UNIVERSITY OF CAMBRIDGE

MAT2

MATHEMATICAL TRIPOS Part II

Tuesday 9 June 2026 2:00pm to 5:00pm

PAPER 2

Before you begin read these instructions carefully.

The examination paper is divided into two sections. Each question in Section II carries twice the number of marks of each question in Section I. Section II questions also carry an alpha or beta quality mark and Section I questions carry a beta quality mark.

Candidates may obtain credit from attempts on **at most six questions** from Section I and from any number of questions from Section II.

Write on **one side** of the paper only and begin each answer on a separate sheet.

Write legibly; otherwise you place yourself at a grave disadvantage.

At the end of the examination:

Separate your answers to each question.

Complete a gold cover sheet **for each question** that you have attempted, and place it at the front of your answer to that question.

Complete a green main cover sheet listing **all the questions** that you have attempted.

Every cover sheet must also show your Blind Grade Number and desk number.

Tie up your answers and cover sheets into **a single bundle**, with the main cover sheet on the top, and then the cover sheet and answer for each question, in the numerical order of the questions.

STATIONERY REQUIREMENTS
Gold cover sheets
Green main cover sheet
Script paper
Rough paper

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

SECTION I

1G Number Theory

Define the *Möbius function* μ and the *Riemann zeta function* ζ.

Show that

$$\mu(n)^2 = \sum_{d^2|n} \mu(d).$$

[You may assume, without proof, properties of the Möbius function covered in lectures.]

Deduce that for $s \in \mathbb{C}$ with $\Re(s) > 1$,

$$\sum_{n=1}^{\infty} \frac{\mu(n)^2}{n^s} = \frac{\zeta(s)}{\zeta(2s)}.$$

[*Hint: You may wish to use the fact that*

$$\zeta(s) \cdot \sum_{n=1}^{\infty} \frac{\mu(n)}{n^s} = 1$$

for $s \in \mathbb{C}$ *with* $\Re(s) > 1$.]

2I Topics in Analysis

State Runge's theorem on uniform approximation of holomorphic functions by polynomials.

Let $\Delta = \{z \in \mathbb{C} : |z| < 1\}$ and let $L^+ = \{it : t \in \mathbb{R}, t \geqslant 0\} \subset \mathbb{C}$. Prove that there is a sequence of complex polynomials which converges to $1/z$ uniformly on each compact subset of $\Delta \setminus L^+$.

3H Coding & Cryptography

Let (X_n) be a source of random variables taking values in some finite alphabet Σ.

(a) What does it mean for the source to be *Bernoulli*?

(b) What does it mean for the source to be *reliably encodable at rate* r?

(c) What is the *information rate* of the source?

(d) What does it mean for the source to satisfy the *asymptotic equipartition property*?

Stating clearly any results you use, show that a Bernoulli source has information rate $H(X)$, where $H(X)$ denotes the entropy of X.

4I Automata and Formal Languages

(a) For a deterministic finite state automaton (DFA) $D = (Q, \Sigma, \delta, q_0, F)$ with extended transition function $\widehat{\delta}$, define the notion of an *inaccessible state*.

(b) Given a DFA D without inaccessible states, give the definition of the equivalence relation $\sim$, the function δ', and the set F' such that

$$D/\sim\ = (Q/\sim, \Sigma, \delta', [q_0], F')$$

is the minimal DFA for D. Show that δ' is well-defined.

(c) State (without proof) a theorem from the course relating the minimal DFA for D defined in part (b) to the smallest size of a DFA E such that $\mathcal{L}(D) = \mathcal{L}(E)$.

(d) Suppose that $\Sigma = \{a\}$ and fix $n \geqslant 1$. Let $L = \{a^{2m} : m \geqslant 0\}$. Construct an example of a DFA D without inaccessible states with $\mathcal{L}(D) = L$ that has $2n$ states. Prove that $D/\sim$ has two states.

5K Statistical Modelling

Write down the *logistic regression* model for $(X_i, Y_i) \in \mathbb{R}^p \times \{0, 1\}$, $i = 1, \ldots, n$. Then show that the logit function

$$\text{logit}(\pi) = \log \frac{\pi}{1-\pi}, \ 0 < \pi < 1,$$

is the canonical link function for the binomial generalised linear model. Name an algorithm that you can use for fitting logistic regression with a dataset. [You do not need to further describe the algorithm.]

Recall that linear discriminant analysis assumes

$$X_i \mid Y_i = 0 \sim \mathrm{N}(\mu_0, \Sigma), \quad X_i \mid Y_i = 1 \sim \mathrm{N}(\mu_1, \Sigma),$$

where $\mu_0, \mu_1 \in \mathbb{R}^p$ and $\Sigma \in \mathbb{R}^{p \times p}$ is positive definite. Show that this is a submodel of logistic regression with an intercept term (that is, all assumptions of the logistic regression are satisfied).

6C Mathematical Biology

Spruce budworms have a population density $n(x,t)$ and are restricted to lie in an interval $x \in [0, L]$. The population is described by the diffusion equation

$$\frac{\partial n}{\partial t} = D\frac{\partial^2 n}{\partial x^2} \ ,$$

where D is a constant.

(a) What boundary conditions at $x = 0$ and $x = L$ would ensure that the total population is conserved?

(b) The population grows at a rate α per capita. What change should we make to the diffusion equation to capture this?

(c) Suppose the surface of the domain is hostile, so that we now apply the boundary conditions $n(0,t) = n(L,t) = 0$. Derive a condition on the parameters D, α, and L that is necessary if the population is not to die out.

7D Further Complex Methods

Consider a function $f(t)$ such that $f(t) = 0$ for $t < 0$ and $f(t) \neq 0$ for $t \geqslant 0$.

Define the *Laplace transform* $\hat{f}(p)$ and the *inverse Laplace transform.* Explain what a *Bromwich contour* is. Explain how the Bromwich inversion integral is evaluated by closing the contour for $t > 0$ and for $t < 0$.

Let $u(x,t)$ be the concentration of a chemical compound on a half-line $x > 0$ for $t > 0$. It obeys the partial differential equation

$$u_t = D\, u_{xx} + e^{-t}, \qquad x > 0,\ t > 0,$$

for $D > 0$ with initial condition $u(x,0) = 0$ for $x > 0$ and boundary condition $u(0,t) = e^{-t}$ for $t > 0$.

Write down an integral representation for $u(x,t)$ using the Bromwich contour, identifying clearly any singularities and explaining how one would choose a contour around them.

8A Classical Dynamics

A pendulum consists of a light rod of length ℓ with a mass m at the end. It oscillates in the xy-plane making an angle $\theta(t)$ to the downward vertical, along which gravity acts. Assume the constant gravitational acceleration g points in the negative y direction. The pendulum's pivot is attached to a lift that descends with constant acceleration a, so that the position of the pivot as a function of time is $x_{\text{pivot}}(t) = 0$ and $y_{\text{pivot}}(t) = -\frac{1}{2}at^2$.

(a) Derive the Lagrangian for the system, and hence obtain the equation of motion for $\theta(t)$. What is the motion when $a = g$?

(b) Find the equilibrium configurations for arbitrary a. Determine which configuration is stable when $a < g$, and when $a > g$.

9D Cosmology

Consider a gas of particles in a cubic box of side length L and volume $V = L^3$. Let $\bar{n}(p)\,dp$ be the mean number of particles in the interval $[p, p+dp]$, where p is the magnitude of the momentum. The total particle number and energy are

$$N = \int_0^\infty \bar{n}(p)\,dp\,, \qquad E = \int_0^\infty \mathcal{E}(p)\,\bar{n}(p)\,dp\,,$$

where the single-particle energy is $\mathcal{E}(p) = \sqrt{p^2c^2 + m^2c^4}$.

(a) Assuming the allowed momentum modes scale as $p \propto L^{-1}$, show that an infinitesimal change in the volume implies

$$\frac{dp}{dV} = -\frac{p}{3V}.$$

Consider a quasistatic change in volume at fixed particle number N and entropy S. Using the induced change in the momenta to compute the corresponding change in the total energy and, identifying it with the thermodynamic relation $dE = -P\,dV$, obtain the pressure

$$P = \frac{1}{3V}\int_0^\infty p\,\frac{d\mathcal{E}}{dp}\,\bar{n}(p)\,dp\,. \qquad (\dagger)$$

(b) Specialize to an ultrarelativistic gas with $\mathcal{E}(p) = pc$ and show that the equation of state is $P = \frac{1}{3}\rho$, where the energy density is $\rho = E/V$. In an expanding universe, assume the gas remains in equilibrium with energy density $\rho \propto T^4$ and number density $n \propto T^3$. Briefly explain why the temperature falls inversely with the scale factor $T \propto a^{-1}$.

(c) In the non-relativistic limit for a gas at temperature T, the typical momentum is given by $\langle p^2/2m\rangle \approx \frac{3}{2}kT \ll mc^2$. Use ($\dagger$) and expand $\mathcal{E}(p)$ in this limit to obtain the equation of state $P = \frac{2}{3}u$, where $u \equiv U/V$ with the internal energy U given by $U = E - N\,mc^2$. Briefly explain how the temperature T is expected to fall with the scale factor a.

10E Quantum Information and Computation

Consider a quantum database whose $N = 2^n$ items are labelled by the computational basis states $\{|x\rangle : x \in \{0,1\}^n\}$. Let $|\psi_0\rangle$ denotes the uniform superposition state.

(a) If the database has a single marked item, say $|x_0\rangle$, then Grover's search algorithm uses the Grover iteration operator $Q := -I_{|\psi_0\rangle} I_{|x_0\rangle}$. For any $|\psi\rangle \in (\mathbb{C}^2)^{\otimes n}$, give the definition of the reflection operator $I_{|\psi\rangle}$, and specify the action of $I_{|x_0\rangle}$ on any computational basis state.

(b) Next, suppose the database has $1 < M < N$ marked items, which form the set

$$G = \{|g_1\rangle, |g_2\rangle, \ldots, |g_M\rangle \, : \, g_i \in \{0,1\}^n \text{ for } i = 1, 2, \ldots, M\}.$$

The Grover iteration operator used to search for a marked item is now $Q_M := -I_{|\psi_0\rangle} I_G$, for a suitable reflection operator I_G.

(i) Write the reflection operator I_G in terms of the states in G.

(ii) Show that Q_M preserves the two-dimensional subspace $\mathcal{S}$ spanned by $|\psi_0\rangle$ and the state $|\psi_G\rangle$, the equal superposition of all marked items.

(iii) Construct an orthonormal basis for $\mathcal{S}$ and express Q_M as a 2×2 matrix in this basis.

(iv) Show that Q_M can be interpreted as a rotation in the subspace $\mathcal{S}$. Defining the angle α through $\sin^2 \alpha := M/N$, determine the angle that a vector in $\mathcal{S}$ is rotated after one application of Q_M.

(v) Assuming that $M \ll N$ and starting from $|\psi_0\rangle$, find the number of repeated applications of Q_M needed to find a marked item with high probability.

SECTION II

11I Topics in Analysis

(a) Let $\gamma : [0,1] \to \mathbb{C} \setminus \{0\}$ be a continuous map with $\gamma(0) = \gamma(1)$. Define the *winding number* $w(\gamma; 0)$ about the origin.

Show that the complex conjugate path $\overline{\gamma}$ satisfies $w(\overline{\gamma}\,; 0) = w(\gamma^{-}; 0)$, where $\gamma^{-}(t) = \gamma(1-t)$.

(b) State precisely and prove a theorem about homotopy invariance of the winding number. [You may assume that if $\sigma : [0,1] \to \mathbb{C}$ is continuous, $\sigma(0) = \sigma(1)$ and $|\sigma(t)| < |\gamma(t)|$ for each $0 \leqslant t \leqslant 1$, then $w(\gamma + \sigma; 0) = w(\gamma; 0)$.]

(c) Let $D = \{z \in \mathbb{C} : |z| \leqslant 1\}$ and $C = \{z \in \mathbb{C} : |z| = 1\}$. If $g : D \to C$ is continuous, prove that for each non-zero integer n, there exists a point $z \in C$ such that $z^n = -g(z)$.

12H Coding & Cryptography

(a) State and prove Shannon's noiseless coding theorem. [You may use Gibbs' and Kraft's inequalities as long as they are clearly stated].

(b) Consider a random variable with m equiprobable outcomes.

(i) What is the entropy of this information source?

(ii) Describe an optimal binary code for this source when $m = 2^b$.

(iii) For arbitrary $m \geqslant 1$, show that in an optimal binary code the codeword lengths differ by at most 1. Hence, or otherwise, find the expected word length for such an optimal code.

13D Further Complex Methods

Consider a multivalued function $f(z) : \mathbb{C} \to \mathbb{C}$. Define a *branch point*, a *branch cut*, and a *branch* of $f(z)$.

Explain how the function $\arcsin(z)$ can be defined as a complex integral and why it is a multivalued function. How can a single-valued function $\mathrm{Arcsin}(z)$ be constructed such that $\mathrm{Arcsin}(0) = 0$?

Let

$$\arccos(z) = \int_z^1 \frac{dt}{\sqrt{1-t^2}}.$$

Show, using branch cut integration around a suitably chosen branch cut, that for a suitably chosen path $\gamma : 0+ \to 1$

$$\arccos(0+) = \frac{\pi}{2}$$

defining carefully the point $z = 0+$.

Define a single-valued branch of $\mathrm{Arccos}(z)$ such that

$$\mathrm{Arcsin}(z) + \mathrm{Arccos}(z) = \frac{\pi}{2}$$

on its domain. State the domain on which your definition is intended to hold.

Consider now

$$\arctan(z) := \int_0^z \frac{dt}{1+t^2}.$$

By calculating values of $\arctan(1)$ show that it is a multi-valued function. How can it be made to a single-valued function $\mathrm{Arctan}(z)$?

For z in the domain of $\mathrm{Arctan}(z)$, show that

$$\mathrm{Arctan}(z) = \mathrm{Arcsin}\left(\frac{z}{\sqrt{1+z^2}}\right).$$

14A Classical Dynamics

A funnel consists of a cylindrical tube of radius $2a$, opening out into a smooth cone whose surface makes an angle $\alpha \in (0, \pi/2)$ with the axis of the cylinder. The funnel is fixed in place with its symmetry axis being vertical and the cylinder at the bottom. A spherical marble of uniform density, mass m, radius a and moment of inertia $(2/5)ma^2$ rolls on the conical surface of the funnel without slipping or twisting (that is, it has no spin about the normal to the surface).

(a) Using appropriate generalised coordinates for unconstrained degrees of freedom, write down the Lagrangian of the system, solving any constraints explicitly.

(b) Identify and interpret two independent conserved quantities.

(c) The marble is launched along the cone horizontally with initial speed v_0 at a height where the cone has radius $8a$ at its point of contact with the marble. Show that the marble never enters the cylinder provided

$$v_0^2 \geqslant v_{0\,\mathrm{min}}^2 = \frac{4}{7} ga \cot\alpha \,,$$

and briefly explain physically why the marble stays out of the cylinder for any $v_0 > v_{0\,\mathrm{min}}$.

(d) In Lagrangian mechanics, what is meant by a *holonomic constraint*? If the constraint is implemented using a Lagrange multiplier, define the *constraint force*. Show that when the constraint has no explicit time dependence, the constraint force does no work during the motion.

(e) The marble is now relaunched horizontally from the same height, but with a speed ensuring it is at the minimum of its effective potential for the radial motion. Calculate the magnitude of the constraint force associated to keeping the particle on the cone's surface.

15E Quantum Information and Computation

This question concerns quantum key distribution using two different protocols, both in the absence and presence of noise, as described below.

(a) As in the standard BB84 quantum key distribution protocol, Alice and Bob are spatially separated and can communicate through both classical and quantum channels. An eavesdropper, Eve, performs an intercept–resend attack by measuring each qubit (sent by Alice) in the *Breidbart basis* $\{|b_0\rangle, |b_1\rangle\}$, where

$$|b_0\rangle = \cos\left(\frac{\pi}{8}\right)|0\rangle + \sin\left(\frac{\pi}{8}\right)|1\rangle, \quad |b_1\rangle = \sin\left(\frac{\pi}{8}\right)|0\rangle - \cos\left(\frac{\pi}{8}\right)|1\rangle.$$

and subsequently sending the post-measurement state on to Bob. Assume there is no noise in transmission.

Briefly describe the procedure Alice and Bob take to generate a private encryption key and to test whether eavesdropping has occurred.

Given that Bob chooses his basis randomly, calculate the probability (for a single sent qubit) that Alice and Bob can detect the eavesdropping. [*Hint: First determine the probability that Bob gets a different outcome from Alice's bit when he measures in the same basis as Alice.*]

(b) Now consider a six-state quantum key distribution protocol which is a generalisation of BB84 in the following way. Alice and Bob use the following *three* bases to respectively send and measure a qubit: the computational basis, the Hadamard basis $\{|+\rangle, |-\rangle\}$, and the circular basis $\{|R\rangle, |L\rangle\}$, where $|R\rangle = \frac{1}{\sqrt{2}}(|0\rangle + i|1\rangle)$, and $|L\rangle = \frac{1}{\sqrt{2}}(|0\rangle - i|1\rangle)$. Alice and Bob choose their bases uniformly at random for each qubit. Eavesdropper Eve performs an intercept–resend attack, but she can *only* measure in the computational, Hadamard, and circular bases. [You may use without proof that these bases are *mutually unbiased*: that is, for any pair of distinct bases $\{|e_0\rangle, |e_1\rangle\}$ and $\{|e'_0\rangle, |e'_1\rangle\}$, one has $|\langle e_i|e'_j\rangle|^2 = \frac{1}{2}$ for $i, j \in \{0, 1\}$. Again assume that there is no noise in transmission.]

If Bob measures in the same basis that Alice transmits, calculate the probability that Bob gets a measurement outcome that is different from Alice's encoded bit, and hence determine the probability that, with a single transmitted qubit, Alice and Bob can detect Eve's attack.

(c) Now consider the possibility that on the way from Alice to Bob, transmitted qubits are affected by noise which acts in the following way. Let $p \in [0, 1]$. With probability $p/4$ an X gate gets applied to the qubit, with probability $p/4$ a Y gate gets applied to the qubit, with probability $p/4$ a Z gate gets applied to the qubit, and with probability $1 - 3p/4$ the qubit gets sent through unchanged.

Suppose that Alice transmits state $|\psi\rangle$ and that Bob measures in the computational basis. Assuming that there is no eavesdropper (no Eve), what is the probability that Bob's measurement outcome is 0? [You may use without proof that, for any pair of qubit states $|\phi\rangle$ and $|\psi\rangle$, $|\langle\phi|X\psi\rangle|^2 + |\langle\phi|Y\psi\rangle|^2 + |\langle\phi|Z\psi\rangle|^2 = 2 - |\langle\phi|\psi\rangle|^2$.]

Now consider parts (a) and (b) in the presence of noise. In each case, determine the range of p for which Alice and Bob can detect Eve's eavesdropping. State any additional assumptions you must make.

16J Logic & Set Theory

Let $n \geqslant 2$ be a natural number and let $H_n = \{0\} \cup \{1/k;\ 1 \leqslant k < n\}$. For $a, b \in H_n$, let

$$a \to b = \begin{cases} 1 & \text{if } a \leqslant b, \text{ and} \\ b & \text{if } a > b. \end{cases}$$

Let $\mathbf{H}_n = (H_n, \to, 0)$. We say that a map v from the set of implicative formulas (i.e., formulas that do not contain the symbols $\wedge$, $\vee$, $\neg$, or $\top$) to H_n is an *n-valuation* if it is a homomorphism into $\mathbf{H}_n$. An implicative formula φ is called *n-valid* if for all n-valuations v, we have that $v(\varphi) = 1$.

(a) Let $a, b \in H_n$ and $P(a,b) = ((a \to b) \to a) \to a$. Show that

$$P(a,b) = \begin{cases} 1 & \text{if } a \leqslant b, \text{ and} \\ a & \text{if } a > b. \end{cases}$$

(b) Let $a, b, c \in H_n$ and $T(a,b,c) = (b \to c) \to a$. Show that

$$T(a,b,c) = \begin{cases} 1 & \text{if } b > c \text{ and } c \leqslant a, \text{ and} \\ a & \text{otherwise.} \end{cases}$$

(c) State the axioms of types 1, 2, and 3 used in proofs in propositional logic.

(d) Show that all axioms of type 1 and 2 are n-valid for all $n \geqslant 2$ and that, for any $n \geqslant 3$, there are axioms of type 3 that are not n-valid.

(e) Using parts (a) and (b), or otherwise, find a formula that is 3-valid, but not 4-valid. Justify your claim.

17J Graph Theory

(a) State and prove Ramsey's theorem for $R(s,t)$ for $s, t \geqslant 2$. Deduce that the Ramsey number $R(t)$ satisfies $R(t) \leqslant 2^{2t}$.

(b) Let the positive integers be 2-coloured. Show that, for any integer $n \geqslant 3$, there exists a monochromatic set $\{x_1, x_2, \ldots, x_n\}$, where $x_1, \ldots, x_n$ are not necessarily distinct, such that

$$x_1 + x_2 + \ldots + x_{n-1} = x_n.$$

(c) Show that we can 2-colour $\mathbb{R}^2$ such that there is no equilateral triangle with sidelength 1 whose vertices have the same colour. [*Hint: Consider colouring* $\mathbb{R}^2$ *in vertical strips of a certain width.*]

Show that if we 3-colour $\mathbb{R}^2$, we can always find two points of the same colour at distance 1 apart.

18F Galois Theory

(a) Define the nth *cyclotomic polynomial* $\Phi_n(X) \in \mathbb{C}[X]$. Prove that $\Phi_n(X) \in \mathbb{Z}[X]$. For which integers $n \geqslant 1$ is $\Phi_n(X)$ irreducible over $\mathbb{Q}$? Prove your assertion when $n = p$ is a prime number.

(b) Show that if $n > 1$ is odd, then $\Phi_{2n}(X) = \Phi_n(-X)$.

(c) Let $L = \mathbb{Q}(\zeta_{15})$ where ζ_{15} is a primitive 15th root of unity. Show that $G = \mathrm{Gal}(L/\mathbb{Q})$ can be generated by two elements, say σ and τ. Write down the lattice of subgroups of G, and the corresponding lattice of subfields of L. [You should write each intermediate field F explicitly in terms of generators, for example $F = \mathbb{Q}(\sqrt{-3}, \sqrt{5})$.] Which of these fields are contained in $\mathbb{R}$?

19F Representation Theory

Let G be a finite group and let Z be its centre.

(a) Let V be a representation of G. Define the *kernel* of V, and show that it is a normal subgroup. The derived subgroup G' is the subgroup of G generated by the elements $ghg^{-1}h^{-1}$, for $g, h \in G$. Show that the 1-dimensional representations of G are precisely the irreducible representations of G with G' in their kernel.

(b) Let $\Theta : Z \to \mathbb{C}^*$ be a 1-dimensional representation, and V an irreducible summand of $\mathrm{Ind}_Z^G \Theta$. Describe how Z acts on V, and determine the number of times V occurs in the decomposition of $\mathrm{Ind}_Z^G \Theta$ into irreducibles.

(c) Let p be a prime number and let

$$G = \left\{ \begin{pmatrix} 1 & x & c \\ & 1 & y \\ & & 1 \end{pmatrix} \middle| \, x, y, c \in \mathbb{F}_p \right\}.$$

Compute $|G|$ and deduce that G has no irreducible representation V with $1 < \dim V < p$. Compute Z and G'.

Show that if $\Theta : Z \to \mathbb{C}^*$ is a 1-dimensional representation, then $\mathrm{Ind}_Z^G \Theta$ is either a direct sum of 1-dimensional representations, or of the form $V_\Theta \oplus \ldots \oplus V_\Theta$ where V_Θ is an irreducible representation of dimension p.

Hence, or otherwise, list all the irreducible representations of G. You need not determine the characters, but you must prove that your list is complete.

[You may use any theorems from lectures, provided you state them clearly.]

20G Number Fields

Let $K = \mathbb{Q}(\alpha)$ where α is an algebraic integer. Let R be a ring with $\mathbb{Z}[\alpha] \subset R \subset \mathcal{O}_K$. Let $y_1, \ldots, y_n$ be a $\mathbb{Q}$-basis for K with $\mathrm{Tr}_{K/\mathbb{Q}}(\alpha^{i-1} y_j) = \delta_{ij}$ for all $1 \leqslant i, j \leqslant n$.

(a) Show that $\mathcal{O}_K \subset \mathbb{Z}y_1 + \ldots + \mathbb{Z}y_n$. Deduce that R is Noetherian.

(b) Let I be an R-submodule of K. Show that the following are equivalent.

(i) There exists $0 \neq c \in K$ such that $cI \subset R$.

(ii) I is finitely generated as an R-module.

If these conditions hold then we say that I is a *fractional ideal.* It is *invertible* if there exists a fractional ideal J with $IJ = R$.

(c) Show that the following are equivalent.

(i) For every non-zero ideal $I \subset R$ there exists a non-zero ideal $J \subset R$ with IJ principal.

(ii) Every non-zero fractional ideal is invertible.

(d) Suppose that the conditions in part (c) hold. Show that if $\beta \in \mathcal{O}_K$ then $I = R[\beta]$ is a fractional ideal. By considering I^2, or otherwise, show that $R = \mathcal{O}_K$.

(e) Find explicit non-zero ideals I, J_1, J_2 in $R = \mathbb{Z}[\sqrt{5}]$ with $IJ_1 = IJ_2$, yet $J_1 \neq J_2$.

21H Algebraic Topology

What does it mean for two spaces to be *homotopy equivalent*? Show that $\mathbb{C}^* \times \mathbb{C}$ is homotopy equivalent to S^1.

Let $X = \mathbb{C}^2 \setminus \{(x, x) : x \in \mathbb{C}\}$, and let $x_0 = (1, 0) \in X$. Find the fundamental group $\pi_1(X, x_0)$.

Let Y be the set of degree two, monic complex polynomials with distinct roots, equipped with the subspace topology from the complex vector space of their coefficients. Let $f : X \to Y$ be the function taking (a, b) to the polynomial $(x - a)(x - b)$. Show that $f : X \to Y$ is a covering map.

Let $p_0 \in Y$ be given by $p_0(x) = x(x - 1)$. Find the fundamental group $\pi_1(Y, p_0)$.

22I Linear Analysis

(a) State and prove the Riesz representation theorem.

(b) Let H be a complex Hilbert space. A *sesquilinear form on* H is a map $\theta\colon H \times H \to \mathbb{C}$ such that for each $y \in H$ the map $H \to \mathbb{C}$ given by $x \mapsto \theta(x, y)$ is linear, and for each $x \in H$ the map $H \to \mathbb{C}$ given by $y \mapsto \theta(x, y)$ is conjugate-linear. We say θ is *bounded* if

$$\|\theta\| = \sup\{|\theta(x, y)| : x, y \in H, \|x\| \leqslant 1, \|y\| \leqslant 1\} < \infty .$$

Show that if θ is a bounded sesquilinear form on H, then there is a unique map $S\colon H \to H$ such that $\theta(x, y) = \langle x, Sy\rangle$ for all $x, y \in H$. Show further that $S \in \mathcal{B}(H)$ and $\|S\| = \|\theta\|$.

(c) Define the *adjoint* of an operator $T \in \mathcal{B}(H)$. Using part (b), or otherwise, prove the existence and uniqueness of the adjoint of T, and compute its norm.

(d) Define *hermitian, unitary* and *normal* operators. Let $T \in \mathcal{B}(H)$ be hermitian. Show that every approximate eigenvalue of T is real. Deduce that every element of the spectrum of T is a real approximate eigenvalue.

23H Analysis of Functions

(a) Define the *Sobolev space* $H^s(\mathbb{R}^d)$ for $s \in \mathbb{R}$ and $d \in \mathbb{Z}_{\geqslant 1}$. When $s \in \mathbb{Z}_{\geqslant 0}$, give an equivalent Sobolev norm without using the Fourier transform. [You do not need to prove the equivalence.]

(b) State and prove the Sobolev embedding theorem relating $H^s(\mathbb{R}^d)$ to $L^\infty(\mathbb{R}^d)$ for suitable s.

(c) Define the space $H^1_0(\Omega)$ for a domain $\Omega \subset \mathbb{R}^d$.

(d) Show that $H^1_0(\mathbb{R}^3\backslash\{0\}) = H^1(\mathbb{R}^3)$. [*Hint: Work with the equivalent Sobolev norm you gave in part (a), and make use of* $\varphi_n(x) = \varphi(nx)$ *where* $\varphi \in C^\infty(\mathbb{R}^d)$ *is such that* $\varphi(x) = 1$ *for* $|x| \leqslant 1$ *and* $\varphi(x) = 0$ *for* $|x| \geqslant 2$*. You may assume without proof that such* φ *exists.*]

(e) Is it true that $H^1_0(\mathbb{R}\backslash\{0\}) = H^1(\mathbb{R})$? Justify your answer.

24G Riemann Surfaces

(a) State and prove the Riemann–Hurwitz formula. [You may assume the valency theorem.]

(b) Let $g(z) \in \mathbb{C}[z]$ be a monic polynomial of degree n with no repeated roots, and $d > 1$ an integer dividing n. Let X be the Riemann surface (with branch points) of the function $\sqrt[d]{g}$ and $f\colon X \to \mathbb{C}$ the analytic map given by z. Show that there exists a compact Riemann surface $\widehat{X}$, containing X as a dense open subset, and an analytic map $\hat{f}\colon \widehat{X} \to \mathbb{C}_\infty$ extending f.

What is the cardinality of $\widehat{X} \setminus X$? Compute the genus of $\widehat{X}$ in terms of n and d.

25F Algebraic Geometry

Let $V \subset \mathbb{A}^n$ be an irreducible affine variety and $p \in V$. Define the *tangent space* $T_{V,p}$, the *dimension* of V, and what it means for p to be a *smooth* point of V. Prove that the set of smooth points of V is a non-empty Zariski open subset of V.

Prove that the set of lines $L \subset \mathbb{P}^2$ can be realized as a projective variety by identifying the set with a projective plane, which we denote by $(\mathbb{P}^2)^*$. Let P be a fixed point in $\mathbb{P}^2$. Show that the set of lines in $\mathbb{P}^2$ containing P is a line in $(\mathbb{P}^2)^*$.

Let $X \subset \mathbb{P}^2$ be a (smooth irreducible projective) planar curve. Show that the dual map $P \mapsto T_{X,P}$ defines a morphism from X to $(\mathbb{P}^2)^*$.

The dual curve $X^* \subset (\mathbb{P}^2)^*$ is the set of lines $L \subset \mathbb{P}^2$ tangent to X; that is, the image of X under the dual map. Let X be defined by $x^3 + y^3 + z^3 = 0$. Find the degree 6 equation defining X^* in the coordinates $(u : v : w)$ on $(\mathbb{P}^2)^*$. Is X^* smooth? What is the geometric interpretation of any points on X that map to singular points on X^*? [You may use without proof the fact that X is smooth and irreducible.]

26I Differential Geometry

(a) Define the *exponential map* $\exp_p$ for a surface Σ and point $p \in \Sigma$, describing carefully its domain.

(b) Consider the unit sphere $S^2 = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 1\}$. Describe explicitly the exponential map at the point $(0, 0, 1) \in S^2$, and the associated geodesic polar coordinate system. What is the largest region of S^2 which may be covered by such a coordinate system?

(c) Show that for any two distinct points $p, q \in S^2$ there exists a geodesic $\gamma : [a, b] \to S^2$ such that $\gamma(a) = p$, $\gamma(b) = q$, and such that for any other regular curve $\widetilde{\gamma} : [\tilde{a}, \tilde{b}] \to S^2$ with $\widetilde{\gamma}(\tilde{a}) = p$, $\widetilde{\gamma}(\tilde{b}) = q$ one has $\ell(\widetilde{\gamma}) \geqslant \ell(\gamma)$, where ℓ denotes length.

Show, moreover, that if p and q are not antipodal, then γ is unique when expressed in arc length parametrisation $\gamma : [0, \ell] \to S^2$, and that the above inequality is necessarily strict unless $\widetilde{\gamma}$ is a reparametrisation of γ.

(d) Now consider $\Sigma = S^2 \setminus \{(0, 0, 1)\}$. Given any two distinct points $p, q \in \Sigma$, does there still necessarily exist some geodesic $\gamma : [a, b] \to \Sigma$ such that $\gamma(a) = p$, $\gamma(b) = q$?

Does there necessarily exist a regular curve $\gamma : [a, b] \to \Sigma$ with $\gamma(a) = p$, $\gamma(b) = q$ which is length minimising in the above sense, i.e. for any other regular curve $\widetilde{\gamma} : [\tilde{a}, \tilde{b}] \to \Sigma$ with $\widetilde{\gamma}(\tilde{a}) = p$, $\widetilde{\gamma}(\tilde{b}) = q$ one has $\ell(\widetilde{\gamma}) \geqslant \ell(\gamma)$? Justify your answers.

27L Probability and Measure

Let E be a set and let $\mathcal{A}$ be a σ-algebra on E.

(a) What does it mean to say that μ is a *measure* on $(E, \mathcal{A})$?

Let λ denote Lebesgue measure on $\mathbb{R}^d$.

(b) Show that λ has the following dilatation property: for all $r > 0$ and all $A \in \mathcal{B}(\mathbb{R}^d)$, for $rA = \{rx : x \in A\}$, we have

$$\lambda(rA) = r^d \lambda(A).$$

[*Hint: You may find it helpful to consider a characterization of λ in terms of its values on sets of a simple form.*]

Given a Borel measurable function $f : \mathbb{R}^d \to [0, \infty)$, consider for $k \in \mathbb{Z}$ the set

$$E_k = \{x \in \mathbb{R}^d : 2^k < f(x) \leqslant 2^{k+1}\}$$

and let

$$F_n(x) = \sum_{k=-n}^{n} 2^k \mathbf{1}_{E_k}(x).$$

(c) Show that $F_n(x)$ converges for all x, with limit $F(x)$ satisfying

$$F(x) \leqslant f(x) \leqslant 2F(x).$$

(d) Show that f is Lebesgue integrable if and only if the following condition holds

$$\sum_{k=-\infty}^{\infty} 2^k \lambda(E_k) < \infty.$$

(e) Use part (d) to determine for which $b \in (0, \infty)$ the following function f on $\mathbb{R}^d$ is Lebesgue integrable

$$f(x) = \mathbf{1}_{\{|x|<1\}} |x|^{-b}.$$

28K Applied Probability

(a) For $\lambda > 0$, define what is meant by a *simple birth process* $(X_t)_{t\geqslant 0}$ with $X_0 = 1$ and parameter λ. Find the probability generating function $\mathbb{E}[z^{X_t}]$. Hence find the distribution of X_t and $\mathbb{E}X_t$. Now if $X_0 = i$ for some $i \in \mathbb{N}$, find the distribution of X_t and $\mathbb{E}X_t$. [You may assume that $\mathbb{E}[z^{X_t}]$ is differentiable in t. You may also use that the probability generating function of a Geometric(p) random variable is $\frac{zp}{1-(1-p)z}$.]

(b) Consider a right-continuous continuous-time Markov chain $(X_t)_{t\geqslant 0}$ on the state space $\{0, 1, 2, \ldots\}$ such that

$$q_{i,i+1} = \frac{1}{i+2} \text{ for } i \geqslant 0, \quad q_{i,0} = \frac{1}{(i+1)(i+2)} \text{ for } i \geqslant 1.$$

All other off-diagonal elements of Q are 0. Is $(X_t)_{t\geqslant 0}$ recurrent? Is $(X_t)_{t\geqslant 0}$ explosive? Is $(X_t)_{t\geqslant 0}$ positive recurrent? Does $(X_t)_{t\geqslant 0}$ have an invariant distribution? Is the jump chain positive recurrent? Justify your answers.

(c) Let $(X_t)_{t\geqslant 0}$ be an irreducible non-explosive right-continuous continuous-time Markov chain with generator Q and transition semigroup P. Suppose there exists a distribution π such that $\pi P(t) = \pi$ for all $t > 0$. Does it follow that $(X_t)_{t\geqslant 0}$ is recurrent? Does it follow that $(X_t)_{t\geqslant 0}$ is positive recurrent? Does it follow that π is invariant for $(X_t)_{t\geqslant 0}$? [Clearly state all results you use.]

29L Principles of Statistics

Let $X_1, \ldots, X_n$ be an independent random sample from the uniform distribution on $[0, \theta]$, where $\theta > 0$ is an unknown parameter. Throughout this question, we consider the scale-invariant loss function

$$L(\delta, \theta) = \theta^{-2}(\delta - \theta)^2.$$

(a) Find the maximum likelihood estimator $\hat{\theta}_n$ and calculate its risk. [*Hint: The first and second moments of a Beta*$(n, 1)$ *random variable* Y *are given by*

$$\mathbb{E}(Y) = \frac{n}{n+1}, \quad \mathbb{E}(Y^2) = \frac{n}{n+2}. \quad]$$

(b) Is the maximum likelihood estimator admissible? Justify your answer. [*Hint: Consider estimators of the form* $c\hat{\theta}_n$.]

(c) State the meaning of the following notions in decision theory:

(i) the *Bayes risk* $r(\pi, \delta)$ of an estimator δ with respect to a prior π,

(ii) a *Bayes estimator.*

For $\alpha, \ell \in (0, \infty)$, the Pareto$(\alpha, \ell)$ density $\pi_{\alpha,\ell}$ is given by

$$\pi_{\alpha,\ell}(\theta) = \frac{\alpha \ell^\alpha}{\theta^{\alpha+1}} \mathbf{1}_{\{\theta \geqslant \ell\}}.$$

(d) Suppose we take $\pi_{\alpha,\ell}$ as prior for θ. Find the Bayes estimator $\hat{\theta}_{\alpha,\ell}$ for the scale-invariant loss function $L(\delta, \theta)$. [*Hint: The negative moments of a Pareto*(α, ℓ) *random variable* T *are given by*

$$\mathbb{E}(T^{-k}) = \frac{\alpha \ell^{-k}}{\alpha + k}, \quad k \geqslant 1. \quad]$$

30L Stochastic Financial Models

(a) What does it mean to say that an adapted integrable process $X = (X_n)_{n \geqslant 0}$ is a *supermartingale*?

(b) Let $R_0 = 0$ and $R_n = \xi_1 + \cdots + \xi_n$ for $n \geqslant 1$, where $(\xi_n)_{n \geqslant 1}$ are independent and identically distributed with $\mathbb{P}(\xi_1 = 1) = p$ and $\mathbb{P}(\xi_1 = -1) = 1 - p$ for a constant $p \in (1/2, 1)$. Determine for which values of $c \in (0, \infty)$ the process defined by $X_n = c^{R_n}$ is a supermartingale in the filtration generated by $(R_n)_{n \geqslant 0}$.

(c) Suppose that X is a supermartingale and U is an increasing concave function. Set $Y_n = U(X_n)$. Assuming that Y is integrable, show that Y is a supermartingale.

Let $H = (H_n)_{n \geqslant 1}$ be a bounded previsible process and let $X = (X_n)_{n \geqslant 0}$ be an integrable adapted process. The adapted process $H \bullet X$ is defined by $(H \bullet X)_0 = 0$ and, for $n \geqslant 1$,

$$(H \bullet X)_n = \sum_{k=1}^{n} H_k (X_k - X_{k-1}).$$

(d) In the case where H is non-negative and X is a supermartingale, show that $\mathbb{E}[(H \bullet X)_n] \leqslant 0$ for all $n \geqslant 0$.

(e) Conversely, in the case where $\mathbb{E}[(H \bullet X)_n] \leqslant 0$ for all $n \geqslant 0$ whenever H is non-negative, show that X is a supermartingale.

31K Mathematics of Machine Learning

Consider a classification setting with i.i.d. input–output pairs $(X_1, Y_1), \ldots, (X_n, Y_n) \in [-1,1]^p \times \{-1,1\}$.

(a) What is meant by the *empirical ϕ-risk* $\hat{R}_\phi(h)$ of hypothesis $h : \mathbb{R}^p \to \mathbb{R}$ given convex surrogate loss ϕ?

(b) Given a class of hypotheses $\mathcal{H}$, what is the *Rademacher complexity* $\mathcal{R}_n(\mathcal{H})$?

(c) Show that the empirical ϕ-risk minimiser $\hat{h}$ and the minimiser h^* of the ϕ-risk R_ϕ satisfy

$$\mathbb{E}R_\phi(\hat{h}) - R_\phi(h^*) \leqslant \mathbb{E}\sup_{h\in\mathcal{H}}\{R_\phi(h) - \hat{R}_\phi(h)\}.$$

Suppose ϕ is the hinge loss. Quoting or naming any results that you need, argue that

$$\mathbb{E}R_\phi(\hat{h}) - R_\phi(h^*) \leqslant 2\mathcal{R}_n(\mathcal{H}).$$

(d) Let $\varepsilon_1, \ldots, \varepsilon_n$ be i.i.d. Rademacher random variables, independent of the data. Show that

$$\mathbb{E}\left(\max_{j=1,\ldots,p}\left|\frac{1}{n}\sum_{i=1}^{n}\varepsilon_i X_{ij}\right|\right) \leqslant \sqrt{2\log(2p)/n}.$$

[Standard properties of sub-Gaussian random variables may be used without proof.]

Suppose now that the hypothesis class $\mathcal{H}$ is formed from single hidden layer neural networks with no bias terms of the form

$$x \mapsto h(x) = \sum_{k=1}^{m}\alpha_k\psi(\beta_k^\top x).$$

Here ψ is the reLU activation function and $\alpha \in \mathbb{R}^m$ and $\beta_k \in \mathbb{R}^p$ for all k. Suppose moreover that we have the additional constraints $\|\beta_k\|_1 \leqslant 1$ for all k and $\|\alpha\|_1 \leqslant \lambda$ for some $\lambda > 0$.

(e) Show that

$$\mathcal{R}_n(\mathcal{H}) \leqslant \lambda\mathbb{E}\left(\sup_{b:\|b\|_1\leqslant 1}\left|\frac{1}{n}\sum_{i=1}^{n}\psi(b^\top X_i)\varepsilon_i\right|\right).$$

(f) Finally show that

$$\mathbb{E}R_\phi(\hat{h}) - R_\phi(h^*) \leqslant \frac{4\lambda\sqrt{2\log(2p)}}{\sqrt{n}}.$$

32D Asymptotic Methods

(a) Consider the behaviour of the exponential integral function

$$E(x) = \int_x^\infty \frac{e^{-t}}{t}\, dt, \qquad (\dagger)$$

as $x \to \infty$.

(i) Use integration by parts to show that

$$E(x) \sim e^{-x} x^{-1} \sum_{n=0}^{\infty} a_n x^{-n}, \qquad (\star)$$

as $x \to \infty$, where the coefficients a_n are to be determined.

(ii) Verify that Watson's lemma applied to ($\dagger$) also produces ($\star$).

(iii) Determine the number of terms one must retain in ($\star$) so that the asymptotic approximation is optimally truncated. Show that the resulting relative error is exponentially small.

[You may quote, without proof, the result $\Gamma(x+1) \sim \sqrt{2\pi x}(x/e)^x$, as $x \to \infty$.]

(b) Consider

$$J_n = \int_0^\pi \cos\left[n \cos(nt)\right] dt,$$

for n a large positive integer. Show that its leading order asymptotic approximation is $A \cos \alpha$, where A and α are functions of n you need to find. Without calculation, estimate the size of the next term.

[You may quote, without proof, any relevant results concerning the method of stationary phase.]

33B Dynamical Systems

Consider the nonlinear oscillator

$$\begin{aligned}\dot{x} &= \ \ 2y + \mu x(1 - ax^2 - y^2), \\ \dot{y} &= -2x,\end{aligned}$$

for some constants $a > 1$ and $\mu > 0$. If μ were zero the system would be Hamiltonian.

(a) Find a forward-invariant region $\mathcal{D}$ that encircles, but does not contain, the origin and hence show that there is at least one periodic orbit in $\mathcal{D}$, referring explicitly to any results that you use.

(b) Define the *Floquet multiplier* λ for a periodic orbit with period T. The *Floquet exponent* σ is defined as $T^{-1}\log\lambda$. If $(x_p(t), y_p(t))$ is a periodic orbit of the above system with period T then show, clearly explaining any general results that you use, that the Floquet exponent σ is given by

$$\sigma = \frac{\mu}{T}\int_0^T dt\,(1 - 3ax_p(t)^2 - y_p(t)^2).$$

(c) Describe the *energy balance method*. Use it to show that for $\mu \ll 1$ there is a single periodic orbit and find its distance from the origin. Confirm that your answer is indeed in $\mathcal{D}$.

(d) Calculate the Floquet exponent at leading order in μ to confirm that the orbit is stable when $0 < \mu \ll 1$.

(e) Show that the Floquet exponent can be deduced directly from the expression for the change in the Hamiltonian over one orbit as derived by the energy balance method. Verify that the value obtained by this approach is equal to that obtained from part (d).

34B Integrable Systems

Let $L = -\partial_x{}^2 + u(x,t)$ and let A be a differential operator involving ∂_x such that

$$\partial_t L = [L, A]. \qquad (\dagger)$$

Show that eigenvalues of L corresponding to normalisable eigenfunctions do not depend on t.

Let $A = \partial_x{}^3 + a(x,t)\partial_x + b(x,t)$. By constructing an eigenfunction f of L such that $\partial_t f + Af = 0$, find 2×2 matrices M_A, M_L such that the system of PDEs (†) for u, a, b admits a zero–curvature representation

$$\partial_t M_L - \partial_x M_A - [M_A, M_L] = 0.$$

[You may assume that the eigenvalues of L are non–degenerate.]

Now assume that the functions u, a and b do not depend on x. Show that $\mathrm{Tr}\{(M_L)^k\}$ does not depend on t for any positive integer k.

35A Principles of Quantum Mechanics

Consider two qubits A and B and the following Bell states, written in the basis $\{|+\rangle, |-\rangle\}$ for each qubit:

$$\begin{aligned}
|\mathcal{B}_1\rangle &= \frac{1}{\sqrt{2}}|+\rangle_A|+\rangle_B + \frac{1}{\sqrt{2}}|-\rangle_A|-\rangle_B, \\
|\mathcal{B}_2\rangle &= \frac{1}{\sqrt{2}}|+\rangle_A|+\rangle_B - \frac{1}{\sqrt{2}}|-\rangle_A|-\rangle_B, \\
|\mathcal{B}_3\rangle &= \frac{1}{\sqrt{2}}|+\rangle_A|-\rangle_B + \frac{1}{\sqrt{2}}|-\rangle_A|+\rangle_B, \\
|\mathcal{B}_4\rangle &= \frac{1}{\sqrt{2}}|+\rangle_A|-\rangle_B - \frac{1}{\sqrt{2}}|-\rangle_A|+\rangle_B.
\end{aligned}$$

Here $|+\rangle$ and $|-\rangle$ are eigenstates of the spin σ_z with eigenvalues $+1$ and -1 respectively.

(a) Compute $\langle\mathcal{B}_1|\mathcal{B}_i\rangle$ for $i = 1, 2, 3, 4$.

(b) Compute the reduced density matrix $\rho_A = \mathrm{Tr}_B(|\mathcal{B}_1\rangle\langle\mathcal{B}_1|)$ of the qubit A in the state $|\mathcal{B}_1\rangle$. Compute the von Neumann entropy of ρ_A and determine whether it is pure or mixed.

(c) Can $|\mathcal{B}_1\rangle$ be written as a product state $|\psi\rangle_A \otimes |\chi\rangle_B$? Briefly justify.

(d) Compute $\langle\mathcal{B}_i|\, \sigma_z^{(A)}\sigma_z^{(B)}\, |\mathcal{B}_i\rangle$ for $i = 1, \ldots, 4$. Discuss the implication of this result for measurements of the spin of the two qubits.

(e) Show that the Bell states are eigenstates of

$$T^{AB} := \sigma_x^{(A)}\sigma_x^{(B)} + \sigma_y^{(A)}\sigma_y^{(B)} + \sigma_z^{(A)}\sigma_z^{(B)},$$

and find the corresponding eigenvalues. Here σ_x, σ_y and σ_z are Pauli matrices and the superscript indicates which qubit they act on.

(f) Consider local unitary transformations of the form $U = U_A \otimes U_B$, with $U_{A,B}$ acting on qubit A or B. Show that all four Bell states are locally equivalent, that is, for any i, j there exist U_A, U_B such that

$$|\mathcal{B}_j\rangle = (U_A \otimes U_B)\,|\mathcal{B}_i\rangle.$$

[*Hint: Consider* $U = 1_A \otimes \sigma_i^{(B)}$ *acting on* $|\mathcal{B}_1\rangle$.]

36E Applications of Quantum Mechanics

Consider a Hamiltonian H with a discrete spectrum whose ground state energy is denoted E_0.

(a) Briefly describe how one uses the variational method to provide an upper bound on E_0.

(b) If the trial state differs from the ground state of H by a term of order ϵ, prove that this leads to an error of order ϵ^2 in the energy.

A particle of mass m and charge $-q$ is placed in an infinite one-dimensional potential well of width a, that is

$$V(x) = \begin{cases} 0 & \text{for } 0 \leqslant x \leqslant a\,, \\ +\infty & \text{elsewhere}\,. \end{cases}$$

In addition, the particle is subject to a perturbatively weak, uniform electric field $\mathcal{E}$ so that the following term is added to the potential

$$V_{\mathcal{E}}(x) = \left(x - \frac{a}{2}\right) q\,\mathcal{E}\,.$$

(c) Briefly explain why the perturbation to the ground state energy due to $V_{\mathcal{E}}(x)$ only appears at order $\mathcal{E}^2$.

(d) In order to give an upper bound to the shift in energy of the ground state due to $V_{\mathcal{E}}(x)$, one might consider the following trial function

$$\psi_\alpha(x) = \sqrt{\frac{2}{a}}\sin\left(\frac{\pi x}{a}\right)\left[1 + \left(x - \frac{a}{2}\right)\alpha\, q\,\mathcal{E}\right]\,,$$

where α is the variational parameter. Explain briefly why this is a reasonable choice of trial function.

(e) Determine the optimal value of α which gives the best bound on ground state energy to quadratic order in $\mathcal{E}$. [You do not need to determine the corresponding energy.]

[*Hint: You may use without proof the following:*

$$\int_0^a \cos\left(\frac{2\pi x}{a}\right)\left(x - \frac{a}{2}\right)^2 dx = \frac{a^3}{2\pi^2}\,, \qquad \int_0^a \sin\left(\frac{2\pi x}{a}\right)\left(x - \frac{a}{2}\right) dx = -\frac{a^2}{2\pi}\,.\ ]$$

37E Statistical Physics

(a) Consider the grand Gibbs canonical potential, defined as

$$\Gamma := \Phi + pV \; = \; E - TS + pV - \mu N,$$

for any substance made of a single type of molecule (with only short-range interactions). Write down a formula for the variation $d\Gamma$. Show that

$$\Gamma = 0\,,$$

regardless of the nature of the substance.

(b) A large number N of atoms are trapped in a two-dimensional plane, with no ability to move or vibrate in the third dimension. At low temperatures, these atoms form a planar crystal, arranged in a square grid with area A. Let the speed of sound be given by c_s. (Although it is not realistic, assume that c_s is the same for both transverse and longitudinal modes.)

(i) Calculate the density of states $g(\omega)$, where $g(\omega)d\omega$ is the density of one-phonon states available in a small frequency window $[\omega, \omega + d\omega)$.

(ii) Estimate the Debye frequncy ω_D for the system.

(iii) Show that the partition function of the gas of phonons at low temperature T is given by

$$Z = \exp\left(\frac{XAk_B^2T^2}{c_s^2\hbar^2}\right),$$

where X is a numerical coefficient you should express as an infinite series. For what temperatures is your answer valid?

(iv) For the same low temperature regime, calculate the entropy $S(T)$.

38A General Relativity

(a) Define the *Einstein tensor* $G_{\mu\nu}$ in terms of the Ricci tensor $R_{\mu\nu}$. State, without proof, the property of the Einstein tensor that is required for consistency of Einstein's equations

$$G_{\mu\nu} = 8\pi T_{\mu\nu} ,$$

where $T_{\mu\nu}$ is the energy-momentum tensor. Units with $G = c = 1$ are used throughout.

(b) Consider a FLRW metric in coordinates $x^0 = t$, x^i $(i = 1, 2, 3)$:

$$ds^2 = -dt^2 + a(t)^2 h_{ij}\, dx^i dx^j ,$$

where h_{ij} is a time-independent spatially homogeneous metric depending on a discrete parameter $k = 1, 0$ or -1. The components of the Ricci tensor for this metric are:

$$R_{00} = -3\frac{\ddot{a}}{a} , \qquad R_{ij} = (\, a\ddot{a} + 2\dot{a}^2 + 2k\,)\, h_{ij} , \qquad R_{0i} = R_{i0} = 0 ,$$

where dots denote derivatives with respect to t. Given that the energy-momentum tensor takes the form

$$T_{00} = \rho(t) , \qquad T_{ij} = p(t)\, a(t)^2 h_{ij} , \qquad T_{0i} = T_{i0} = 0 ,$$

where $\rho(t)$ and $p(t)$ are independent of x^i, show that

$$\dot{a}^2 = \alpha\, a^2 + \beta , \qquad \ddot{a} = \gamma\, a , \qquad (*)$$

where α, β, γ are coefficients depending on ρ, p, k that you should determine.

(c) In the Einstein static universe the energy-momentum tensor has contributions from a cosmological constant $\Lambda > 0$ and dust with constant density ρ_m:

$$\rho = \rho_0 := \frac{\Lambda}{8\pi} + \rho_m , \qquad p = p_0 := -\frac{\Lambda}{8\pi} .$$

Show that there is a static solution to $(*)$ with $a = a_0$, a constant, for a certain choice of the discrete parameter k, and find the values of a_0 and ρ_m in terms of Λ.

(d) Now consider a time-dependent perturbation of the static solution in part (c) corresponding to the addition of a small amount of radiation:

$$a = a_0 + \delta a(t) , \quad \rho = \rho_0 + \delta\rho(t) , \quad p = p_0 + \delta p(t) , \quad \text{with} \quad \delta p(t) = \frac{1}{3}\delta\rho(t) ,$$

where $\delta a(t)$, $\delta\rho(t)$ are small and independent of x^i. Determine whether the Einstein static universe is stable or unstable under such a perturbation.

39C Fluid Dynamics II

(a) A uniform rigid cylinder of radius a and finite length falls through unbounded and very viscous fluid. The axis of the cylinder is tilted at some angle to the vertical. Write down the Stokes equations (with no body force) and explain what is meant by the *reversibility* of Stokes flow. Making use of suitable diagrams, use reversibility and symmetries to show that the orientation of the cylinder's axis does not change in any direction as it falls.

(b) Suppose now that the same cylinder falls with its axis vertical through viscous fluid towards a rigid horizontal plane. The gap width $h(t)$ between the closest end of the cylinder and the plane is uniform and satisfies $h(t) \ll a$.

For a prescribed value of $V \equiv dh/dt$, use lubrication theory to find the radial velocity and fluid pressure in the gap. [Assume that the fluid surrounding the gap has constant pressure p_0 and that any end effects near the circular edge of the gap are negligible.]

Deduce that

$$\frac{dh}{dt} = -\frac{2Wh^3}{3\pi\mu a^4}\,,$$

where W is the weight of the cylinder, adjusted for buoyancy. How does h vary asymptotically at large time?

40C Waves

Small-amplitude disturbances in a homogeneous elastic solid, with density ρ and Lamé moduli λ and μ, are governed by the equation

$$\rho \frac{\partial^2 \mathbf{u}}{\partial t^2} = (\lambda + 2\mu)\nabla(\nabla \cdot \mathbf{u}) - \mu\nabla \times (\nabla \times \mathbf{u}) \,,$$

where $\mathbf{u}(\mathbf{x}, t)$ is the displacement.

(a) Consider harmonic plane-wave solutions of the form

$$\mathbf{u}(\mathbf{x}, t) = \mathrm{Re}[\mathbf{A} \exp(i\mathbf{k} \cdot \mathbf{x} - i\omega t)] \,,$$

with constant amplitude vector $\mathbf{A}$. Show that

$$\omega^2 \mathbf{A} = c_P^2 \, \mathbf{k}(\mathbf{k} \cdot \mathbf{A}) - c_S^2 \, \mathbf{k} \times (\mathbf{k} \times \mathbf{A}) \,,$$

giving expressions for the P-wave and S-wave speeds c_P and c_S respectively. Explain mathematically how such waves can be classified into longitudinal P-waves and transverse SV- and SH-waves.

(b) The half-space $y < 0$ is filled with the elastic solid described above, while the slab $0 < y < h$ is filled with a homogeneous elastic solid with Lamé moduli λ' and μ'. The boundary at $y = h$ is rigid. A harmonic plane SH-wave propagates from $y < 0$ towards the interface $y = 0$, with displacement

$$(0, 0, 1)\mathrm{Re}[\exp(ilx + imy - i\omega t)] \,.$$

Write down the relationship between l, m and ω, and determine the complex amplitude of the SH-wave reflected back into $y < 0$. Show that this complex amplitude has unit modulus, and comment on this result physically.

41B Numerical Analysis

Consider the Chebyshev polynomials $\{T_n\}_{n=0}^{\infty}$ defined on the interval $[-1,1]$, given by the three-term recurrence relation

$$T_0(x) = 1, \quad T_1(x) = x,$$
$$T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x), \quad n \geqslant 1.$$

In your answers to the following questions clearly state any properties of the Chebyshev polynomials that you assume.

(a) Consider the inner product $\langle f, g\rangle_w := \int_{-1}^{1} f(x)g(x)w(x)dx$ for functions $f, g : [-1,1] \to \mathbb{R}$, and let $\mathcal{H}$ denote the set of functions $f : [-1,1] \to \mathbb{R}$ such that $\|f\|_w := \langle f, f\rangle_w < \infty$, where $w(x) = (1-x^2)^{-1/2}$. Show that the Chebyshev polynomials are orthogonal with respect to this inner product. Define, using the convention specified in lectures, the *Chebyshev coefficients* $\{c_k\}_{k=0}^{\infty}$ of a function $f \in \mathcal{H}$, and determine the Chebyshev coefficients of $f(x) = \sqrt{1-x^2}$.

(b) Prove that Chebyshev series approximate functions in $\mathcal{H}$ with convergence in the $\|\cdot\|_w$ norm. [You may use the fact that Fourier series converge in the L^2-norm for $L^2[-\pi,\pi]$ functions.]

(c) Derive the equality

$$\frac{\pi}{2}\sum_{k=1}^{\infty}|c_k|^2 + \pi|c_0|^2 = \|f\|_w,$$

where the function $f \in \mathcal{H}$ and $\{c_k\}_{k=0}^{\infty}$ are the Chebyshev coefficients of f.

(d) Prove that Chebyshev series approximations to functions in $\mathcal{H} \cap C^{\infty}[-1,1]$ yield spectral convergence, i.e. the N-term approximation converges faster in the $\|\cdot\|_w$ norm than $\mathcal{O}(N^{-p})$ for any $p = 1, 2, \ldots$.

END OF PAPER