UNIVERSITY OF CAMBRIDGE

MAT2

MATHEMATICAL TRIPOS Part II

Monday, 08 June, 2026 2:00pm to 5:00pm

PAPER 1

Before you begin read these instructions carefully.

The examination paper is divided into two sections. Each question in Section II carries twice the number of marks of each question in Section I. Section II questions also carry an alpha or beta quality mark and Section I questions carry a beta quality mark.

Candidates may obtain credit from attempts on **at most six questions** from Section I and from any number of questions from Section II.

Write on **one side** of the paper only and begin each answer on a separate sheet.

Write legibly; otherwise you place yourself at a grave disadvantage.

At the end of the examination:

Separate your answers to each question.

Complete a gold cover sheet **for each question** that you have attempted, and place it at the front of your answer to that question.

Complete a green main cover sheet listing **all the questions** that you have attempted.

Every cover sheet must also show your Blind Grade Number and desk number.

Tie up your answers and cover sheets into **a single bundle**, with the main cover sheet on the top, and then the cover sheet and answer for each question, in the numerical order of the questions.

STATIONERY REQUIREMENTS
Gold cover sheets
Green main cover sheet
Treasury tag

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

SECTION I

1G Number Theory

Let $N \geqslant 1$ be an integer. Define what it means for a positive integer a to be a *primitive root modulo* N. Let p be an odd prime. Define what it means for an integer a with $a \not\equiv 0 \mod p$ to be a *quadratic residue modulo* p.

Show that when p is an odd prime, no primitive root is a quadratic residue mod p.

Find all primitive roots mod 11. What is the smallest primitive root mod 121?

[*Any results from lectures that you use should be stated clearly.*]

2I Topics in Analysis

State Brouwer's fixed point theorem in the plane and also the equivalent theorem concerning continuous retractions. Explain (without detailed calculations) why the two statements are equivalent.

Let $S^2 = \{(x, y, z) \in \mathbb{R}^3 : x^2+y^2+z^2 = 1\}$. Show that if $f : S^2 \to S^2$ is a continuous map such that $f(p) \neq p$ for each $p \in S^2$, then f is surjective. [You may assume without proof that for each $p \in S^2$ and $0 < \varepsilon < 2$ the set $S^2 \setminus B_\epsilon(p)$ is homeomorphic to a closed disc in $\mathbb{R}^2$, where $B_\epsilon(p) = \{q \in S^2 : d(p, q) < \varepsilon\}$ and d is the Euclidean distance in $\mathbb{R}^3$.]

3H Coding & Cryptography

Consider the discrete random variable $\mathbf{X}$ taking seven values x_i $(1 \leqslant i \leqslant 7)$ with the following probabilities:

$$\mathbf{X} = \begin{pmatrix} x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & x_7 \\ 0.45 & 0.28 & 0.14 & 0.04 & 0.04 & 0.03 & 0.02 \end{pmatrix}.$$

(a) Find a binary Huffman code for $\mathbf{X}$.

(b) Find the expected codeword length for the encoding.

(c) Find a ternary Huffman code for $\mathbf{X}$ (that is the codewords are strings of elements in $\{0, 1, 2\}$). Show that the expected word length in this case is 1.36.

4J Automata and Formal Languages

In this question, we will set $\mathbb{N} = \{0, 1, 2, 3, \dots\}$.

(a) Define the class of *primitive recursive* functions giving precise definitions of the basic functions and the closure operations that define it.

(b) Show that the addition function $(n, m) \mapsto n + m$ is primitive recursive.

(c) Let $f \colon \mathbb{N}^2 \to \mathbb{N}$ be primitive recursive. Show that its diagonal function $d \colon \mathbb{N} \to \mathbb{N}$, given by $d(n) = f(n, n)$, is primitive recursive.

(d) A function $f \colon \mathbb{N}^2 \to \mathbb{N}$ is called a *complete list* if for each primitive recursive function $g \colon \mathbb{N} \to \mathbb{N}$, there is some $n \in \mathbb{N}$ such that for all $k \in \mathbb{N}$, we have $f(n, k) = g(k)$. Show that a complete list cannot be primitive recursive.

5K Statistical Modelling

Define the *normal linear model* with response $Y \in \mathbb{R}^n$, $X \in \mathbb{R}^{n \times p}$, coefficient vector $\beta \in \mathbb{R}^p$, and noise variance $\sigma^2 > 0$. What is the maximum likelihood estimator of β? [You do not need to derive it and can assume X has full column rank.]

Define the *leverage* of the ith data point for $1 \leqslant i \leqslant n$, then express the variance of the residual of the ith data point (treating X as fixed) in terms of its leverage and σ^2. Explain why we pay attention to high-leverage points in regression analysis.

6C Mathematical Biology

A certain sexually transmitted disease is only passed on by heterosexual sex. The populations of susceptible heterosexual males and females are denoted by S and $\widehat{S}$, respectively. The populations of infected males and females are denoted by I and $\widehat{I}$, respectively. The disease is modelled by the equations

$$\frac{d\widehat{S}}{dt} = -\widehat{\beta}\widehat{S}I + \widehat{\nu}\widehat{I} \quad \text{and} \quad \frac{d\widehat{I}}{dt} = \widehat{\beta}\widehat{S}I - \widehat{\nu}\widehat{I},$$
$$\frac{dS}{dt} = -\beta S\widehat{I} + \nu I \quad \text{and} \quad \frac{dI}{dt} = \beta S\widehat{I} - \nu I,$$

with β, $\widehat{\beta}$, ν, and $\widehat{\nu}$ constant.

(a) Give a biological interpretation of each term in the equations.

(b) Define $N = S + I$ and $\widehat{N} = \widehat{S} + \widehat{I}$ and assume that

$$N\widehat{N}\beta\widehat{\beta} > \nu\widehat{\nu}\ .$$

By writing equations involving only I and $\widehat{I}$, show that there are fixed points at $I = \widehat{I} = 0$ and at

$$I = \frac{\widehat{N}N - \widehat{\nu}\nu/(\widehat{\beta}\beta)}{\nu/\beta + \widehat{N}} \quad \text{and} \quad \widehat{I} = \frac{\widehat{N}N - \widehat{\nu}\nu/(\widehat{\beta}\beta)}{\widehat{\nu}/\widehat{\beta} + N}\ .$$

(c) Show that the fixed point at the origin is unstable. Why must the other fixed point be stable?

7D Further Complex Methods

Define the *Cauchy principal value* $\mathcal{P}\int_{-\infty}^{\infty} f(x)\,dx$ of an improper integral.

Let $\omega \in \mathbb{R} \setminus \{0\}$. Consider

$$\int_{-\infty}^{\infty} \frac{e^{i\omega x}}{x^2 - x - 6}\,dx.$$

Explain why this integral does not exist in the usual sense.

Show that the Cauchy principal value

$$I(\omega) = \mathcal{P}\int_{-\infty}^{\infty} \frac{e^{i\omega x}}{x^2 - x - 6}\,dx$$

exists, and evaluate it by considering an appropriate contour integral. State clearly any standard results involving contour integrals that you use.

Hence deduce the values of

$$\mathcal{P}\int_{-\infty}^{\infty} \frac{\cos(\omega x)}{x^2 - x - 6}\,dx \qquad \text{and} \qquad \mathcal{P}\int_{-\infty}^{\infty} \frac{\sin(\omega x)}{x^2 - x - 6}\,dx.$$

8A Classical Dynamics

(a) A dynamical system has a single generalized coordinate q, Lagrangian L and action

$$S = \int_{t_1}^{t_2} L(q, \dot{q}, t)\, dt\,.$$

State the principle of least action. Define the *generalized momentum* p and explain how the Hamiltonian H is related to the Lagrangian L. What condition(s) on L are necessary for this Hamiltonian to be well-defined?

(b) The Lagrangian for a particle of charge e and mass m moving in three dimensions in the presence of an electromagnetic field is

$$L = \frac{m}{2}\dot{\mathbf{r}}^2 - e(\phi - \dot{\mathbf{r}} \cdot \mathbf{A})\,,$$

where $\phi(\mathbf{r}, t)$ and $\mathbf{A}(\mathbf{r}, t)$ are the scalar and vector potentials describing the electromagnetic field.

(i) Write down the particle's Hamiltonian.

(ii) Consider the case $\phi = 0$ and $\mathbf{A} = (0, 0, -Bx)$, where B is a constant. Obtain Hamilton's equations for the particle.

9D Cosmology

A system of N point particles of equal mass m interacts only through Newtonian gravity. Let $\mathbf{r}_i(t)$ denote the position of the ith particle and $\mathbf{p}_i = m\dot{\mathbf{r}}_i$ its momentum, then the kinetic energy T and potential energy V are given by

$$T = \frac{1}{2}\sum_{i=1}^{N} m\dot{\mathbf{r}}_i^2, \qquad V = -\sum_{i<j} \frac{Gm^2}{|\mathbf{r}_i - \mathbf{r}_j|}.$$

Assume the system is gravitationally bound with particles remaining in a confined region.

(a) The scalar moment $\mathcal{G}$ is defined by

$$\mathcal{G}(t) = \sum_{i=1}^{N} \mathbf{p}_i \cdot \mathbf{r}_i.$$

Show that

$$\frac{d\mathcal{G}}{dt} = 2T + \sum_{i=1}^{N} \mathbf{F}_i \cdot \mathbf{r}_i\,,$$

where the total force on particle i is $\mathbf{F}_i = \sum_{j\neq i} \mathbf{f}_{ij}$, with $\mathbf{f}_{ij}$ the force exerted on particle i by particle j.

(b) Split and rearrange sums over pairwise forces using Newton's third law to express

$$\sum_{i=1}^{N} \mathbf{F}_i \cdot \mathbf{r}_i = \sum_{i}^{N} \sum_{j<i} \mathbf{f}_{ij} \cdot (\mathbf{r}_i - \mathbf{r}_j).$$

Substituting the Newtonian gravitational force described by $\mathbf{f}_{ij}$, show that

$$\sum_{i=1}^{N} \mathbf{F}_i \cdot \mathbf{r}_i = V\,.$$

(c) Deduce the virial theorem by taking a time average over the quantity $d\mathcal{G}/dt$:

$$2\langle T\rangle = -\langle V\rangle.$$

(d) A rich cluster of galaxies can be modeled as a virialized gravitational system. Briefly discuss how redshift measurements of velocity dispersions $\langle v^2\rangle$ of galaxies in a cluster can be combined with the cluster radius $\langle R\rangle$ to estimate the mass of the cluster. How can discrepancies with the estimated mass from luminous matter be resolved?

10E Quantum Information and Computation

(a) State the no-signalling principle for quantum states of a bipartite system AB.

Given the orthonormal basis $\mathcal{B}(\theta) = \{|\mu(\theta)\rangle, |\nu(\theta)\rangle\}$, with

$$|\mu(\theta)\rangle = \cos\theta\,|0\rangle + \sin\theta\,|1\rangle \quad \text{and} \quad |\nu(\theta)\rangle = \sin\theta\,|0\rangle - \cos\theta\,|1\rangle\,,$$

and the 2-qubit state $|\phi^+\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$, show that

$$\frac{1}{\sqrt{2}}\Big(|\mu(\theta)\rangle\,|\mu(\theta)\rangle + |\nu(\theta)\rangle\,|\nu(\theta)\rangle\Big) = |\phi^+\rangle \quad \text{for all } \theta.$$

Alice and Bob share the state $|\phi^+\rangle_{AB}$. Alice measures her qubit in the basis $\mathcal{B}(\theta)$, and then Bob measures his qubit in the computational basis. Compute Bob's measurement outcome probabilities in terms of θ and state how your result relates to the no-signalling principle.

(b) State the Helstrom–Holevo theorem on distinguishing two quantum states, $|\alpha_0\rangle$ and $|\alpha_1\rangle$.

Suppose that for any states $|\alpha_0\rangle$ and $|\alpha_1\rangle$ with $|\langle\alpha_0|\alpha_1\rangle| = \cos\theta$, there is a quantum measurement process that will correctly identify the state with probability $p(\theta) = \frac{1}{2}(1+\sin\theta)$. Show that if Alice could clone states from the set $\{|0\rangle, |+\rangle\}$, then the Helstrom–Holevo theorem could be violated.

SECTION II

11H Coding & Cryptography

(a) Let H be the binary code with the following parity check matrix:

$$\begin{pmatrix} 1 & 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 \end{pmatrix}.$$

Show that $|H| = 16$ and the minimum distance of H is 3.

(b) Let H be as above and let K be the reverse of H (that is $x_1 \ldots x_7 \in H$ if and only if $x_7 \ldots x_1 \in K$). You may assume that K is a linear code and that the only codewords K and H have in common are 0000000 and 1111111. Let H' be the parity check extension of H and K' the parity check extension of K.

(i) Show that H' and K' have length 8, size 16 and minimum distance 4. Further, show that the only codewords H' and K' have in common are 00000000 and 11111111.

Let the linear code $G \subseteq \mathbb{F}_2^{24}$ be defined as follows:

$$G = \{(a + x, b + x, a + b + x) : a, b \in H', x \in K'\}.$$

(ii) Show that a codeword $(a+x, b+x, a+b+x) \in G$ uniquely determines a, b and x. Hence show that $|G| = 2^{12}$.

(iii) Show that G has minimum distance 8. [You may use without proof that every codeword in G has weight a multiple of 4.]

(iv) Puncture G in the last bit and denote this new code by G^-. Show that G^- is a perfect code.

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12I Automata and Formal Languages

(a) Define the notion of a *(variable based) grammar* G and the *language* $\mathcal{L}(G)$ generated by G.

(b) Define *type 2* and *context free* rules, grammars, and languages. State the pumping lemma for context free languages.

In the following, let $\Sigma = \{a, b, c\}$ be a set of terminal symbols and $N = \{S, A, B, C\}$ be a set of non-terminal symbols.

(c) Determine, with justification, whether the following languages over Σ are context free.

(i) The set of words of the form $a^m b^n c^m$ for any $m, n \geqslant 1$.

(ii) The set of words w containing the same number of a's, b's and c's.

(d) Let G be the grammar (N, Σ, P, S) where P is given by

$$S \to abC \,|\, aSBC, \qquad ABC \to cBa \,|\, Cab, \qquad CB \to BC, \qquad B \to b, \qquad C \to c.$$

Determine, with justification, whether $\mathcal{L}(G)$ is a type 2 language.

13K Statistical Modelling

Let Poisson(λ) denote the Poisson distribution with rate parameter λ.

(a) Show that Poisson(λ), $\lambda > 0$ forms an exponential family with natural parameter $\theta = \log \lambda$ by identifying its sufficient statistic and cumulant function.

(b) Let $Y_i \sim \text{Poisson}(\lambda_i), i = 1, \ldots, n$ be independent and $M = \sum_{i=1}^n Y_i$. Show that the conditional distribution of $(Y_1, \ldots, Y_n)$ given $M = m$ is

$$(Y_1, \ldots, Y_n) \mid M = m \sim \text{Multinomial}(m, (\pi_1, \ldots, \pi_n)),$$

where $\pi_i = \lambda_i/\lambda_+, i = 1, \ldots, n$ for $\lambda_+ = \lambda_1 + \cdots + \lambda_n$.

(c) The `lizards` dataset records the species (`Sagrei` or `Distichus`), diameter of the branch they were perched on (`narrow` or `wide`), and the height of that same branch (`high` or `low`) for a sample of 409 lizards.

```
> (data <- as.data.frame(table(lizards)))
    Species Diameter Height Freq
1     Sagrei   narrow   high   86
2 Distichus   narrow   high   73
3     Sagrei     wide   high   35
4 Distichus     wide   high   70
5     Sagrei   narrow    low   32
6 Distichus   narrow    low   61
7     Sagrei     wide    low   11
8 Distichus     wide    low   41
```

Consider the two models fitted by the R code below. Why does the statistical model in `fit1` imply that the variables `Species`, `Diameter`, and `Height` are independent? What independence relations between these three variables are implied by the model in `fit2`?

```
> fit1 <- glm(Freq ~ Species + Diameter + Height, family = poisson, data)
> fit2 <- glm(Freq ~ Species * Diameter + Height, family = poisson, data)
```

What can you conclude from the analysis of deviance below? How would you test the independence relations implied by the model in `fit2`?

```
> anova(fit1, fit2, test = "LR")
Analysis of Deviance Table

Model 1: Freq ~ Species + Diameter + Height
Model 2: Freq ~ Species * Diameter + Height
  Resid. Df Resid. Dev Df Deviance  Pr(>Chi)
1         4     25.037
2         3     12.431  1   12.606 0.0003845 ***
---
Signif. codes:  0 ‘***’ 0.001 ‘**’ 0.01 ‘*’ 0.05 ‘.’ 0.1 ‘ ’ 1
```

14D Further Complex Methods

Let $I(z) = \int_0^\infty t^{z-1}e^{-t}\,dt$ be Euler's integral. You may assume it converges and is analytic for $\mathrm{Re}\,(z) > 0$.

Show how the Gamma function $\Gamma(z)$ can be defined in terms of $I(z)$ such that $\Gamma(n) = (n-1)!$.

Prove the reflection formula $\Gamma(z)\Gamma(1-z) = \dfrac{\pi}{\sin(\pi z)}$.

The expressions $\Gamma(z)\Gamma\left(z+\frac{1}{2}\right)$ and $\Gamma(2z)$ define two meromorphic functions with the same poles, which are simple. Hence there exists an analytic entire function $g : \mathbb{C} \to \mathbb{C}$ with

$$\Gamma(z)\Gamma\left(z+\frac{1}{2}\right) = e^{g(z)}\Gamma(2z).$$

Using the reflection formula, show that $g(z) = -2\,z\,\log 2 + h(z)$ with $h(z+1) = h(z)$. Why is $h(z)$ entire? The entire 1-periodic function $h(z)$ has the expansion $h(z) = \sum_{n\in\mathbb{Z}} a_n e^{2\pi i n z}$ and has at most exponential growth in any vertical strip $\{s \leqslant \mathrm{Re}\,z \leqslant t\}$, i.e. $|h(z)| \leqslant Ce^{\alpha|\mathrm{Im}\,z|}$ for suitable constants $C, \alpha > 0$. Deduce that $h(z) = b$ is constant and by considering a suitable z, determine the value of b. Deduce

$$\Gamma(z)\Gamma\left(z+\frac{1}{2}\right) = 2^{1-2z}\sqrt{\pi}\,\Gamma(2z).$$

Find $\Gamma(\frac{1}{6})$ in terms of $\Gamma(\frac{1}{3})$.

15D Cosmology

The Friedmann equation and fluid conservation equation for a homogeneous and isotropic universe are given by

$$\left(\frac{\dot{a}}{a}\right)^2 + \frac{kc^2}{a^2} = \frac{8\pi G}{3c^2}\rho + \frac{\Lambda c^2}{3}, \qquad \dot{\rho} = -3\frac{\dot{a}}{a}(\rho + P),$$

where a is the scale factor, ρ is the energy density, P is the pressure and k is the curvature.

(a) Consider matter with a linear equation of state $P = w\rho$ where w is a constant. Show that the energy density behaves as $\rho(t) = \rho_0\, a(t)^{-3(1+w)}$, where $\rho_0 \equiv \rho(t_0)$ is the density measured today at $t = t_0$ (assuming the usual normalisation $a(t_0) = 1$). Define the density parameter Ω_i for a given matter component ρ_i using the ratio with the critical energy density of a flat ($k = 0$) universe and expressed using the Hubble parameter H.

(b) The standard concordance cosmology has had great success describing our Universe using just three key constituents: radiation $\rho_{\rm R}$, non-relativistic matter $\rho_{\rm M}$, and a cosmological constant Λ, with the curvature measured to be negligible ($k \approx 0$). Show that the Friedmann equation for this model can be expressed as

$$\left(\frac{\dot{a}}{a}\right)^2 = H_0^2\left(\frac{\Omega_{R0}}{a^4} + \frac{\Omega_{M0}}{a^3} + \Omega_{\Lambda 0}\right),$$

where H_0 is the Hubble parameter today and you should define and justify each term.

(c) Recent observations have suggested the concordance model needs an additional ingredient to explain time-varying acceleration. Consider a flat universe dominated by a cosmological constant today with $\Omega_{\Lambda 0} \leqslant 1$ (for brevity $\Omega \equiv \Omega_{\Lambda 0}$) plus a small additional component of 'phantom matter' $\Omega_{P0} = 1 - \Omega \ll 1$, which satisfies a linear equation of state with $w = -4/3$. Show that the Friedmann equation becomes $\dot{a} = H_0\sqrt{\Omega}\, a\sqrt{1 + \beta^2 a}$, where $\beta^2 = (1 - \Omega)/\Omega$. Solve this equation by considering $b^2(t) = 1 + \beta^2 a(t)$, that is, you should find the following and then specify the constant κ using $a(t_0) = 1$:

$$\log(b - 1) - \log(b + 1) = H_0\sqrt{\Omega}\,(t - t_0) + \log \kappa\,.$$

Hence, or otherwise, find the late-time concordance solution with Λ and phantom matter:

$$a(t) = \frac{4\Omega \exp\left[H_0\sqrt{\Omega}\,(t - t_0)\right]}{\left((1 + \sqrt{\Omega}) - (1 - \sqrt{\Omega})\exp\left[H_0\sqrt{\Omega}\,(t - t_0)\right]\right)^2}.$$

Verify that the behaviour of $a(t)$ near $t \approx t_0$ for $1 - \Omega \ll 1$ agrees with expectations for a Λ-dominated universe. However, note that even a small component of phantom matter will lead to the catastrophic 'Big Rip' demise of this universe; find the time $t_{\rm BR}$ when the scale factor becomes singular and briefly comment on how quickly this happens. What will be the fate of loosely-bound gravitational objects as $t \to t_{\rm BR}$?

16J Logic & Set Theory

(a) State the axiom of choice and Zermelo's wellordering theorem, providing definitions of concepts used in the statements.

(b) Prove Zermelo's wellordering theorem assuming all axioms of Zermelo-Fraenkel set theory with the axiom of choice.

(c) State a version of Hessenberg's theorem that can be proved without the axiom of choice. [You do not need to argue that it can be proved without the axiom of choice.]

(d) Work in Zermelo-Fraenkel set theory without the axiom of choice and prove that the axiom of choice is equivalent to the statement "for every infinite set X, there is a bijection between X and $X \times X$". [*Hint: For one of the implications, consider sets of the form $X \cup \alpha$ where α is an ordinal disjoint from the set X.*]

[*Throughout this question, you may use that the following can all be proved in Zermelo-Fraenkel set theory without the axiom of choice: the recursion theorem, the representation theorem for wellorders, Hartogs's theorem, and the fact that Zermelo's wellordering theorem implies the axiom of choice.*]

17J Graph Theory

(a) Let G be a bipartite graph with vertex classes X and Y. What does it mean to say that G *contains a matching from X to Y*? State and prove Hall's marriage theorem.

(b) Let $G = (V, E)$ be a bipartite graph with vertex classes X and Y. Suppose that every $x \in X$ has degree $d(x) \geqslant 1$, and that if $x \in X$ and $y \in Y$ with $xy \in E$, then $d(x) \geqslant d(y)$. Show that G contains a matching from X to Y. [*Hint: If not, consider a minimal subset of X that fails Hall's condition.*]

(c) Let $n \geqslant 1$ and $k \geqslant 1$ be integers. Assume that we are given a sequence $A_1, A_2, \ldots, A_k$ of subsets of $\{1, 2, \ldots, n\}$ that is an *antichain*, which means that $A_i \not\subseteq A_j$ for all $i \neq j$. Suppose further that $|A_1 \cup \cdots \cup A_k| > k$. Show that there exist i and $x \in A_i$ such that if we replace A_i in the sequence by $A_i \setminus \{x\}$, then the resulting sequence is still an antichain. [*Hint: Argue by contradiction.*]

18F Galois Theory

Let $f \in K[X]$ be a separable polynomial of degree n. Define the *Galois group* $\mathrm{Gal}(f/K)$ of f, and show that it is a transitive subgroup of S_n if and only if f is irreducible. Give an example of two polynomials $f, g \in \mathbb{Q}[X]$ of the same degree n which have the same splitting field, but such that $\mathrm{Gal}(f/\mathbb{Q})$ is not conjugate to $\mathrm{Gal}(g/\mathbb{Q})$ in S_n.

Let $f \in \mathbb{F}_p[X]$ be a separable polynomial whose irreducible factors have degrees $n_1, \ldots, n_r$. Show that $\mathrm{Gal}(f/\mathbb{F}_p)$ is generated by an element of cycle type $(n_1, \ldots, n_r)$. [Standard facts about the Galois theory of finite fields may be used without proof.]

Find the Galois group of the polynomial $X^4 + X^3 + 2X^2 + 1$ (i) over $\mathbb{F}_2$, (ii) over $\mathbb{Q}$.

19F Representation Theory

(a) Compute the character table of the alternating group A_5.

[You may use without proof that there are 5 conjugacy classes, and they have sizes $(1, 15, 20, 12, 12)$ and consist of elements of order $(1, 2, 3, 5, 5)$, respectively, and any other structure of the group and its subgroups that you need. You may also use without proof that there is an irreducible representation of dimension 3, with character values $(3, -1, 0, \frac{1+\sqrt{5}}{2}, \frac{1-\sqrt{5}}{2})$, respectively.]

If V is any irreducible complex representation of A_5 of dimension 3, show that V is not the restriction of a representation of the symmetric group S_5.

(b) Let G be a finite group acting on a finite set X.

Define the *permutation representation* $\mathbb{C}[X]$, and compute its character. Compute $\dim \operatorname{Hom}_G(\mathbf{1}, \mathbb{C}[X])$ in terms of the orbits for G acting on X.

Let $d = |X|$. Let $\Gamma_n = \{\Sigma \subseteq X : |\Sigma| = n\}$ be the set of all n-element subsets of X. Show that $\mathbb{C}[\Gamma_n]$ is isomorphic to $\mathbb{C}[\Gamma_{d-n}]$ for all $0 \leqslant n \leqslant d$.

Now take $G = A_5$ acting on $X = \{1, \ldots, 5\}$ in the natural way. Is it true that $\mathbb{C}[\Gamma_n]$ is isomorphic to $\wedge^n \mathbb{C}[X]$ for all $0 \leqslant n \leqslant 5$? Give a proof or counterexample.

20G Number Fields

(a) Let $d \neq 0, 1$ be a square-free integer. Compute an integral basis for $K = \mathbb{Q}(\sqrt{d})$ and compute the discriminant D_K.

(b) Let $m \neq 0, \pm 1$ be an odd square-free integer. Let $L = \mathbb{Q}(i, \sqrt{m})$. Show that if $\theta \in \mathcal{O}_L$ then $\theta = \frac{1}{2}(a + bi + c\sqrt{m} + di\sqrt{m})$ where $a, b, c, d \in \mathbb{Z}$ satisfy

$$\begin{aligned} a^2 - b^2 &\equiv m(c^2 - d^2) \pmod{4}, \\ 2ab &\equiv 2mcd \pmod{4}. \end{aligned}$$

Compute an integral basis for L and compute the discriminant D_L.

[*You should split into cases according to the values of d or m mod* 4.]

21H Algebraic Topology

Let K be a simplicial complex, and $f : K \to K$ a simplicial map. State two equivalent formulas for the Lefschetz number of f. State and prove the Lefschetz fixed point theorem. [Results about simplicial approximation and barycentric subdivision may be used without proof.]

Let $S^n \subset \mathbb{R}^{n+1}$ be the standard sphere. Let $a : S^n \to S^n$ be the antipodal map. For n even, can a be homotopic to the identity?

Suppose $v : S^n \to \mathbb{R}^{n+1}$ is a continuous function such that the dot product $\mathbf{x} \cdot v(\mathbf{x})$ vanishes for all $\mathbf{x} \in S^n$. For n even, show that v must have at least one zero.

22I Linear Analysis

(a) State and prove the open mapping lemma. State the open mapping theorem.

(b) Let $T\colon X \to Y$ be a linear map between normed spaces. We say that T is *bounded below* if there exists $\delta > 0$ such that $\|Tx\| \geqslant \delta\|x\|$ for all $x \in X$. Assume now that X and Y are Banach spaces and that T is bounded and injective. Show that T is bounded below if and only if the image of T is closed in Y.

(c) Let X and Y be Banach spaces. Show that each of the following properties of a bounded linear map $T \in \mathcal{B}(X, Y)$ are stable under perturbation, *i.e.* that for some $\varepsilon > 0$, any $S \in \mathcal{B}(X, Y)$ with $\|S - T\| < \varepsilon$ has the same property.

(i) T is bounded below.

(ii) T is surjective.

(iii) T is an isomorphism.

23H Analysis of Functions

(a) State and prove a formula for the Fourier transform of the convolution of two functions in $L^1(\mathbb{R}^d)$ in terms of the Fourier transforms of the individual functions.

(b) State and prove Plancherel's formula for Schwartz functions. [This is also known as Parseval's formula. You may use without proof the Fourier inversion formula, and properties of Schwartz functions.]

(c) Show that the formula in part (a) is valid for the convolution of a function in $L^1(\mathbb{R}^d)$ and a function in $L^2(\mathbb{R}^d)$.

(d) Fix a function $f \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ such that $\widehat{f}(\xi) = 0$ if and only if $\xi = 1$. Denote by Y the vector space of functions that are finite linear combinations of translates of f, that is, the elements of Y are the functions $a_1 f(x - t_1) + \ldots + a_n f(x - t_n)$, where $n \in \mathbb{Z}_{\geqslant 1}$, $a_1, \ldots, a_n \in \mathbb{C}$ and $t_1, \ldots, t_n \in \mathbb{R}$ are arbitrary numbers.

(i) Prove that Y is dense in $L^2(\mathbb{R})$.

(ii) Prove that Y is not dense in $L^1(\mathbb{R})$.

[*Hint: In part (d), you may find it useful to recall a characterisation of linear subspaces of a Banach space X that are not dense in X in terms of the existence of non-zero functionals on X with appropriate properties.*]

[*In parts (c) and (d), you may use any results from the course provided they are stated clearly.*]

24G Riemann Surfaces

(a) Let $\pi\colon X \to Y$ be a continuous map of path-connected Hausdorff topological spaces. What does it mean to say that π is a *covering map*? What does it mean to say that π is a *regular covering map*? Give an example of a covering map between Riemann surfaces which is surjective and not regular.

(b) Let (f, U) and (g, V) be function elements on domains U, $V \subset \mathbb{C}$. What does it mean to say that (g, V) is an *analytic continuation* of (f, U)? What is a *complete analytic function*?

Let $\mathcal{F}$ be a complete analytic function. Explain how to construct the Riemann surface R of $\mathcal{F}$ along with its canonical covering map to $\mathbb{C}$, and describe how $\mathcal{F}$ gives rise to an analytic function $R \to \mathbb{C}$. [Proofs are not required.]

(c) Let (f, U) be a function element on a domain $U \subset \mathbb{C}$ and $F\colon U \to \mathbb{C}$ an analytic function with $F' = f$.

(i) Let (g, V) be an analytic continuation of (f, U), with $U \cap V$ connected and V simply-connected. Show that there exists an analytic function $G\colon V \to \mathbb{C}$ with $G' = g$ such that (G, V) is an analytic continuation of (F, U).

(ii) Let $\mathcal{F}$ and $\widetilde{\mathcal{F}}$ be the complete analytic functions determined by (f, U) and (F, U), respectively. Show that the Riemann surface of $\widetilde{\mathcal{F}}$ is a surjective regular cover of the Riemann surface of $\mathcal{F}$.

25F Algebraic Geometry

Define what it means for two irreducible varieties X and Y to be *birational*. State a criterion for X and Y to be birational in terms of their function fields $k(X)$ and $k(Y)$.

Let $X \subset \mathbb{P}^2$ be the curve defined by $y^2z = x^3 + x^2z$. Show that X is irreducible. Show that X has exactly one singular point P. By considering lines through P or otherwise, construct a birational map $\varphi : \mathbb{P}^1 \dashrightarrow X$.

Let $\widetilde{X} \subset X \times \mathbb{P}^1$ be the closure of the set of pairs

$$\{(Q, t) \mid Q \in \{y = tx\} \cap X, Q \neq P\}.$$

Let $\pi : \widetilde{X} \to X$ be projection onto the first factor. Show that the preimage $\pi^{-1}(Q)$ contains a unique point for all $Q \neq P$, while $\pi^{-1}(P)$ consists of exactly two points. Briefly explain the geometric significance of this fact.

Let $Y \subset \mathbb{P}^n$ be a smooth projective variety. Suppose there exists a dominant rational map $\psi : \mathbb{P}^m \dashrightarrow Y$. Must Y be isomorphic to $\mathbb{P}^k$ for some k? Justify your answer.

26I Differential Geometry

(a) Let γ be a regular curve. Define the *curvature* and *torsion* of γ, carefully describing any conditions necessary for these to be well defined.

(b) Suppose the image of γ lies on an oriented surface $S \subset \mathbb{R}^3$. Define the *normal* and *geodesic curvature* of γ. What does it mean for γ to be a *geodesic*?

(c) Can a surface $S \subset \mathbb{R}^3$ of positive Gaussian curvature contain a line segment? Can a connected compact surface $S \subset \mathbb{R}^3$ of positive Gaussian curvature contain two non-intersecting closed geodesics?

(d) Consider a regular curve $\gamma : (a,b) \to \mathbb{R}^3$, with $-\infty \leqslant a < b \leqslant \infty$, not necessarily parametrised by arc length. We say that γ is *regularly extendible* if there exists a regular curve $\widetilde{\gamma} : I \to \mathbb{R}^3$, where $I \subset \mathbb{R}$ is a open interval, and an open interval $(c,d) \subset \mathbb{R}$ with $(c,d] \subset I$ or $[c,d) \subset I$ such that $\widetilde{\gamma}|_{(c,d)} : (c,d) \to \mathbb{R}^3$ is a reparametrisation of $\gamma : (a,b) \to \mathbb{R}^3$.

Let $\gamma : (a,b) \to \mathbb{R}^3$ now be a regular curve parametrised by arc length, where we assume $-\infty < a < b < \infty$. Assume that the curvature $k(s)$ satisfies $k(s) \geqslant \epsilon > 0$ for some $\epsilon > 0$ and all $s \in (a,b)$ and let $\tau(s)$ denote the torsion. Show that if

$$\limsup_{s \to a} \left(k(s) + |\tau(s)|\right) = \infty, \qquad \text{and} \qquad \limsup_{s \to b} \left(k(s) + |\tau(s)|\right) = \infty,$$

then γ is not regularly extendible.

27L Probability and Measure

(a) State and prove the first and second Borel–Cantelli lemmas.

Let $(X_n)_{n \geqslant 1}$ be a sequence of independent identically distributed real random variables with density p and characteristic function Φ given by

$$p(x) = \frac{1}{\pi(1+x^2)}, \quad \Phi(\xi) = e^{-|\xi|}.$$

Set $S_n = X_1 + \cdots + X_n$.

(b) Show that the sequence of random variables S_n/n converges in law.

(c) By considering the convergence of the series $\sum_{n \geqslant 1} \mathbb{P}(X_n/n > 1)$ or otherwise, show that almost surely S_n/n does not converge to a finite limit.

28K Applied Probability

(a) Let $(X_t)_{t \geqslant 0}$ be a Poisson process with rate λ. Show that conditional on the event $X_t = n$ for some $t > 0$, the jump times are distributed as the order statistics of n i.i.d. Uniform $[0, t]$ random variables.

(b) Buses arrive according to a Poisson process with rate λ, indexed by time $t \geqslant 0$. Let T be an exponentially distributed random variable with mean $1/\nu$ independent of the arrival process of the buses. Find the distribution of the number of buses that arrive in the interval $[0, T]$.

(c) Suppose particles arrive according to a Poisson process on $[0, \infty)$ with rate λ. Each particle emits radiation for a time that is uniformly distributed on $[0, 1]$. The emission durations of different particles are independent of each other and independent of the arrival process. A detector records the number of particles that are emitting radiation at time t. Find the distribution of this number.

[*Hint: You may use that the density of a Gamma distribution with parameter n is given by $e^{-x}x^{n-1}/(n-1)!$, for $x > 0$.*]

29L Principles of Statistics

Let $f(x; \theta)$ be a regular statistical model for a real-valued parameter θ.

(a) Explain what is meant by the *maximum likelihood estimator* $\hat{\theta}_n$ based on a sample $X_1, \ldots, X_n$.

(b) State the standard asymptotic result on the distribution of the maximum likelihood estimator in terms of the Fisher information $I(\theta)$. [You are not expected to give precise conditions for the validity of this result.]

Consider the model defined by the density

$$f(x; \theta) = e^{-(x-\theta)} \mathbf{1}_{\{x \geqslant \theta\}}, \quad x \in \mathbb{R}.$$

(c) Find the maximum likelihood estimator $\hat{\theta}_n$ in this case and explain briefly why the standard asymptotics for the distribution of the maximum likelihood estimator may fail to apply.

(d) Show that there exists a constant $\alpha > 0$ such that the sequence of random variables $n^{\alpha}(\hat{\theta}_n - \theta)$ converges in distribution, with limit distribution of non-zero variance, where α and the limiting distribution are to be determined.

(e) Consider the alternative estimator $\tilde{\theta}_n = \bar{X}_n - 1$, where $\bar{X}_n$ is the sample mean. Compare the efficiency of $\hat{\theta}_n$ relative to $\tilde{\theta}_n$ by calculating the limit of the ratio of their mean squared errors as $n \to \infty$.

30L Stochastic Financial Models

Consider a one-period market with d risky assets, with time-t price vector S_t for $t \in \{0,1\}$ and interest rate r. An investor has utility function U, assumed concave, differentiable and strictly increasing, and initial wealth x. Let

$$\mathcal{X} = \{(1+r)x + \theta^\top (S_1 - (1+r)S_0) : \theta \in \mathbb{R}^d\}$$

be the set of possible time-1 wealths attainable by the investor. For a random variable Z, let

$$u(Z) = \sup_{X \in \mathcal{X}} \mathbb{E}[U(X+Z)]$$

be the indirect utility of the payout Z, where $U(X+Z)$ is assumed integrable for all choices of X and Z. [You may assume that this supremum is attained at some $X_Z \in \mathcal{X}$, and in particular that $u(0) = \mathbb{E}[U(X_0)]$ for some $X_0 \in \mathcal{X}$.]

(a) Show that

(i) if $Z_0 \leqslant Z_1$ almost surely and $\mathbb{P}(Z_0 < Z_1) > 0$, then $u(Z_0) < u(Z_1)$,

(ii) $u(pZ_1 + (1-p)Z_0) \geqslant pu(Z_1) + (1-p)u(Z_0)$ for all $p \in (0,1)$.

(b) In this set-up, how is an *indifference price* $\pi(Y)$ of a contingent claim with payout Y defined? Show that the indifference price exists and is unique. [You may use standard properties of concave functions without proof.]

(c) Show that

$$\mathbb{E}[U'(X_0)S_1] = (1+r)\mathbb{E}[U'(X_0)]S_0.$$

[You may interchange differentiation and expectation without justification.]

(d) How is the *marginal utility price* $\pi_0(Y)$ of a contingent claim with payout Y defined? Show that

$$\pi_0(Y) \geqslant \pi(Y).$$

[You may use standard properties of concave functions.]

(e) Suppose that S_1 and Y are jointly Gaussian, and let $\mu = \mathbb{E}(S_1)$ and $V = \mathrm{cov}(S_1)$. Assume that V is positive definite. Show that

$$\pi_0(Y) = \frac{1}{1+r}\left(\mathbb{E}(Y) - (\mu - (1+r)S_0)^\top V^{-1}\mathrm{cov}(S_1, Y)\right).$$

[You may use standard properties of Gaussian random vectors.]

31K Mathematics of Machine Learning

(a) Suppose $\mathcal{H}$ is a class of functions $h : \mathbb{R}^p \to \{-1, 1\}$ with $|\mathcal{H}| \geqslant 2$. Define the *shattering coefficient* $s(\mathcal{H}, n)$ and the *VC dimension* $\mathrm{VC}(\mathcal{H})$ of $\mathcal{H}$.

Let $\mathcal{A}$ be the set of all partitions $\mathcal{I}$ of $\mathbb{R}$ of the form

$$\mathcal{I} := \{(-\infty, a_1), [a_1, a_2), [a_2, a_3), \ldots, [a_{m-2}, a_{m-1}), [a_{m-1}, \infty)\}$$

for some $a_1 < \cdots < a_{m-1}$. We denote $(-\infty, a_1)$ by I_1, $[a_{k-1}, a_k)$ by I_k for $k = 2, \ldots, m-1$, and $[a_{m-1}, \infty)$ by I_m. For a given input set $\mathcal{X}$ and a class $\mathcal{F}$ of functions $f : \mathcal{X} \to \mathbb{R}$, we write $\mathcal{H}_{\mathcal{F}}$ to denote the class of functions $\{\mathrm{sgn} \circ f : f \in \mathcal{F}\}$.

(b) Given fixed $\beta_1, \ldots, \beta_m \in \mathbb{R}$, let $\mathcal{F}_1$ be the class of functions from $\mathbb{R}$ to $\mathbb{R}$ given by

$$\mathcal{F}_1 := \left\{ x \mapsto \sum_{k=1}^{m} \beta_k \mathbb{1}_{I_k}(x) : \mathcal{I} = \{I_1, \ldots, I_m\}, \mathcal{I} \in \mathcal{A} \right\}.$$

Show that $s(\mathcal{H}_{\mathcal{F}_1}, n) \leqslant (n+1)^{m-1}$. [*Hint: Given* $x_1 < x_2 < \cdots < x_n$, *consider the partition* $(-\infty, x_1], (x_1, x_2], (x_2, x_3], \ldots, (x_{n-1}, x_n], (x_n, \infty)$.]

(c) Given fixed $\mathcal{I}_1, \ldots, \mathcal{I}_p \in \mathcal{A}$, let $\mathcal{F}_2$ be a class of functions from $\mathbb{R}^p \to \mathbb{R}$ given by

$$\mathcal{F}_2 = \mathcal{F}_2(\mathcal{I}_1, \ldots, \mathcal{I}_p) := \left\{ x \mapsto \sum_{j=1}^{p} \sum_{I \in \mathcal{I}_j} \beta_{j,I} \mathbb{1}_I(x_j) \ : \beta_{j,I} \in \mathbb{R} \text{ for all } j, I \right\}.$$

Show that $\mathrm{VC}(\mathcal{H}_{\mathcal{F}_2}) \leqslant 1 + p(m-1)$.

(d) Now let

$$\mathcal{F}_3 := \left\{ x \mapsto \sum_{j=1}^{p} \sum_{I \in \mathcal{I}_j} \beta_{j,I} \mathbb{1}_I(x_j) \ : \beta_{j,I} \in \mathbb{R} \text{ for all } j, I \text{ and } \mathcal{I}_1, \ldots, \mathcal{I}_p \in \mathcal{A} \right\},$$

so the difference compared to $\mathcal{F}_2$ is that the partitions are not fixed. Show that

$$s(\mathcal{H}_{\mathcal{F}_3}, n) \leqslant (n+1)^{1+2p(m-1)}.$$

32B Dynamical Systems

(a) What are the properties required of a *strict Lyapunov function* on a domain $\mathcal{D}$ for a dynamical system $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$ in $\mathbb{R}^n$ with a fixed point $\mathbf{x}_0 \in \mathcal{D}$? Define the *domain of stability* of $\mathbf{x}_0$.

By considering the function $V = \frac{1}{2}(x^2+y^2)$ show that the origin is an asymptotically stable fixed point of

$$\begin{aligned}\dot{x} &= -2x + y + x^3 - xy^2,\\ \dot{y} &= -x - 2y + 6x^2y + 4y^3.\end{aligned}$$

Show also that the domain of stability of the origin is contained in $V \leqslant 1$ and includes $V < \frac{1}{4}$.

(b) Find the fixed points of the dynamical system

$$\begin{aligned}\dot{x} &= x(\mu^2 - y),\\ \dot{y} &= x^2 - y,\end{aligned}$$

and determine their type, including the dependence on μ, for $\mu \neq 0$. Find the stable and unstable manifolds of the origin up to 4th order.

33B Integrable Systems

Let M be a six–dimensional phase–space with canonical coordinates (x, y, z, p_x, p_y, p_z), with the Hamiltonian

$$H = \frac{1}{2}({p_x}^2 + {p_y}^2 + {p_z}^2) + V(x, y, z). \qquad (\dagger)$$

(a) Write down Hamilton's equations for (M, H), define a first integral of the system and state what it means to say that the system is *integrable.*

(b) State the Arnold–Liouville theorem.

In the remainder of the question restrict attention to the case where $V(x, y, z) = U(r)$, with $r^2 = x^2 + y^2 + z^2$.

(c) Show that $m_z = xp_y - yp_x$ and $m_y = zp_x - xp_z$ Poisson commute with H. Consider $m_x \equiv \{m_y, m_z\}$ and find a non–constant function $G = G(m_x, m_y, m_z)$ that Poisson commutes with m_x. Deduce that Hamilton's equations, with the Hamiltonian given by ($\dagger$) with $V(x, y, z) = U(r)$, are integrable. [You do not need to show that (H, G, m_x) are independent everywhere on M, but you should show that (H, G, m_x) are independent in a neighbourhood of each point $\mathbf{x} = (1, 0, 0), \mathbf{p} = (p_x, p_y, p_z)$ with $p_y \neq 0, p_z \neq 0$.]

34A Principles of Quantum Mechanics

(a) Define the *uncertainty* of a Hermitian operator Q in the normalised state $|\psi\rangle$. Then state the *generalised uncertainty principle* for two Hermitian operators A and B.

(b) A one-dimensional quantum harmonic oscillator with Hamiltonian

$$H = \frac{1}{2}(P^2 + \omega^2 X^2)$$

is in the coherent state $|\alpha\rangle = e^{\alpha A^\dagger - \bar{\alpha} A}|0\rangle$, where α is a complex number, $\bar{\alpha}$ its complex conjugate, and A is the annihilation operator. Compute the norm of $|\alpha\rangle$ and of $A|\alpha\rangle$.

(c) Compute the expectation value and uncertainty of the position X and the Hamiltonian H in the state $|\alpha\rangle$.

(d) When is the generalised uncertainty principle for these two operators saturated?

(e) Compute the time evolution $|\alpha(t)\rangle$ given that $|\alpha(0)\rangle = |\alpha\rangle$. When is the generalised uncertainty principle for X and H in the state $|\alpha(t)\rangle$ saturated?

35E Applications of Quantum Mechanics

Consider a particle of mass m in three dimensions under the influence of a potential $V(\mathbf{r})$. One approach to solve scattering in this system is to use the Lippmann–Schwinger equation, which reads

$$\psi(\mathbf{k};\mathbf{r}) = e^{i\mathbf{k}\cdot\mathbf{r}} + \int d^3r'\, G_0^+(k;\mathbf{r}-\mathbf{r}')V(\mathbf{r}')\psi(\mathbf{k};\mathbf{r}')\ ,$$

where

$$G_0^+(k;\mathbf{r}-\mathbf{r}') = -\frac{2m}{\hbar^2}\frac{1}{4\pi}\frac{e^{ik|\mathbf{r}-\mathbf{r}'|}}{|\mathbf{r}-\mathbf{r}'|}\ ,$$

is a Green's function with appropriate boundary conditions for waves moving outwards.

(a) Provide a derivation of the Born approximation to the differential cross section, such that

$$\frac{d\sigma}{d\Omega} \approx \left(\frac{m}{2\pi\hbar^2}\right)^2 |\widetilde{V}(\mathbf{q})|^2\ ,$$

where $\widetilde{V}(\mathbf{q})$ and $\mathbf{q}$ are quantities you should specify.

(b) A certain potential $V(\mathbf{r})$ falls as r^{-n} at large distances. Show that the Born approximation to the cross-section is finite if $n > 2$. Briefly comment on the case when $n = 1$.

(c) Using the Born approximation, compute the differential cross section for the potential

$$V(\mathbf{r}) = V_0 \exp\left(-\frac{r^2}{2r_0^2}\right)\ ,$$

with V_0 and r_0 real constants. How must the particle's energy compare to V_0 in order for the Born approximation to be well-justified?

$\Big[$*Hint: You may use without proof that, for $a > 0$,*

$$\int_{-\infty}^{\infty} e^{-ar^2}\,dr = \sqrt{\frac{\pi}{a}}\ .\Big]$$

36E Statistical Physics

Throughout this problem you may assume that all gasses are ideal; that air is a combination of 80% nitrogen gas (N_2 molecules composed of the N-14 isotope) and 20% oxygen gas (O_2 molecules composed of the O-16 isotope); that helium gas is He composed of the He-4 isotope; that nuclear binding energies and electrons may be neglected so that atomic masses (in atomic units) are given by their isotope numbers.

(a) Consider a spherical, rubber party balloon, in a room at room temperature and standard pressure, which is filled with a fixed number N of helium atoms at room temperature T. Neglecting in this part the additional pressure on the internal gas caused by the rubber tension, calculate

(i) the mass density ρ of the He gas relative to the surrounding air,

(ii) the specific heat capacity c_p (at fixed pressure) of the helium, relative to the surrounding air, i.e. $c_p^{\mathrm{He}}/c_p^{\mathrm{air}}$.

(b) In this part, you will take the tension of the rubber into account. Suppose that the rubber has a free energy $F = tA$, where A is its area, and t is the (assumed to be constant) tension per unit length. (Since rubber forces are mostly entropic, it is necessary to add this to the free energy F rather than to the energy E.)

(i) Write down an expression for the free energy F of the balloon, including both the rubber and the internal gas.

(ii) Write down an equation for the equilibrium of the system with respect to changes of the balloon radius r, taking into account the surrounding air pressure.

(iii) At room temperature T, calculate the effect of tension on the radius r to $O(t)$ in a small-t expansion.

37D Electrodynamics

(a) A point particle of mass m and charge q moves in a background electromagnetic field with field-strength tensor $F_{\mu\nu}$. The velocity 4-vector $u^\mu = dx^\mu/d\tau$, with τ the particle's proper time, satisfies

$$m\frac{du^\mu}{d\tau} = qF^\mu{}_\nu u^\nu\,, \qquad (*)$$

where $F^\mu{}_\nu = \eta^{\mu\rho}F_{\rho\nu}$ and $\eta_{\mu\nu} = \mathrm{diag}(-1,+1,+1,+1)$ the Minkowski metric. In some inertial frame, let the particle's velocity be $\mathbf{v}$, with Lorentz factor $\gamma = (1-|\mathbf{v}|^2/c^2)^{-1/2}$, and the electric and magnetic fields $\mathbf{E}$ and $\mathbf{B}$, respectively. Show that the components of $(*)$ reduce to

$$\frac{d\mathbf{p}}{dt} = q(\mathbf{E}+\mathbf{v}\times\mathbf{B})\,,$$
$$mc^2\frac{d\gamma}{dt} = q\mathbf{E}\cdot\mathbf{v}\,,$$

where the momentum $\mathbf{p} = \gamma m\mathbf{v}$. Give a physical interpretation of the second equation.

[*You may assume the following relations between the components of* $F_{\mu\nu}$ *and those of the electric and magnetic fields,* $\mathbf{E} = (E_1,E_2,E_3)$ *and* $\mathbf{B} = (B_1,B_2,B_3)$*:*

$$F_{i0} = -F_{0i} = E_i/c \qquad \textit{and} \qquad F_{ij} = \varepsilon_{ijk}B_k\,,$$

for $i,j = 1,2,3$.]

(b) Maxwell's equations in vacuum admit wavelike solutions with propagation speed c. A circularly polarized wave propagating along the x-axis of some inertial frame has electric field

$$\mathbf{E} = E_0(0,\cos\omega\xi,\sin\omega\xi)\,,$$

where E_0 and ω are constants and $\xi = t - x/c$. Determine the corresponding magnetic field $\mathbf{B}(\xi)$. [You should neglect any constant term.]

The particle in part (a) moves under the influence of this electromagnetic wave. Show that circular motion in the plane $x = 0$ with constant speed is possible, and that the Lorentz factor satisfies

$$\gamma^2 = 1 + \left(\frac{qE_0}{m\omega c}\right)^2\,.$$

Give the radius of the circle in terms of q, E_0, m, ω and γ.

38A General Relativity

The Schwarzschild metric in units with $c = 1$ is given by

$$ds^2 = -\Big(1 - \frac{r_s}{r}\Big)dt^2 + \Big(1 - \frac{r_s}{r}\Big)^{-1}dr^2 + r^2(\, d\theta^2 + \sin^2\theta\, d\phi^2\,) \;,$$

where r_s is a positive constant, the Schwarzschild radius.

(a) For a photon moving in the Schwarzschild metric (with $r > 0$) explain why the trajectory can be assumed to lie in the equatorial plane $\theta = \pi/2$. With this assumption, show that the photon trajectory satisfies

$$\Big(\frac{dr}{d\lambda}\Big)^2 = F(r)\,,$$

where λ is an affine parameter and $F(r)$ is a function of r that is to be determined.

Setting $u = 1/r$, show that

$$\Big(\frac{du}{d\phi}\Big)^2 = \frac{1}{b^2} - u^2 + r_s u^3\,,$$

where b is a constant, to be defined.

(b) Now consider solutions of the form

$$u(\phi) = \frac{1}{b}(\, v_0(\phi) + \varepsilon\, v_1(\phi) + O(\varepsilon^2)\,)\,,$$

where $\varepsilon = r_s/b \ll 1$ and where the following symmetry conditions are imposed:

$$v_0(\pi - \phi) = v_0(\phi) \quad \text{and} \quad v_1(\pi - \phi) = v_1(\phi)\,.$$

Show that to zeroth order in ε the trajectories are straight lines and explain, with reference to a sketch, the geometrical significance of the constant b.

Show that there is a solution for the first order correction of the form

$$v_1(\phi) = A + B\cos 2\phi\,,$$

where A and B are constants, to be determined. Hence deduce an approximate expression for the total angle through which the photon is deflected. Comment briefly on how this result can be tested experimentally.

39C Fluid Dynamics II

The two-dimensional steady boundary-layer equations can be written as

$$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=U\frac{dU}{dx}+\nu\frac{\partial^2 u}{\partial y^2},\qquad \frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0,$$

where U is the velocity far from the layer, and the other symbols take their usual meanings.

State the approximations that have been made to the two-dimensional Navier–Stokes equations to arrive at the above equations.

Calculate the components in plane polar coordinates of the potential flow $\mathbf{u}=\boldsymbol{\nabla}\phi$ for $\phi=-Ak^{-1}r^k\cos(k\theta)$, where $1<k<2$ and $A>0$ are constants. Explain why this could represent an inviscid flow past an infinite wedge with rigid boundaries at $\theta=\pm\pi/k$, but would require boundary layers if the viscosity was nonzero but small.

Consider the boundary layer on $\theta=\pi/k$, and let x denote radial distance along the boundary. Write down $U(x)$ and use a scaling argument to derive the estimate

$$\delta(x)=(\nu x^{2-k}/A)^{1/2},$$

for the thickness of the boundary layer.

Show that the boundary-layer equations are satisfied by a self-similar stream function of the form

$$\psi(x,y)=U(x)\delta(x)f(\eta),\quad \text{where}\quad \eta=\frac{y}{\delta(x)},$$

provided that the dimensionless function $f(\eta)$ satisfies an equation of the form

$$f'''=\alpha f'^2+\beta f f''+\gamma\,,$$

where the numerical coefficients α, β and γ are to be determined in terms of k. What are the boundary conditions satisfied by $f(\eta)$?

Show that the boundary-layer assumption is invalid sufficiently close to the tip of the wedge, but is self-consistent for $x\gg(\nu/A)^a$, where a is to be found.

40C Waves

An infinite rigid wavy wall moves with constant speed V in the positive x direction, such that the equation of the wall is given by

$$y = y_{\text{wall}}(x,t) \equiv \text{Re}[\epsilon \exp(ik(x - Vt))] \,,$$

where $0 < \epsilon \ll 1$ and k is a constant.

(a) The region $y > y_{\text{wall}}$ is filled with a uniform acoustic medium in which the steady sound speed and mean density are c_1 and ρ_1 respectively. You may assume that the wall imposes a boundary condition on the medium at $y = 0$.

For the case $V > c_1$, show that the y component of the time-averaged acoustic energy flux is

$$\frac{\rho_1 k^2 V^3 \epsilon^2}{2\beta_1} \,,$$

where $\beta_1 = \sqrt{(V/c_1)^2 - 1}$. Explain briefly, without further detailed calculation, how your answer would change if $V < c_1$.

[You may use without proof the fact that the acoustic energy flux vector, $\mathbf{I}$, is given by $\mathbf{I} = p'\mathbf{u}'$, where p' is the acoustic pressure and $\mathbf{u}'$ is the acoustic velocity.]

(b) Now suppose that the uniform medium is replaced by a two-layer medium, in which the steady sound speed and density in $h > y > y_{\text{wall}}$ are now c_0 and ρ_0 respectively, while in $y \geqslant h$ the steady sound speed and density still take the values c_1 and ρ_1 respectively.

For the case $c_1 < V < c_0$, calculate the y component of the time-averaged acoustic energy flux in $y \geqslant h$. Simplify your answer in the limiting cases (i) $k\beta_0 h \gg 1$ and (ii) $k\beta_0 h \ll 1$, where $\beta_0 = \sqrt{1 - (V^2/c_0^2)}$. Comment briefly on your answers in these two limits.

41B Numerical Analysis

(a) Describe the *power method* for computing the largest eigenvalue (in modulus) of a square matrix.

(b) Let $A \in \mathbb{R}^{n \times n}$ be symmetric. Show that all eigenvectors of A are real and that eigenvectors corresponding to different eigenvalues are orthogonal. Then use Householder reflections to show that there exists a real orthogonal matrix $Q \in \mathbb{R}^{n \times n}$ such that

$$Q^T A Q = \Lambda = \operatorname{diag}(\lambda_1, \ldots, \lambda_n),$$

and that, for $k = 1, \ldots, n$, the kth column of Q is an eigenvector, i.e.

$$A\, Q e_k = \lambda_k\, Q e_k,$$

where e_k is the kth canonical basis vector of $\mathbb{R}^n$.

(c) Suppose $A \in \mathbb{R}^{n \times n}$ is symmetric and that

$$Q^T A Q = \operatorname{diag}(\lambda_1, \ldots, \lambda_n),$$

where $Q = \big[\, q_1 \mid \cdots \mid q_n \,\big]$ is orthogonal and $|\lambda_1| > |\lambda_2| \geqslant \cdots \geqslant |\lambda_n|$. Let the vectors $q^{(k)}$ be the k-th iteration vectors of the power method, for $k = 0, 1, 2, \ldots$, and the scalars $\lambda^{(k)}$ be defined by $\lambda^{(k)} = \big(q^{(k)}\big)^T A\, q^{(k)}$. Define $\theta_k \in [0, \pi/2]$ by

$$\cos\theta_k = \big|q_1^T q^{(k)}\big|.$$

Show that if $\cos\theta_0 \neq 0$, then for $k = 0, 1, \ldots$ we have

$$|\sin\theta_k| \;\leqslant\; \left|\frac{\lambda_2}{\lambda_1}\right|^k \tan\theta_0,$$

$$\big|\lambda^{(k)} - \lambda_1\big| \;\leqslant\; \max_{2 \leqslant i \leqslant n} |\lambda_1 - \lambda_i| \left|\frac{\lambda_2}{\lambda_1}\right|^{2k} \tan^2\theta_0.$$

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