fokker–Planck equation,

$$\frac{\partial P}{\partial t} = -\frac{\partial (uP)}{\partial x} + \frac{\partial^2 (DP)}{\partial x^2}\,, \qquad (\dagger)$$

for a probability distribution $P(x,t)$, where $u(x)$ and $D(x)$ are functions that you should define in terms of $W(x,r)$. Explain how the Fokker–Planck equation $(\dagger)$ ensures that probability is conserved and give a physical interpretation of $u(x)$ and $D(x)$. Give an example of probability rates $W(x,r)$ for which $(\dagger)$ reduces to the diffusion equation.

(c) Compute the time derivatives $d\langle x\rangle/dt$ and $d(\mathrm{var}(x))/dt$, where $\langle\cdot\rangle$ denotes an expectation value and $\mathrm{var}(x)$ is the variance of x.

(d) Consider the rates $W(x,1) = e^{-\lambda x}$ and $W(x,-1) = e^{\lambda x}$, with $W(x,r) = 0$ for $r \neq \pm 1$. Use the Fokker–Planck equation $(\dagger)$ to derive the steady-state probability distribution $P(x,t)$. [You need not compute the overall normalisation constant of this distribution.]

Sketch the distribution.

(e) Now take the rates to be $W(x,1) = a$ and $W(x,-1) = b$, where a and b are unequal, positive constants. As before, $W(x,r) = 0$ for $r \neq \pm 1$.

(i) Change variables from x to $z = x - ut$, and from $(\dagger)$ deduce the partial differential equation for $\tilde{P}(z,t) = P(x,t)$.

(ii) Given $\tilde{P}(z,0) \geqslant 0$ for all z, argue that, if there exists any time $T > 0$ at which $\tilde{P}(z_*,T) < 0$ for some z_*, then the equation deduced in part (e)(i) is contradictory. [*Hint: Consider the earliest such time T.*]

(iii) Let us add a term to $(\dagger)$, obtaining

$$\frac{\partial P}{\partial t} = -\frac{\partial (uP)}{\partial x} + \frac{\partial^2 (DP)}{\partial x^2} - \frac{\partial^3 (FP)}{\partial x^3}\,. \qquad (\star)$$

Amending your answer to part (b), express $F(x)$ in terms of a and b. Explain why the argument of part (e)(ii) does not hold for $(\star)$.

14D Cosmology

(a) Consider a massive particle at position $\mathbf{r} = a\,\mathbf{x}$ with scale factor a and comoving coordinate $\mathbf{x}$ in an expanding universe. The equation of motion in cosmic time $\ddot{\mathbf{r}} = -\nabla_{\mathbf{r}}\Phi = -\frac{1}{a}\nabla_{\mathbf{x}}\Phi$ is governed by the gravitational potential Φ, satisfying the Poisson equation

$$\nabla^2_{\mathbf{r}}\Phi = \frac{4\pi G}{c^2}\rho_{\mathrm{M}}\,, \qquad (*)$$

where $\rho_{\mathrm{M}}(\mathbf{r},\tau)$ is the dominant matter density. The Friedmann equation in conformal time τ is $\mathcal{H}^2 = \frac{8\pi G}{3c^2}\rho_{\mathrm{M}}a^2$, where $\frac{dt}{d\tau} = a$ and $\mathcal{H} = \frac{1}{a}\frac{da}{d\tau} = \frac{a'}{a}$. We will expand this mildly perturbed universe about a homogeneous background using $\rho_{\mathrm{M}}(\mathbf{r},\tau) \equiv \bar{\rho}_{\mathrm{M}}(\tau)\big(1 + \delta_{\mathrm{M}}(\mathbf{r},\tau)\big)$, where $\bar{\rho}_{\mathrm{M}}$ is the mean density and δ is the density perturbation. Similarly, take $\Phi \equiv \bar{\Phi} + \phi$ where you may assume that the background potential $\bar{\Phi} = -\frac{1}{2}\mathcal{H}'|\mathbf{x}|^2$ satisfies $(*)$ with $\rho_{\mathrm{M}} = \bar{\rho}_{\mathrm{M}}$ and $\Phi = \bar{\Phi}$.

(i) Introduce Lagrangian coordinates $\mathbf{r} = a(\tau)\big(\mathbf{q} + \boldsymbol{\psi}(\mathbf{q},\tau)\big)$ in which $\mathbf{q}$ is the unperturbed comoving position and $\psi(\mathbf{q},\tau)$ is the comoving displacement. Show that the perturbed part of the equation of motion for $\ddot{\mathbf{r}}$ takes the form

$$\mathbf{x}'' + \mathcal{H}\mathbf{x}' \;\simeq\; \boldsymbol{\psi}'' + \mathcal{H}\,\boldsymbol{\psi}' \;\simeq\; -\nabla_{\mathbf{q}}\,\phi\,. \qquad (\dagger)$$

Relate the perturbed potential ϕ to the density perturbation δ_{M} with the Poisson equation.

(ii) Explain why mass conservation implies we can relate matter in a small perturbed comoving volume $d^3\mathbf{r}$ to the unperturbed background $d^3\mathbf{q}$ through $\rho_{\mathrm{M}}(\mathbf{r},\tau)\,d^3\mathbf{r} = \bar{\rho}_{\mathrm{M}}(\tau)\,a(\tau)^3\,d^3\mathbf{q}$. Change coordinates using the Jacobian $|\partial r_i/\partial q_j|^{-1}$ to show that

$$\rho_{\mathrm{M}}(\mathbf{r},\tau) \;\simeq\; \bar{\rho}_{\mathrm{M}}(\tau)\,(1 - \nabla_{\mathbf{q}}\cdot\boldsymbol{\psi} + \ldots)\,.$$

Combine with the divergence of $(\dagger)$ to obtain the density perturbation evolution equation for δ_{M} in a matter-dominated universe:

$$\delta''_{\mathrm{M}}(\mathbf{x},\tau) + \mathcal{H}(\tau)\,\delta'_{\mathrm{M}}(\mathbf{x},\tau) - \frac{4\pi G}{c^2}\,\bar{\rho}_{\mathrm{M}}(\tau)\,a^2(\tau)\,\delta_{\mathrm{M}}(\mathbf{x},\tau) = 0\,.$$

(b) Now consider the evolution of radiation perturbations δ_{R} deep in the radiation-dominated era ($\tau \ll \tau_{\mathrm{eq}}$) for which you are given the Fourier space evolution equation:

$$\delta''_{\mathrm{R}}(\mathbf{k},\tau) + \frac{\alpha\,k}{\mathcal{H}}\,\delta'_{\mathrm{R}}(\mathbf{k},\tau) + \frac{c^2}{3}k^2\,\delta_{\mathrm{R}}(\mathbf{k},\tau) = 0\,, \qquad (k = |\mathbf{k}|)\,,$$

where the last term accounts for relativistic pressure effects and we have neglected the (diminishing) gravitational potential term. Here, the middle term incorporates phenomenological damping due to the flow (or free-streaming) of relativistic particles out of overdense regions (or into underdense regions). For slowly varying damping with $\alpha \ll 1/\tau < k$, seek radiation era solutions of the form $\delta_{\mathrm{R}} = e^{-S}u$ to find the solution

$$\delta_{\mathrm{R}}(\mathbf{k},\tau) \;\simeq\; \exp\big[-S(\mathbf{k},\tau)\big]\,\Big[A_{\mathbf{k}}\cos(ck\tau/\sqrt{3}) + B_{\mathbf{k}}\sin(ck\tau/\sqrt{3})\Big]\,,$$

where you should specify the $S(\mathbf{k},\tau)$ and state the approximations made. Briefly describe your solution and compare to matter perturbation behaviour during the matter era.

15E Quantum Information and Computation

Alice (A), Bob (B), and Charlie (C) are separated in space. They can perform quantum operations on systems held locally in their possession, and they can communicate classically. They are unable to directly transmit quantum systems between each other. Each also has a supply of local qubit ancillas all in state $|0\rangle$. This scenario is referred to as 'local operations and classical communication,' or LOCC.

(a) Consider the 3-qubit state

$$|\varphi\rangle = \alpha\,|000\rangle + \beta\,|111\rangle + \gamma\,|100\rangle\,, \text{ with } |\alpha|^2 + |\beta|^2 + |\gamma|^2 = 1\,.$$

Show that $|\varphi\rangle$ is entangled (that is, not a product state of three qubits) for all $\alpha, \beta \neq 0$ and any γ.

(b) Suppose that Alice, Bob, and Charlie share the state

$$|\Omega\rangle_{ABC} = \frac{1}{\sqrt{2}}\Big(\,|000\rangle_{ABC} + |111\rangle_{ABC}\Big)\,.$$

Alice also possesses a 1-qubit state $|\xi\rangle$ and wishes to transfer it to Bob.

(i) Show how this may be achieved by a suitable LOCC protocol, stating clearly the sequence of individual, local operations and classical communications used. [*Hint: It may be useful to consider Charlie first performing a suitable measurement.*]

(ii) Can the transfer be done with an LOCC protocol that does not involve any classical communication between parties A, B, and C? Give a reason for your answer.

(c) Now instead suppose that Alice and Bob possess two copies of the state

$$|\psi\rangle_{AB} = \alpha\,|00\rangle_{AB} + \beta\,|11\rangle_{AB}\,, \text{ with } \alpha, \beta \neq 0\,;$$

that is, they have the 4-qubit state $|\psi\rangle_{A_1B_1} \otimes |\psi\rangle_{A_2B_2}$. They wish to obtain the state

$$|\phi^+\rangle_{AB} = \frac{1}{\sqrt{2}}\Big(\,|00\rangle_{AB} + |11\rangle_{AB}\Big)\,.$$

Due to soaring energy costs, for this task Alice's and Bob's local quantum capability is restricted to performing only CX and Pauli gates, and to making (possibly incomplete) measurements in the computational basis. Show how they may obtain the state $|\phi^+\rangle_{AB}$ with a nonzero probability of success by a LOCC protocol, and give the success probability of your protocol. [*Hint: It may be useful to consider the expression for the 4-qubit state* $|\psi\rangle_{A_1B_1} \otimes |\psi\rangle_{A_2B_2}$ *with qubit labels reordered as* A_1, A_2, B_1, B_2*.*]

(d) Alice and Bob now possess one copy of $|\phi^+\rangle_{AB}$ and wish to convert it back to $|\psi\rangle_{AB}$, with these states defined as in part (c). Describe an LOCC protocol, now with general local quantum actions allowed, that will achieve this task with certainty.

16J Logic & Set Theory

Work in the language of ordered fields with binary function symbols $+$ and $\cdot$, a binary relation symbol $<$, and two constant symbols 0 and 1. Let T be the axioms of ordered fields (i.e., the usual field axioms, the usual axioms for strict total orders, and the additional axioms $\forall x \forall y \forall z (x < y \Rightarrow x + z < y + z)$ and $\forall x \forall y (x > 0 \wedge y > 0 \Rightarrow x \cdot y > 0)$). Let $\mathbf{F} = (F, +, \cdot, 0, 1, <) \models T$. Let t_n, $n \in \mathbb{N}$, be terms defined recursively by $t_0 = 0$, $t_1 = 1$, and $t_{n+1} = t_n + 1$. We say that $\mathbf{F}$ is *Archimedean* if for all $x \in F$ there is an n such that $x < t_n^{\mathbf{F}}$.

(a) State what it means that a class of ordered fields is *axiomatisable*.

(b) State the compactness theorem for predicate logic.

(c) Show that the class of Archimedean ordered fields is not axiomatisable.

For $x, y \in F$ (we are allowing the case $x = y$), sets of the form (x, y), $[x, y)$, $(x, y]$, $[x, y]$, $(-\infty, x)$, $(-\infty, x]$, (x, ∞), or $[x, \infty)$ are called *intervals*. Finite unions of intervals are called *basic sets*.

(d) Let $\varphi(x)$ be a quantifier-free formula with one free variable. Show that $\{x : \mathbf{F} \models \varphi(x)\}$ is a basic set. [You may use without proof the fact that for a polynomial function p, the sets $\{x : p(x) = 0\}$ and $\{x : p(x) < 0\}$ are basic sets.]

(e) Assume that an ordered field $\mathbf{F}$ has the following property. For each formula $\varphi(x)$ with one free variable, there is a quantifier-free formula $\psi(x)$ such that for all $x \in F$, we have that $\mathbf{F} \models \varphi(x)$ if and only if $\mathbf{F} \models \psi(x)$.

Show that every non-negative element of F is a square, i.e., for each $x > 0$, there is a y such that $y^2 = x$. [You may use without proof the following statements that hold in all ordered fields: if $x > 0$ and $y > 1$, then $x \cdot y > x$, and if $x > 0$ and $y < 1$, then $x \cdot y < x$.]

17J Graph Theory

(a) Define the *adjacency matrix* A of a graph G. What are the *eigenvalues* of G?

(i) Show that if λ is an eigenvalue of G, then $|\lambda| \leqslant \Delta$, where Δ is the maximum degree of G.

(ii) Show that if G is connected, then Δ is an eigenvalue of G if and only if G is regular. In this case show that Δ has multiplicity 1 and eigenvector $(1, 1, \ldots, 1)^T$.

(b) Let G have adjacency matrix A. For each $r \in \mathbb{N}$, give a characterisation of the rth power A^r of A in terms of walks in G. Assume now that G is d-regular and that for all $r \in \mathbb{N}$, the number of circuits in G of length r is

$$5^r + 3 \cdot 5^{r/2} + 3 \cdot (-1)^r \cdot 5^{r/2} + 5 \cdot (-1)^r.$$

Find the order and size of G. Compute d.

(c) What is meant by a graph G of order n being *strongly regular with parameters* (d, a, b)? If such a graph G exists and $b > 0$ and G is not complete, write down the rationality condition.

Let G be a graph of order at least 2 containing no triangles, in which every pair of non-adjacent vertices has exactly three common neighbours. Show that G must be d-regular of order $1 + d(d+2)/3$ for some $d \in \{1, 3, 21\}$. Show that such a graph exists for $d = 3$.

18F Galois Theory

Let L/K be an extension of fields.

(a) What does it mean to say that L/K is *finite*? What does it mean to say that L/K is *algebraic*? Show that any finite extension is algebraic, and give an example of an extension L/K which is algebraic but not finite.

(b) Now assume L/K is algebraic. Let $x \in L$. What does it mean to say that x is *separable* over K? Show that if $\operatorname{char} K = 0$ then x is separable over K, but if instead $\operatorname{char} K = p > 0$ then there exists an integer $n \geqslant 0$ such that x^{p^n} is separable over K.

(c) Let $L = \mathbb{Q}(\sqrt{2}, \sqrt[3]{2}, \zeta_3)$ where ζ_3 is a primitive cube root of unity. Compute the degree $[L : \mathbb{Q}]$ and decide whether $L/\mathbb{Q}$ is Galois. For which integers $n \geqslant 1$ does there exist $x \in L$ with $x^n = 2$? Justify your answers.

19F Representation Theory

(a) Define the topological group SU_2, and show that it is homeomorphic to S^3. Write down a subgroup T of SU_2 that is isomorphic to the circle group S^1.

Let $\mathcal{O}$ be a conjugacy class in SU_2, and suppose $|\mathcal{O}| > 1$. Show $\mathcal{O}$ is homeomorphic to the 2-sphere S^2, and $\mathcal{O}$ meets T at exactly 2 points. Parameterise all the conjugacy classes in SU_2.

Let $\rho : SU_2 \to \mathrm{GL}(V)$ be an n-dimensional complex representation of SU_2. Show that $\det \rho(g) = 1$ for all $g \in SU_2$.

(b) Let $G = Q_8$ be the quaternion group of order 8. Construct the 2-dimensional irreducible complex representation V of G, and use this to show that $G \leqslant SU_2$.

Decompose $V \otimes V$ and $V \otimes V \otimes V$ as representations of SU_2, carefully stating any results from lectures that you use. For each irreducible summand W of $V \otimes V$ as a representation of SU_2, decompose $\mathrm{Res}_G^{SU_2} W$ as a representation of G.

20H Algebraic Topology

State the Mayer-Vietoris theorem for a simplicial complex K which is the union of subcomplexes M and N.

Let $T = S^1 \times S^1$ be the torus, and fix distinct points $p, q \in T$. Let X be the quotient topological space $T/\sim$, where $\sim$ is the equivalence relation generated by $p \sim q$.

(a) Show that X is triangulable.

(b) Calculate the homology groups of X. [You may use any result from lectures without proof.]

(c) Are all maps from X to S^2 homotopic to constant maps? Briefly justify your answer. [*Hint: Consider second homology groups.*]

21I Linear Analysis

(a) Let X be a non-zero, complex Banach space and $T \in \mathcal{B}(X)$. Define the *spectrum* $\sigma(T)$ and the *spectral radius* $r(T)$ of T. Briefly explain why $r(T)$ is well-defined. Show that $r(T) \leqslant \inf_{n \in \mathbb{N}} \|T^n\|^{1/n}$.

(b) State the complex version of the Stone–Weierstrass theorem and deduce it from the real version. Let K and L be compact Hausdorff spaces. Show that finite sums of functions on the product space $K \times L$ of the form $(x, y) \mapsto f(x)g(y)$ ($f \in C(K)$, $g \in C(L)$) are dense in $C(K \times L)$.

(c) Let $h \in C([0,1]^2)$. For $f \in C[0,1]$ define

$$T_h f(x) = \int_0^1 h(x,y) f(y) \, \mathrm{d}y, \quad x \in [0,1] \, .$$

Show that $T_h \in \mathcal{B}(C[0,1])$. [You do not need to show that $T_h f$ is continuous.] Show that the map $T \colon h \mapsto T_h$ from $C([0,1]^2) \to \mathcal{B}(C[0,1])$ is a bounded linear map.

(d) Now fix $h \in C([0,1]^2)$. Using part (b), or otherwise, show that T_h is the limit of a sequence of finite-rank operators. Explain briefly why this means that T_h is compact. By considering T_h^n, or otherwise, show that if $h(x,y) = 0$ whenever $0 \leqslant x \leqslant y \leqslant 1$, then $\sigma(T_h) = \{0\}$.

22H Analysis of Functions

(a) Give the definition of a *Lebesgue point* of a function $f \in L^1(\mathbb{R}^d)$.

(b) Let $f \in L^1(\mathbb{R}^d)$. Prove that almost every point in $\mathbb{R}^d$ is a Lebesgue point of f. [You may use without proof the Hardy–Littlewood maximal inequality and any results in the course about the density of continuous functions, provided they are stated clearly.]

(c) Let $f \in L^1(\mathbb{R})$ and let $g \in L^\infty(\mathbb{R})$. Suppose that g is supported on a compact set and $\int g(x)dx = 1$. For $n \in \mathbb{Z}_{\geqslant 1}$, let $g_n(x) = ng(nx)$. Prove that $\lim_{n\to\infty} f * g_n(x) = f(x)$ for all Lebesgue points x of f.

(d) Does the statement in part (c) remain true if we replace the assumption that g is supported on a compact set with the assumption that $g \in L^1(\mathbb{R})$, keeping all other assumptions unchanged? Give a proof or a counterexample.

23G Riemann Surfaces

Let $\Lambda = \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2 \subset \mathbb{C}$ be a lattice, where $\omega_1, \omega_2 \in \mathbb{C} \setminus \{0\}$ and $\mathrm{Im}(\omega_2/\omega_1) > 0$.

What is an *elliptic function* with respect to Λ? Show that if f is an elliptic function with respect to Λ, and $\mathcal{P}$ is a period parallelogram, then f has as many poles in $\mathcal{P}$ as it has zeroes (counted with multiplicity).

Define the *Weierstrass* $\wp$*-function* attached to Λ and show that it is an elliptic function. Write $\omega_3 = -\omega_1 - \omega_2$ and $e_j = \wp(\omega_j/2)$ $(1 \leqslant j \leqslant 3)$. Show that the e_j are distinct and that $\wp(z)$ satisfies the differential equation

$$\wp'(z)^2 = 4(\wp(z) - e_1)(\wp(z) - e_2)(\wp(z) - e_3).$$

Prove that

$$\frac{\wp'(z + \omega_1/2)}{\wp'(z)} = -\left(\frac{\wp(\omega_1/4) - e_1}{\wp(z) - e_1}\right)^2.$$

[*You may assume that the series*

$$\sum_{0 \neq \omega \in \Lambda} \frac{1}{|\omega|^\alpha}$$

converges for every real $\alpha > 2$.]

24F Algebraic Geometry

Let C be a (smooth irreducible projective) curve of genus g. State the Riemann–Roch theorem. Define the *canonical class* K_C and prove that $\deg(K_C) = 2g - 2$.

Suppose $g > 1$, and suppose there exists an effective divisor D of degree 2 on C with $\ell(D) = 2$. Prove that C is hyperelliptic. Deduce that every genus 2 curve is hyperelliptic.

Now suppose that $g = 3$ and C is not hyperelliptic. Show that the canonical map $\varphi_{K_C} : C \to \mathbb{P}^2$ is an embedding.

Determine how many points Q of a genus 3 hyperelliptic curve satisfy $\ell(2Q) = 2$. [*Hint: You may wish to apply Riemann–Roch to a suitably chosen divisor of degree 3.*]

25I Differential Geometry

Let $S \subset \mathbb{R}^3$ denote a surface.

(a) State what it means for a smooth map $\phi : S \to S$ to be an *isometry*.

(b) We call a smooth function $f : S \to \mathbb{R}$ an *isometry invariant* if given any isometry $\phi : S \to S$, we have $f(x) = f(\phi(x))$ for all $x \in S$. Deduce that if f is an isometry invariant and ϕ is an isometry, then ϕ must preserve the level sets of f.

(c) By using the Theorema Egregium, show that $K : S \to \mathbb{R}$ is an isometry invariant, where K denotes the Gaussian curvature of S. If a smooth diffeomorphism $\phi : S \to S$ preserves the level sets of K, does it necessarily follow that ϕ is an isometry?

(d) We say that two isometry invariants $f : S \to \mathbb{R}$, $g : S \to \mathbb{R}$ are *independent at a point* $x \in S$ if $\dim \ker df_x = 1$, $\dim \ker dg_x = 1$ and $\ker df_x + \ker dg_x = T_xS$. Show that if there exist two isometry invariants $f : S \to \mathbb{R}$, $g : S \to \mathbb{R}$ which are independent at some $x \in S$, then there exists a non-empty open neighbourhood $\tilde{S} \subset S$ of x such that any isometry $\phi : \tilde{S} \to \tilde{S}$ must equal the identity.

(e) Suppose the Gaussian curvature $K : S \to \mathbb{R}$ is such that $\dim \ker dK_x = 1$ for all $x \in S$. Show that at each $x \in S$, there exists a unique vector $e_x \in T_xS$ orthogonal to the level set of K through x and such that $dK_x(e_x) > 0$. Defining $g(x) = dK_x(e_x)$, show that g is an isometry invariant of S. [You may assume without proof that g is smooth.]

26L Probability and Measure

(a) State Fatou's lemma and deduce Lebesgue's dominated convergence theorem.

Let $(X_n)_{n \geqslant 1}$ be a sequence of real-valued random variables.

(b) Show that $X_n \to X$ almost surely implies $X_n \to X$ in probability.

For the rest of the question, assume that $X_n \to 0$ almost surely and $\mathbb{E}[X_n^2] \leqslant 1$ for all n.

(c) Does the sequence $(X_n)_{n \geqslant 1}$ converge in L^1? Justify your answer.

(d) Give an example to show that $(X_n)_{n \geqslant 1}$ does not necessarily satisfy the assumptions of the dominated convergence theorem.

27K Applied Probability

Let $(\xi_i)_{i \geqslant 1}$ be a sequence of i.i.d. non-negative random variables with $\mathbb{E}(\xi_1) = 1/\lambda$ for some $\lambda > 0$.

(a) Let M be a non-negative integer-valued random variable such that $\mathbb{E}(M) < \infty$ and suppose that for any $i \in \mathbb{N}$, the event $\{M \leqslant i\}$ depends only on the random variables $\{\xi_1, \ldots, \xi_i\}$. Show that

$$\mathbb{E}\left(\sum_{i=1}^{M} \xi_i\right) = \lambda^{-1}\mathbb{E}(M)\,.$$

Let $T_n = \sum_{i=1}^{n} \xi_i$ for $n \in \mathbb{N}$. Let $(N_t)_{t \geqslant 0}$ be the renewal process associated with the interarrival times (ξ_i).

(b) Show that

$$\mathbb{E}(T_{N_t+1}) = \lambda^{-1}(\mathbb{E}(N_t) + 1)\,.$$

(c) Show that $N_t/t \to \lambda$ almost surely as $t \to \infty$.

(d) Assume that ξ_1 is a bounded random variable. Show that $\mathbb{E}(N_t)/t \to \lambda$ as $t \to \infty$. [*Hint: Consider showing* $T_{N_t+1} - \xi_{N_t+1} \leqslant t < T_{N_t+1}$.]

28L Principles of Statistics

Let $X_1, \ldots, X_n$ be an independent random sample from an unknown distribution. It is desired to estimate a scalar parameter $\theta = g(\mathbb{E}(X_1))$. Denote the sample mean by $\bar{X}_n$ and set

$$\hat{\theta}_n = g(\bar{X}_n), \quad G(t) = \mathbb{P}(\sqrt{n}(\hat{\theta}_n - \theta) \leqslant t).$$

Assume here that G is strictly increasing and continuous and does not depend on n or θ.

(a) In the case where G is a known function, describe the construction of a $(1-\alpha)$-confidence interval C_n for θ.

(b) Explain what is meant by a *bootstrap sample* $X_1^*, \ldots, X_n^*$ constructed from $X_1, \ldots, X_n$.

Denote the sample mean of the bootstrap sample by $\bar{X}_n^*$ and set

$$\hat{\theta}_n^* = g(\bar{X}_n^*), \quad \hat{G}_n(t) = \mathbb{P}\left(\sqrt{n}(\hat{\theta}_n^* - \hat{\theta}_n) \leqslant t \,\middle|\, X_1, \ldots, X_n\right).$$

Assume that $\sup_{t \in \mathbb{R}} |\hat{G}_n(t) - G(t)| \to 0$ in probability.

(c) Describe the construction of an approximate $(1-\alpha)$-confidence interval $\hat{C}_n$ for θ using $\hat{G}_n$ and show that indeed

$$\mathbb{P}(\theta \in \hat{C}_n) \to 1 - \alpha.$$

(d) Why might you want to simulate a number of independent bootstrap samples $X_1^{*j}, \ldots, X_n^{*j}$ for $j = 1, \ldots, B$ in order to implement a similar procedure in practice? Explain how you would use the B samples. [You are not expected to justify convergence of the procedure you propose.]

29L Stochastic Financial Models

Consider a market with one risky asset with time-n price S_n and zero interest rate $r = 0$. Assume that the price evolves as

$$S_n = S_{n-1}(1 + R_n)$$

where the random variables $(R_n)_{n \geqslant 1}$ are independent and identically distributed. Let X_n denote an investor's wealth at time n, and suppose it evolves as

$$X_n = X_{n-1} - C_n + \eta_n R_n$$

where C_n denotes the investor's consumption in the time interval $(n-1, n]$, and η_n denotes the total amount invested in the market in the same time interval. Assume that the process $(C_n, \eta_n)_{n \geqslant 1}$, taking values in $\mathbb{R}^2$, is previsible with respect to the filtration generated by $(R_n)_{n \geqslant 1}$.

Given initial wealth X_0 and a non-random time horizon $N \geqslant 1$, the investor aims to maximise

$$\mathbb{E}\left(\sum_{n=1}^{N} U(C_n) + U(X_N)\right)$$

for a given utility function U, assumed to be bounded from above.

(a) Formulate the investor's problem as a stochastic optimal control problem: define its *value function* V and write down its Bellman equation.

From now on assume that $U(x) = -e^{-x}$ for all x. Assume also that R_1 takes both positive and negative values with positive probability.

(b) Rewrite the Bellman equation in terms of $G = -U$ and $F = -V$.

(c) Show that there is a solution of the form $F(n, x) = B_n e^{-A_n x}$ for some constants $A_n, B_n > 0$.

(d) Determine A_n and show that, for $\gamma = \inf_h \mathbb{E}(e^{-hR_1})$,

$$B_n = (N - n + 1)\gamma^{(N-n)/2}.$$

(e) Find the optimal consumption C_n^* over $(n-1, n]$ in terms of the optimized time-$(n-1)$ wealth X_{n-1}^* and γ.

30D Asymptotic Methods

Consider the behaviour of the function

$$I(x) = \int_{-1}^{\infty} \exp\left[x\left(t + it - \tfrac{1}{2}t^2\right)\right] dt,$$

as $x \to \infty$.

(a) Transform $I(x)$ into a complex contour integral, so that $t = z \in \mathbb{C}$, and determine the curve of steepest descent through the endpoint $z = -1$.

(b) Find the saddle point and the curve of steepest descent that passes through it.

(c) Deform the integration contour so that it passes through the saddle and draw the contour in the complex plane. Carefully estimate the contribution to $I(x)$ from any 'bridging' curves.

(d) Provide an argument for why the saddle contribution dominates the endpoint contribution, and then show that

$$I(x) \sim \alpha x^{-1/2} e^{ix},$$

as $x \to \infty$, where α is a constant you need to determine. What is the magnitude of the next-largest term in the asymptotic expansion of $I(x)$?

[You may quote, without proof, results concerning Laplace's method.]

31B Dynamical Systems

Consider the system

$$\ddot{x} = a + bx - \dot{x} - x^2 - \dot{x}^2, \qquad (*)$$

where a and b are real parameters.

(a) Write $(*)$ as a dynamical system in $\mathbb{R}^2$.

(b) Find all the fixed points as a function of a and b, and show the locations of any bifurcations in the (a, b)-plane.

(c) For the case $a = 0$, find the leading-order approximation to the extended centre manifold of the bifurcation of the origin as b varies. Find also the evolution equation on the extended centre manifold to leading order. Deduce the type of bifurcation and sketch the bifurcation diagram in the (b, x)-plane.

(d) Now repeat part (c) for the case $b = 1$ and letting a vary. Sketch the bifurcation diagram in the (a, x)-plane.

(e) Are there any periodic orbits for the case $a = -1$, $b = 1$? Hence deduce the existence or absence of a bounding function $V(x, y)$ in $\mathbb{R}^2$. In either case be careful to explain any theoretical results that you use.

32B Integrable Systems

Let $u : \mathbb{R}^n \to \mathbb{R}$. Find a one-parameter group of transformations G of $\mathbb{R}^{n+1}$ generated by a vector field

$$W = \phi u \frac{\partial}{\partial u} + \sum_{i=1}^{n} \alpha^i x^i \frac{\partial}{\partial x^i} \qquad (\dagger)$$

where $\phi, \alpha^1, \ldots, \alpha^n$ are constants not all equal to zero. State what is meant by an *invariant* of G, and show that $f : \mathbb{R}^{n+1} \to \mathbb{R}$ is an invariant if and only if $W(f) = 0$.

Find two independent vector fields W_1 and W_2 of the form (†) generating Lie point symmetries of a PDE

$$u_{yy} - u_{tx} + (uu_x)_x = 0. \qquad (\ddagger)$$

Find three functionally independent invariants of W_1 and do the same for W_2. Find two independent non-constant functions $f(x, y, t, u)$ and $g(x, y, t, u)$ which are invariant for both W_1 and W_2.

Show that, if solutions of (‡) are to be invariant under a group generated by

$$W = u\partial_u + x\partial_x + \frac{1}{2}y\partial_y, \quad W_3 = \partial_y,$$

they must be of the form $u = xF(t)$. Construct the corresponding group-invariant solutions.

33A Principles of Quantum Mechanics

A quantum system has Hamiltonian $H(t) = H_0 + \lambda V(t)$, where H_0 is time-independent with discrete, non-degenerate spectrum $H_0 |n\rangle = E_n |n\rangle$, and $|\lambda| \ll 1$. The system is prepared at $t = 0$ in a state that is a linear superposition of two normalised eigenstates $|i_1\rangle \neq |i_2\rangle$ of H_0,

$$|\psi(0)\rangle = \alpha_1 \, |i_1\rangle + \alpha_2 \, |i_2\rangle \,, \qquad |\alpha_1|^2 + |\alpha_2|^2 = 1, \qquad H_0 \, |i_k\rangle = E_{i_k} \, |i_k\rangle \;\; (k = 1, 2).$$

Consider the perturbation

$$V(t) = W \, e^{-\gamma t} \cos(\omega t) \, \Theta(t) \,,$$

where W is a time-independent Hermitian operator, ω and γ are positive constants, and Θ is the Heaviside function. For a fixed final eigenstate $|f\rangle$ of H_0 (with $f \neq i_1, i_2$), you are given the matrix elements

$$W_k \equiv \langle f| \, W \, |i_k\rangle \qquad (k = 1, 2) \,.$$

(a) Working in the interaction picture, compute the first-order transition amplitude $a_f^{(1)}(t)$ for $|\psi\rangle \to |f\rangle$.

(b) Take the limit $t \to \infty$ and express $a_f^{(1)}(\infty)$ in terms of α_k, W_k and the complex response factor

$$F_k \equiv \frac{1}{2} \left[\frac{1}{\gamma - i(\omega_{fi_k} + b_1 \omega)} + \frac{b_2}{\gamma - i(\omega_{fi_k} + b_3 \omega)} \right]$$

where b_1, b_2, b_3 and ω_{fi_k} are coefficients you should determine.

(c) Hence compute the first-order transition probability $P_{\psi \to f}(\infty)$ and briefly discuss the result.

(d) Now consider the more general initial state $|\phi_0\rangle = \sum_k \alpha_k \, |i_k\rangle$ for $k = 1, \ldots, N$ with $\sum_k |\alpha_k|^2 = 1$. Assume that W_k and F_k can be approximated by k-independent constants $\overline{W}$ and $\overline{F}$. For fixed λ, $\overline{F}$ and $\overline{W}$, determine the largest value of $P_{\phi \to f}(\infty)$ as function of N that can be obtained by optimally choosing α_k.

34E Applications of Quantum Mechanics

(a) Consider a three-dimensional system, with Hamiltonian H, that is invariant under the discrete translational symmetries of a Bravais lattice Λ. State and prove Bloch's theorem for H.

(b) A crystal has identical atoms at positions given by the vectors

$$a\left[n_1\hat{\mathbf{x}} + n_2\hat{\mathbf{y}} + n_3\hat{\mathbf{z}}\right] ,$$
$$a\left[\left(n_1 + \tfrac{1}{2}\right)\hat{\mathbf{x}} + \left(n_2 - \tfrac{1}{2}\right)\hat{\mathbf{y}} + \left(n_3 + \tfrac{1}{2}\right)\hat{\mathbf{z}}\right] ,$$
$$a\left[\left(n_1 + \tfrac{1}{2}\right)\hat{\mathbf{x}} + \left(n_2 + \tfrac{1}{2}\right)\hat{\mathbf{y}} + \left(n_3 - \tfrac{1}{2}\right)\hat{\mathbf{z}}\right] ,$$
$$a\left[\left(n_1 - \tfrac{1}{2}\right)\hat{\mathbf{x}} + \left(n_2 + \tfrac{1}{2}\right)\hat{\mathbf{y}} + \left(n_3 + \tfrac{1}{2}\right)\hat{\mathbf{z}}\right] ,$$

where (n_1, n_2, n_3) are arbitrary integers, a is a constant, and $\hat{\mathbf{x}}$, $\hat{\mathbf{y}}$, $\hat{\mathbf{z}}$ are mutually orthogonal unit vectors.

(i) Show that these vectors define a Bravais lattice Λ with basis vectors

$$\mathbf{a}_1 = \tfrac{a}{2}\left(-\hat{\mathbf{x}} + \hat{\mathbf{y}} + \hat{\mathbf{z}}\right) , \quad \mathbf{a}_2 = \tfrac{a}{2}\left(\hat{\mathbf{x}} - \hat{\mathbf{y}} + \hat{\mathbf{z}}\right) , \quad \mathbf{a}_3 = \tfrac{a}{2}\left(\hat{\mathbf{x}} + \hat{\mathbf{y}} - \hat{\mathbf{z}}\right) .$$

(ii) Determine the volume of the primitive cell.

(iii) Find basis vectors $\mathbf{b}_1$, $\mathbf{b}_2$, and $\mathbf{b}_3$ for the reciprocal lattice and determine the volume of the unit cell of the reciprocal lattice.

(c) Consider the dynamics of a single electron on the lattice Λ given above, where the Hamiltonian is described by a tight-binding model given by

$$H = \sum_{\mathbf{r}\in\Lambda}\left[E_0|\mathbf{r}\rangle\langle\mathbf{r}| - \mu\sum_i\Big(|\mathbf{r}\rangle\langle\mathbf{r}+\mathbf{a}_i| + |\mathbf{r}+\mathbf{a}_i\rangle\langle\mathbf{r}|\Big)\right],$$

where E_0 and μ are real constants.

(i) Write down the form that the eigenstates of the Hamiltonian take, and determine the energy spectrum as a function of the wave vector in the Brillouin zone.

(ii) What is the width of the energy band, $\Delta E = E_{\max} - E_{\min}$, in this zone?

(iii) Show that, for small crystal momentum, the dispersion relation takes the form of a free particle and determine the effective masses.

[*Hint: You may use without proof the fact that a matrix of the form*

$$\begin{pmatrix} \rho & \tau & \tau \\ \tau & \rho & \tau \\ \tau & \tau & \rho \end{pmatrix}$$

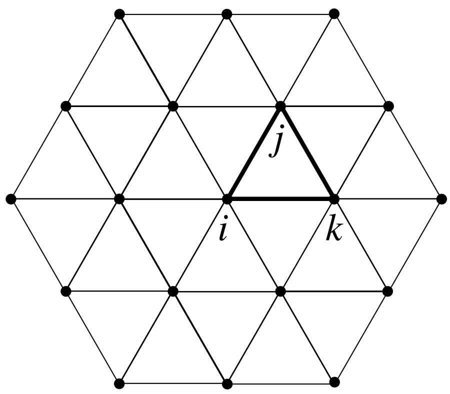
has eigenvalues $\rho - \tau$, $\rho - \tau$, *and* $\rho + 2\tau$.]

35E Statistical Physics

Consider N spins, $s_i \in \{-1, 1\}$, placed on the vertices $i, j, k \ldots$ of a triangular lattice with hexagonal symmetry (see figure) in $d = 2$ spatial dimensions, subject to the following Hamiltonian

$$H = -B \sum_i s_i - J \sum_{\langle ij \rangle} s_i s_j - K \sum_{\langle ijk \rangle} s_i s_j s_k \,,$$

where $\langle ij \rangle$ indicates a sum over nearest neighbour edges and $\langle ijk \rangle$ represents a sum over elemental triangles (as depicted in the figure), and $B, J, K \in \mathbb{R}$ are parameters.



(a) Assuming the validity of mean field theory, calculate the canonical partition function $Z(N, \beta)$, written in the form of a sum over possible values of the total magnetisation $M = \sum_i s_i$.

(b) Using Stirling's formula: $\ln N! \approx N \ln N - N$, show that for large N the partition function can be approximated as an integral

$$Z_N(\beta, h) \approx \int_{-1}^{1} \exp[-N f(m)] \; dm,$$

where the average magnetisation is $m = M/N$, and

$$f(m) = P(m) + \frac{1+m}{2} \ln \frac{1+m}{2} + \frac{1-m}{2} \ln \frac{1-m}{2},$$

where $P(m)$ is a polynomial in m that you should determine.

(c) Show that this integral is dominated at a value of m satisfying the equation

$$m = \tanh[Q(m)] \,,$$

where $Q(m)$ is a polynomial in m that you should determine.

[*Hint: You may wish to use the identity* $\dfrac{1+\tanh(x)}{1-\tanh(x)} = e^{2x}$.]

(d) Suppose now that $B = J = 0$ and $K > 0$ is fixed. Prove that, if the system is gradually cooled off, there must be a phase transition from $m = 0$ to $m \neq 0$ at some critical value β_*. [You do not need to calculate the value of β_*.]

36D Electrodynamics

In the Lorenz gauge, $\partial_\mu A^\mu = 0$, the 4-vector potential A^μ due to a localised charge/current distribution with conserved current density 4-vector J^μ is

$$A^\mu(t,\mathbf{x}) = \frac{\mu_0}{4\pi}\int \frac{J^\mu(t-|\mathbf{x}-\mathbf{x}'|/c,\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|}\, d^3\mathbf{x}'\,.$$

For charges in non-relativistic periodic motion localised near the origin $\mathbf{x} = \mathbf{0}$, use the dipole approximation to show that at sufficiently large distances $r = |\mathbf{x}|$ the magnetic field is given approximately by

$$\mathbf{B}(t,\mathbf{x}) \approx \frac{\mu_0}{4\pi rc}\ddot{\mathbf{p}}(t-r/c)\times\hat{\mathbf{x}}\,,$$

where $\mathbf{p}$ is the electric dipole moment, which you should define, and $\hat{\mathbf{x}}$ is a unit vector in the radial direction. State clearly the conditions under which this approximate result holds.

Given that the electric field at such large distances is $\mathbf{E}(t,\mathbf{x}) \approx c\mathbf{B}(t,\mathbf{x})\times\hat{\mathbf{x}}$, determine the Poynting vector and show that the instantaneous power P radiated per solid angle is

$$\frac{dP}{d\Omega} = \frac{\mu_0}{(4\pi)^2 c}\left[|\ddot{\mathbf{p}}|^2 - (\hat{\mathbf{x}}\cdot\ddot{\mathbf{p}})^2\right]\,.$$

A point charge q moves non-relativistically in a circle of radius $\mathcal{R}$ with angular frequency ω in the plane $z = 0$, so that $(x,y) = \mathcal{R}(\cos\omega t, \sin\omega t)$. Find the total radiated power in the dipole approximation and its time-averaged angular distribution.

Show further that the electric field along the z-axis is circularly polarized with

$$\mathbf{E}(t,\mathbf{x}) \approx \frac{\mu_0 q\omega^2\mathcal{R}}{4\pi r}\left(\cos\omega\tau, \sin\omega\tau, 0\right)\,,$$

where $\tau = t - r/c$, and along the x-axis is linearly polarized with

$$\mathbf{E}(t,\mathbf{x}) \approx \frac{\mu_0 q\omega^2\mathcal{R}}{4\pi r}\left(0, \sin\omega\tau, 0\right)\,.$$

If the angular frequency is increased so that the charge moves relativistically, determine the total radiated power P by considering the acceleration in the instantaneous rest frame, or otherwise.

37A General Relativity

(a) Consider a manifold with metric tensor $g_{\mu\nu}$ and metric connection $\Gamma_{\alpha\ \beta}^{\ \gamma} = \Gamma_{\beta\ \alpha}^{\ \gamma}$. From the Ricci identity

$$\nabla_\alpha \nabla_\beta V^\mu \, - \, \nabla_\beta \nabla_\alpha V^\mu \, = \, R^\mu{}_{\nu\alpha\beta} V^\nu$$

for any vector field V^μ, and using the definition of ∇_α, find an explicit expression for the Riemann curvature tensor $R^\mu{}_{\nu\alpha\beta}$ in terms of the connection.

(b) State clearly the symmetries of the Riemann tensor $R_{\mu\nu\alpha\beta}$. Explain briefly, without carrying out calculations, how your expression for the Riemann curvature tensor allows all symmetries to be easily checked.

(c) Given vector fields V^μ and W^μ, show that the commutator defined by

$$[\,V,\,W\,]^\mu \, = \, V^\alpha \partial_\alpha W^\mu \, - \, W^\alpha \partial_\alpha V^\mu$$

is also a vector field.

(d) A vector field V^μ is defined to be a Killing vector field if

$$\nabla_\alpha V_\beta \, + \, \nabla_\beta V_\alpha \, = \, 0\,.$$

Use this property to obtain an expression for $\nabla_\alpha \nabla_\beta V^\mu$ in terms of the Riemann tensor contracted with the Killing vector field.

(e) Show that if V^μ and W^μ are Killing vector fields, then the commutator $[V, W]^\mu$ is also a Killing vector field.

38C Fluid Dynamics

Viscous incompressible fluid is contained in a very long cylindrical pipe of radius a. The fluid in the central region $0 < r < c < a$ has viscosity μ_c, while that in the surrounding annular region $c < r < a$ has viscosity μ_a. The fluid moves along the pipe due to a fixed pressure difference applied to the ends of the pipe and you may assume that the flow is unidirectional with $\mathbf{u} = u(r)\mathbf{e}_z$ and that the pressure gradient $G \equiv -\partial p/\partial z$ is uniform.

Starting from the steady Navier–Stokes equations, derive the equations satisfied by $u(r)$ in each region, and state all the boundary conditions. Hence determine the flow profiles $u_c(r)$ and $u_a(r)$ in the central and annular regions respectively. Why does $u_a(r)$ not depend on μ_c?

The mechanical energy equation for viscous flow in a fixed volume V with boundary ∂V is given (in standard notation) by

$$\frac{d}{dt}\int_V \tfrac{1}{2}\rho u^2 \, dV + \int_{\partial V} \tfrac{1}{2}\rho u^2 \, \mathbf{u}\cdot\mathbf{n} \, dS = \int_{\partial V} \mathbf{u}\cdot\boldsymbol{\sigma}\cdot\mathbf{n} \, dS - 2\mu \int_V e_{ij}e_{ij} \, dV \, .$$

Verify this equation holds for the annular volume $V = \{c < r < a, \; 0 < z < L\}$ by calculating the various terms.

[*Hint: In cylindrical polar coordinates* (r, θ, z),

$$e_{rz} = \frac{1}{2}\left(\frac{\partial u_r}{\partial z} + \frac{\partial u_z}{\partial r}\right) \qquad \text{and} \qquad \nabla^2 f(r) = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial f}{\partial r}\right) ,$$

for a function $f = f(r)$.]

39C Waves

The response of a linear wave system to a perturbation at time $t = 0$ can be expressed in the form

$$\eta(x,t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(k) \exp(ikx - i\omega(k)t) \, \mathrm{d}k \ ,$$

where $\omega(k)$ and $F(k)$ are known functions of k, and $\omega(k)$ is real for real k.

(a) By using the method of stationary phase, calculate the leading-order behaviour of $\eta(x,t)$ as $t \to \infty$ with x/t fixed.

[*You may need the result,* $\int_{-\infty}^{\infty} e^{iu^2} \, \mathrm{d}u = \sqrt{\pi} e^{i\pi/4}$.]

(b) The surface of deep water is given by $y = \eta(x,t)$ and extends over $-\infty < x < \infty$. At $t = 0$, the surface is disturbed with initial conditions

$$\eta(x,0) = x \exp(-x^2) \ , \qquad \frac{\partial \eta}{\partial t}(x,0) = 0 \ .$$

Given that the motion of the water is incompressible, with velocity potential $\phi(x,y,t)$ satisfying

$$\nabla^2 \phi = 0 \ ,$$

and with boundary conditions

$$\frac{\partial \phi}{\partial y} = \frac{\partial \eta}{\partial t} \ , \qquad \frac{\partial \phi}{\partial t} = \frac{\partial^2 \eta}{\partial x^2},$$

on $y = 0$, in suitable units, show that the dispersion relation is

$$\omega^2 = |k|^3 \ .$$

Determine the behaviour of $\eta(Vt,t)$ as $t \to \infty$ for a fixed value of $V > 0$.

[*You may need the result,* $\int_{-\infty}^{\infty} e^{-u^2} \, \mathrm{d}u = \sqrt{\pi}$.]

Describe briefly the differences between what would be observed on the water surface by observers travelling at different speeds.

40B Numerical Analysis

The Poisson equation $\nabla^2 u = f$, where f is sufficiently smooth, in the unit square $\Omega = [0,1] \times [0,1]$, is equipped with zero boundary conditions on $\partial\Omega$. It is discretised with the 9-point formula:

$$\begin{aligned}\Gamma_9[u_{m,n}] := &-\frac{10}{3}u_{m,n} + \frac{2}{3}\big(u_{m+1,n} + u_{m-1,n} + u_{m,n+1} + u_{m,n-1}\big)\\ &+ \frac{1}{6}\big(u_{m+1,n+1} + u_{m+1,n-1} + u_{m-1,n+1} + u_{m-1,n-1}\big) = h^2 f_{m,n},\end{aligned}$$

where $1 \leqslant m,n \leqslant M$, $h = 1/(M+1)$ is the step size, (mh, nh) are grid points, $f_{m,n} = f(mh, nh)$ and $u_{m,n} \approx u(mh, nh)$.

(a) Find the local error of approximation of the 9-point formula.

(b) Let $\boldsymbol{e} \in \mathbb{R}^{M\times M}$ be the error, that is $e_{m,n} = u_{m,n} - u(mh, nh)$. Determine the global error $\|\boldsymbol{e}\|_2$ as a function of h. [You may assume that the 9-point formula induces a matrix A_h (similar to the 5-point formula discussed in lectures) for which $\|A_h^{-1}\|_2 \leqslant C/h^2$, for some constant $C > 0$, independent of the ordering of the grid.]

(c) Determine the local and global error of the 9-point formula if f satisfies the Laplace equation $\nabla^2 f = 0$.

(d) Show that the modified 9-point scheme

$$\begin{aligned}\Gamma_9[u_{m,n}] &= h^2 f_{m,n} + \frac{1}{12}h^2\Gamma_5[f_{m,n}]\\ &:= h^2 f_{m,n} + \frac{1}{12}h^2\big(f_{m+1,n} + f_{m-1,n} + f_{m,n+1} + f_{m,n-1} - 4f_{m,n}\big)\end{aligned}$$

has the same local error as that obtained in part (c).

END OF PAPER