UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part II 2026

List of Courses

Riemann Surfaces

Statistical Modelling

Statistical Physics

Stochastic Financial Models

Topics in Analysis

Waves

Paper 1, Section II

25F Algebraic Geometry

Define what it means for two irreducible varieties X and Y to be *birational*. State a criterion for X and Y to be birational in terms of their function fields $k(X)$ and $k(Y)$.

Let $X \subset \mathbb{P}^2$ be the curve defined by $y^2 z = x^3 + x^2 z$. Show that X is irreducible. Show that X has exactly one singular point P. By considering lines through P or otherwise, construct a birational map $\varphi : \mathbb{P}^1 \dashrightarrow X$.

Let $\widetilde{X} \subset X \times \mathbb{P}^1$ be the closure of the set of pairs

$$\{(Q,t) \mid Q \in \{y = tx\} \cap X, Q \neq P\}.$$

Let $\pi : \widetilde{X} \to X$ be projection onto the first factor. Show that the preimage $\pi^{-1}(Q)$ contains a unique point for all $Q \neq P$, while $\pi^{-1}(P)$ consists of exactly two points. Briefly explain the geometric significance of this fact.

Let $Y \subset \mathbb{P}^n$ be a smooth projective variety. Suppose there exists a dominant rational map $\psi : \mathbb{P}^m \dashrightarrow Y$. Must Y be isomorphic to $\mathbb{P}^k$ for some k? Justify your answer.

Paper 2, Section II

25F Algebraic Geometry

Let $V \subset \mathbb{A}^n$ be an irreducible affine variety and $p \in V$. Define the *tangent space* $T_{V,p}$, the *dimension* of V, and what it means for p to be a *smooth* point of V. Prove that the set of smooth points of V is a non-empty Zariski open subset of V.

Prove that the set of lines $L \subset \mathbb{P}^2$ can be realized as a projective variety by identifying the set with a projective plane, which we denote by $(\mathbb{P}^2)^*$. Let P be a fixed point in $\mathbb{P}^2$. Show that the set of lines in $\mathbb{P}^2$ containing P is a line in $(\mathbb{P}^2)^*$.

Let $X \subset \mathbb{P}^2$ be a (smooth irreducible projective) planar curve. Show that the dual map $P \mapsto T_{X,P}$ defines a morphism from X to $(\mathbb{P}^2)^*$.

The dual curve $X^* \subset (\mathbb{P}^2)^*$ is the set of lines $L \subset \mathbb{P}^2$ tangent to X; that is, the image of X under the dual map. Let X be defined by $x^3 + y^3 + z^3 = 0$. Find the degree 6 equation defining X^* in the coordinates $(u : v : w)$ on $(\mathbb{P}^2)^*$. Is X^* smooth? What is the geometric interpretation of any points on X that map to singular points on X^*? [You may use without proof the fact that X is smooth and irreducible.]

Paper 3, Section II

24F Algebraic Geometry

Let C be a (smooth irreducible projective) curve of genus g. State the Riemann–Roch theorem. Define the *canonical class* K_C and prove that $\deg(K_C) = 2g - 2$.

Suppose $g > 1$, and suppose there exists an effective divisor D of degree 2 on C with $\ell(D) = 2$. Prove that C is hyperelliptic. Deduce that every genus 2 curve is hyperelliptic.

Now suppose that $g = 3$ and C is not hyperelliptic. Show that the canonical map $\varphi_{K_C} : C \to \mathbb{P}^2$ is an embedding.

Determine how many points Q of a genus 3 hyperelliptic curve satisfy $\ell(2Q) = 2$. [*Hint: You may wish to apply Riemann–Roch to a suitably chosen divisor of degree 3.*]

Paper 4, Section II

24F Algebraic Geometry

Define the *degree* of a divisor D on a (smooth irreducible projective) curve C. What does it mean for a divisor to be *principal*? Define the *divisor class group* $\mathrm{Cl}(C)$ and the subgroup $\mathrm{Cl}^0(C)$.

Compute the canonical class of $\mathbb{P}^1$. Compute the canonical class of a hyperelliptic curve C.

Prove that a curve C has $\mathrm{Cl}^0(C)$ trivial if and only if C is isomorphic to $\mathbb{P}^1$.

Prove that if C has genus $g \geqslant 1$ and $p_0 \in C$ then the map $\phi : C \to \mathrm{Cl}^0(C)$ given by $p \mapsto [p - p_0]$ is injective. Using this, or otherwise, show that for a curve of genus $g \geqslant 1$, which is not hyperelliptic, the divisor $p + q$ is linearly equivalent to the divisor $r + s$ if and only if $\{p, q\} = \{r, s\}$ as sets.

Paper 1, Section II

21H Algebraic Topology

Let K be a simplicial complex, and $f : K \to K$ a simplicial map. State two equivalent formulas for the Lefschetz number of f. State and prove the Lefschetz fixed point theorem. [Results about simplicial approximation and barycentric subdivision may be used without proof.]

Let $S^n \subset \mathbb{R}^{n+1}$ be the standard sphere. Let $a : S^n \to S^n$ be the antipodal map. For n even, can a be homotopic to the identity?

Suppose $v : S^n \to \mathbb{R}^{n+1}$ is a continuous function such that the dot product $\mathbf{x} \cdot v(\mathbf{x})$ vanishes for all $\mathbf{x} \in S^n$. For n even, show that v must have at least one zero.

Paper 2, Section II

21H Algebraic Topology

What does it mean for two spaces to be *homotopy equivalent*? Show that $\mathbb{C}^* \times \mathbb{C}$ is homotopy equivalent to S^1.

Let $X = \mathbb{C}^2 \setminus \{(x, x) : x \in \mathbb{C}\}$, and let $x_0 = (1, 0) \in X$. Find the fundamental group $\pi_1(X, x_0)$.

Let Y be the set of degree two, monic complex polynomials with distinct roots, equipped with the subspace topology from the complex vector space of their coefficients. Let $f : X \to Y$ be the function taking (a, b) to the polynomial $(x - a)(x - b)$. Show that $f : X \to Y$ is a covering map.

Let $p_0 \in Y$ be given by $p_0(x) = x(x - 1)$. Find the fundamental group $\pi_1(Y, p_0)$.

Paper 3, Section II

20H Algebraic Topology

State the Mayer-Vietoris theorem for a simplicial complex K which is the union of subcomplexes M and N.

Let $T = S^1 \times S^1$ be the torus, and fix distinct points $p, q \in T$. Let X be the quotient topological space $T/\sim$, where $\sim$ is the equivalence relation generated by $p \sim q$.

(a) Show that X is triangulable.

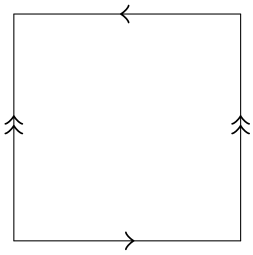
(b) Calculate the homology groups of X. [You may use any result from lectures without proof.]

(c) Are all maps from X to S^2 homotopic to constant maps? Briefly justify your answer. [*Hint: Consider second homology groups.*]

Paper 4, Section II

21H Algebraic Topology

Let K be the Klein bottle obtained by identifying the sides of the unit square as shown in the figure, and let $x_0 \in K$ be the image of the corners of the square.



Give a cell complex description for K, and use it to deduce a presentation for $\pi_1(K, x_0)$.

State the Galois correspondence for based covering spaces. Describe based covering spaces $p : (\widetilde{K}, \widetilde{x}_0) \to (K, x_0)$ such that the corresponding subgroup of $\pi_1(K, x_0)$ is isomorphic to (i) $\mathbb{Z}^2$; and (ii) $\mathbb{Z}$.

Given a based covering map $p : (\widetilde{K}, \widetilde{x}_0) \to (K, x_0)$, show that if $\widetilde{K}$ is compact, then $p_*\pi_1(\widetilde{K}, \widetilde{x}_0)$ must have finite index as a subgroup of $\pi_1(K, x_0)$. Is the converse true?

Paper 1, Section II

23H Analysis of Functions

(a) State and prove a formula for the Fourier transform of the convolution of two functions in $L^1(\mathbb{R}^d)$ in terms of the Fourier transforms of the individual functions.

(b) State and prove Plancherel's formula for Schwartz functions. [This is also known as Parseval's formula. You may use without proof the Fourier inversion formula, and properties of Schwartz functions.]

(c) Show that the formula in part (a) is valid for the convolution of a function in $L^1(\mathbb{R}^d)$ and a function in $L^2(\mathbb{R}^d)$.

(d) Fix a function $f \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ such that $\widehat{f}(\xi) = 0$ if and only if $\xi = 1$. Denote by Y the vector space of functions that are finite linear combinations of translates of f, that is, the elements of Y are the functions $a_1 f(x - t_1) + \ldots + a_n f(x - t_n)$, where $n \in \mathbb{Z}_{\geqslant 1}$, $a_1, \ldots, a_n \in \mathbb{C}$ and $t_1, \ldots, t_n \in \mathbb{R}$ are arbitrary numbers.

(i) Prove that Y is dense in $L^2(\mathbb{R})$.

(ii) Prove that Y is not dense in $L^1(\mathbb{R})$.

[*Hint: In part (d), you may find it useful to recall a characterisation of linear subspaces of a Banach space X that are not dense in X in terms of the existence of non-zero functionals on X with appropriate properties.*]

[*In parts (c) and (d), you may use any results from the course provided they are stated clearly.*]

Paper 2, Section II

23H Analysis of Functions

(a) Define the *Sobolev space* $H^s(\mathbb{R}^d)$ for $s \in \mathbb{R}$ and $d \in \mathbb{Z}_{\geqslant 1}$. When $s \in \mathbb{Z}_{\geqslant 0}$, give an equivalent Sobolev norm without using the Fourier transform. [You do not need to prove the equivalence.]

(b) State and prove the Sobolev embedding theorem relating $H^s(\mathbb{R}^d)$ to $L^\infty(\mathbb{R}^d)$ for suitable s.

(c) Define the space $H_0^1(\Omega)$ for a domain $\Omega \subset \mathbb{R}^d$.

(d) Show that $H_0^1(\mathbb{R}^3 \backslash \{0\}) = H^1(\mathbb{R}^3)$. [*Hint: Work with the equivalent Sobolev norm you gave in part (a), and make use of $\varphi_n(x) = \varphi(nx)$ where $\varphi \in C^\infty(\mathbb{R}^d)$ is such that $\varphi(x) = 1$ for $|x| \leqslant 1$ and $\varphi(x) = 0$ for $|x| \geqslant 2$. You may assume without proof that such φ exists.*]

(e) Is it true that $H_0^1(\mathbb{R} \backslash \{0\}) = H^1(\mathbb{R})$? Justify your answer.

Paper 3, Section II

22H Analysis of Functions

(a) Give the definition of a *Lebesgue point* of a function $f \in L^1(\mathbb{R}^d)$.

(b) Let $f \in L^1(\mathbb{R}^d)$. Prove that almost every point in $\mathbb{R}^d$ is a Lebesgue point of f. [You may use without proof the Hardy–Littlewood maximal inequality and any results in the course about the density of continuous functions, provided they are stated clearly.]

(c) Let $f \in L^1(\mathbb{R})$ and let $g \in L^\infty(\mathbb{R})$. Suppose that g is supported on a compact set and $\int g(x)dx = 1$. For $n \in \mathbb{Z}_{\geqslant 1}$, let $g_n(x) = ng(nx)$. Prove that $\lim_{n\to\infty} f * g_n(x) = f(x)$ for all Lebesgue points x of f.

(d) Does the statement in part (c) remain true if we replace the assumption that g is supported on a compact set with the assumption that $g \in L^1(\mathbb{R})$, keeping all other assumptions unchanged? Give a proof or a counterexample.

Paper 4, Section II

23H Analysis of Functions

(a) State and prove the Banach–Alaoglu theorem for separable Banach spaces. [You do not need to prove metrizability of the topology that appears in the theorem.]

(b) Let $\Phi \subset C_c^\infty(\mathbb{R})$ be a fixed set of functions and let $p \in (1, \infty)$. Write

$$H = \left\{ f \in L^p(\mathbb{R}) : \int \varphi(x)f(x)dx = 0 \text{ for all } \varphi \in \Phi \right\},$$

and let $g_1, g_2, g_3 \in L^p(\mathbb{R})$ be fixed functions. Prove that there is $f_0 \in H$ such that

$$\|f_0 - g_1\|_p + \|f_0 - g_2\|_p + \|f_0 - g_3\|_p = \inf_{f \in H} \left(\|f - g_1\|_p + \|f - g_2\|_p + \|f - g_3\|_p \right),$$

where $\|h\|_p$ denotes the L^p-norm of the function h. [In part (b), you may use without proof any results from the course provided they are stated clearly.]

Paper 1, Section II

35E Applications of Quantum Mechanics

Consider a particle of mass m in three dimensions under the influence of a potential $V(\mathbf{r})$. One approach to solve scattering in this system is to use the Lippmann–Schwinger equation, which reads

$$\psi(\mathbf{k};\mathbf{r}) = e^{i\mathbf{k}\cdot\mathbf{r}} + \int d^3r'\, G_0^+(k;\mathbf{r}-\mathbf{r}')V(\mathbf{r}')\psi(\mathbf{k};\mathbf{r}')\ ,$$

where

$$G_0^+(k;\mathbf{r}-\mathbf{r}') = -\frac{2m}{\hbar^2}\frac{1}{4\pi}\frac{e^{ik|\mathbf{r}-\mathbf{r}'|}}{|\mathbf{r}-\mathbf{r}'|}\ ,$$

is a Green's function with appropriate boundary conditions for waves moving outwards.

(a) Provide a derivation of the Born approximation to the differential cross section, such that

$$\frac{d\sigma}{d\Omega} \approx \left(\frac{m}{2\pi\hbar^2}\right)^2 |\widetilde{V}(\mathbf{q})|^2\ ,$$

where $\widetilde{V}(\mathbf{q})$ and $\mathbf{q}$ are quantities you should specify.

(b) A certain potential $V(\mathbf{r})$ falls as r^{-n} at large distances. Show that the Born approximation to the cross-section is finite if $n > 2$. Briefly comment on the case when $n = 1$.

(c) Using the Born approximation, compute the differential cross section for the potential

$$V(\mathbf{r}) = V_0 \exp\left(-\frac{r^2}{2r_0^2}\right)\ ,$$

with V_0 and r_0 real constants. How must the particle's energy compare to V_0 in order for the Born approximation to be well-justified?

[*Hint: You may use without proof that, for* $a > 0$,

$$\int_{-\infty}^{\infty} e^{-ar^2}\, dr = \sqrt{\frac{\pi}{a}}\ .]$$

Paper 2, Section II

36E Applications of Quantum Mechanics

Consider a Hamiltonian H with a discrete spectrum whose ground state energy is denoted E_0.

(a) Briefly describe how one uses the variational method to provide an upper bound on E_0.

(b) If the trial state differs from the ground state of H by a term of order ϵ, prove that this leads to an error of order ϵ^2 in the energy.

A particle of mass m and charge $-q$ is placed in an infinite one-dimensional potential well of width a, that is

$$V(x) = \begin{cases} 0 & \text{for } 0 \leqslant x \leqslant a\,, \\ +\infty & \text{elsewhere}\,. \end{cases}$$

In addition, the particle is subject to a perturbatively weak, uniform electric field $\mathcal{E}$ so that the following term is added to the potential

$$V_{\mathcal{E}}(x) = \left(x - \frac{a}{2}\right) q\,\mathcal{E}\,.$$

(c) Briefly explain why the perturbation to the ground state energy due to $V_{\mathcal{E}}(x)$ only appears at order $\mathcal{E}^2$.

(d) In order to give an upper bound to the shift in energy of the ground state due to $V_{\mathcal{E}}(x)$, one might consider the following trial function

$$\psi_\alpha(x) = \sqrt{\frac{2}{a}} \sin\left(\frac{\pi x}{a}\right) \left[1 + \left(x - \frac{a}{2}\right) \alpha\, q\,\mathcal{E}\right]\,,$$

where α is the variational parameter. Explain briefly why this is a reasonable choice of trial function.

(e) Determine the optimal value of α which gives the best bound on ground state energy to quadratic order in $\mathcal{E}$. [You do not need to determine the corresponding energy.]

[*Hint: You may use without proof the following:*

$$\int_0^a \cos\left(\frac{2\pi x}{a}\right)\left(x - \frac{a}{2}\right)^2 dx = \frac{a^3}{2\pi^2}\,, \qquad \int_0^a \sin\left(\frac{2\pi x}{a}\right)\left(x - \frac{a}{2}\right) dx = -\frac{a^2}{2\pi}\,.\ ]$$

Paper 3, Section II

34E Applications of Quantum Mechanics

(a) Consider a three-dimensional system, with Hamiltonian H, that is invariant under the discrete translational symmetries of a Bravais lattice Λ. State and prove Bloch's theorem for H.

(b) A crystal has identical atoms at positions given by the vectors

$$\begin{gathered} a\left[n_1\hat{\mathbf{x}} + n_2\hat{\mathbf{y}} + n_3\hat{\mathbf{z}}\right] \ , \\ a\left[\left(n_1 + \tfrac{1}{2}\right)\hat{\mathbf{x}} + \left(n_2 - \tfrac{1}{2}\right)\hat{\mathbf{y}} + \left(n_3 + \tfrac{1}{2}\right)\hat{\mathbf{z}}\right] \ , \\ a\left[\left(n_1 + \tfrac{1}{2}\right)\hat{\mathbf{x}} + \left(n_2 + \tfrac{1}{2}\right)\hat{\mathbf{y}} + \left(n_3 - \tfrac{1}{2}\right)\hat{\mathbf{z}}\right] \ , \\ a\left[\left(n_1 - \tfrac{1}{2}\right)\hat{\mathbf{x}} + \left(n_2 + \tfrac{1}{2}\right)\hat{\mathbf{y}} + \left(n_3 + \tfrac{1}{2}\right)\hat{\mathbf{z}}\right] \ , \end{gathered}$$

where (n_1, n_2, n_3) are arbitrary integers, a is a constant, and $\hat{\mathbf{x}}$, $\hat{\mathbf{y}}$, $\hat{\mathbf{z}}$ are mutually orthogonal unit vectors.

(i) Show that these vectors define a Bravais lattice Λ with basis vectors

$$\mathbf{a}_1 = \tfrac{a}{2}\left(-\hat{\mathbf{x}} + \hat{\mathbf{y}} + \hat{\mathbf{z}}\right) \ , \quad \mathbf{a}_2 = \tfrac{a}{2}\left(\hat{\mathbf{x}} - \hat{\mathbf{y}} + \hat{\mathbf{z}}\right) \ , \quad \mathbf{a}_3 = \tfrac{a}{2}\left(\hat{\mathbf{x}} + \hat{\mathbf{y}} - \hat{\mathbf{z}}\right) \ .$$

(ii) Determine the volume of the primitive cell.

(iii) Find basis vectors $\mathbf{b}_1$, $\mathbf{b}_2$, and $\mathbf{b}_3$ for the reciprocal lattice and determine the volume of the unit cell of the reciprocal lattice.

(c) Consider the dynamics of a single electron on the lattice Λ given above, where the Hamiltonian is described by a tight-binding model given by

$$H = \sum_{\mathbf{r}\in\Lambda}\Big[E_0|\mathbf{r}\rangle\langle\mathbf{r}| - \mu\sum_i\Big(|\mathbf{r}\rangle\langle\mathbf{r}+\mathbf{a}_i| + |\mathbf{r}+\mathbf{a}_i\rangle\langle\mathbf{r}|\Big)\Big] \ ,$$

where E_0 and μ are real constants.

(i) Write down the form that the eigenstates of the Hamiltonian take, and determine the energy spectrum as a function of the wave vector in the Brillouin zone.

(ii) What is the width of the energy band, $\Delta E = E_{\text{max}} - E_{\text{min}}$, in this zone?

(iii) Show that, for small crystal momentum, the dispersion relation takes the form of a free particle and determine the effective masses.

[*Hint: You may use without proof the fact that a matrix of the form*

$$\begin{pmatrix} \rho & \tau & \tau \\ \tau & \rho & \tau \\ \tau & \tau & \rho \end{pmatrix}$$

has eigenvalues $\rho - \tau$, $\rho - \tau$, *and* $\rho + 2\tau$.]

Paper 4, Section II

34E Applications of Quantum Mechanics

Consider a spinless particle of charge q and mass m in three dimensions under the influence of a uniform magnetic field, where the vector potential $\mathbf{A}$ satisfies

$$\mathbf{A} = \frac{B}{2}(-y, x, 0) \ ,$$

with B constant, and electrostatic potential $\phi = 0$.

(a) Write down the Hamiltonian and the Schrödinger equation for this particle.

(b) Consider the operators

$$\boldsymbol{\rho} = \frac{1}{qB}\hat{\mathbf{z}} \times (\mathbf{p} - q\mathbf{A}) \ , \qquad \mathbf{R} = \mathbf{r} - \boldsymbol{\rho} \ ,$$

where $\mathbf{r}$ and $\mathbf{p}$ are the usual position and momentum operators, and $\hat{\mathbf{z}}$ is the unit vector along the z-axis. Show that the only non-zero commutators involving components of $\boldsymbol{\rho}$ and $\mathbf{R}$ are $[\rho_x, \rho_y]$ and $[R_x, R_y]$.

(c) Given the following definitions

$$a = \frac{1}{r_B}\frac{1}{\sqrt{2}}(\rho_x + i\rho_y) \ , \qquad b = \frac{1}{r_B}\frac{1}{\sqrt{2}}(R_x - iR_y) \ ,$$

where $r_B^2 = \hbar/(qB)$, evaluate $[a, a^\dagger]$ and $[b, b^\dagger]$.

(d) When the particle is confined to move in the xy-plane, show that the Hamiltonian can be written as

$$H = \hbar\omega\left(a^\dagger a + \frac{1}{2}\right) \ ,$$

where ω is a parameter you should identify.

(e) Given that there exists a unique state $|\psi\rangle$ satisfying

$$a|\psi\rangle = b|\psi\rangle = 0 \ ,$$

what conclusions can be drawn about the allowed energies of the Hamiltonian and their degeneracies? [*Hint: Consider the expectation value of* $R_x^2 + R_y^2$ *in an eigenstate of the system and how it relates to a typical radius of motion.*]

Paper 1, Section II

28K Applied Probability

(a) Let $(X_t)_{t \geqslant 0}$ be a Poisson process with rate λ. Show that conditional on the event $X_t = n$ for some $t > 0$, the jump times are distributed as the order statistics of n i.i.d. Uniform $[0, t]$ random variables.

(b) Buses arrive according to a Poisson process with rate λ, indexed by time $t \geqslant 0$. Let T be an exponentially distributed random variable with mean $1/\nu$ independent of the arrival process of the buses. Find the distribution of the number of buses that arrive in the interval $[0, T]$.

(c) Suppose particles arrive according to a Poisson process on $[0, \infty)$ with rate λ. Each particle emits radiation for a time that is uniformly distributed on $[0, 1]$. The emission durations of different particles are independent of each other and independent of the arrival process. A detector records the number of particles that are emitting radiation at time t. Find the distribution of this number.

[*Hint: You may use that the density of a Gamma distribution with parameter n is given by $e^{-x} x^{n-1}/(n-1)!$, for $x > 0$.*]

Paper 2, Section II

28K Applied Probability

(a) For $\lambda > 0$, define what is meant by a *simple birth process* $(X_t)_{t \geqslant 0}$ with $X_0 = 1$ and parameter λ. Find the probability generating function $\mathbb{E}[z^{X_t}]$. Hence find the distribution of X_t and $\mathbb{E}X_t$. Now if $X_0 = i$ for some $i \in \mathbb{N}$, find the distribution of X_t and $\mathbb{E}X_t$. [You may assume that $\mathbb{E}[z^{X_t}]$ is differentiable in t. You may also use that the probability generating function of a Geometric(p) random variable is $\frac{zp}{1-(1-p)z}$.]

(b) Consider a right-continuous continuous-time Markov chain $(X_t)_{t \geqslant 0}$ on the state space $\{0, 1, 2, \ldots\}$ such that

$$q_{i,i+1} = \frac{1}{i+2} \text{ for } i \geqslant 0, \quad q_{i,0} = \frac{1}{(i+1)(i+2)} \text{ for } i \geqslant 1.$$

All other off-diagonal elements of Q are 0. Is $(X_t)_{t \geqslant 0}$ recurrent? Is $(X_t)_{t \geqslant 0}$ explosive? Is $(X_t)_{t \geqslant 0}$ positive recurrent? Does $(X_t)_{t \geqslant 0}$ have an invariant distribution? Is the jump chain positive recurrent? Justify your answers.

(c) Let $(X_t)_{t \geqslant 0}$ be an irreducible non-explosive right-continuous continuous-time Markov chain with generator Q and transition semigroup P. Suppose there exists a distribution π such that $\pi P(t) = \pi$ for all $t > 0$. Does it follow that $(X_t)_{t \geqslant 0}$ is recurrent? Does it follow that $(X_t)_{t \geqslant 0}$ is positive recurrent? Does it follow that π is invariant for $(X_t)_{t \geqslant 0}$? [Clearly state all results you use.]

Paper 3, Section II

27K Applied Probability

Let $(\xi_i)_{i\geqslant 1}$ be a sequence of i.i.d. non-negative random variables with $\mathbb{E}(\xi_1) = 1/\lambda$ for some $\lambda > 0$.

(a) Let M be a non-negative integer-valued random variable such that $\mathbb{E}(M) < \infty$ and suppose that for any $i \in \mathbb{N}$, the event $\{M \leqslant i\}$ depends only on the random variables $\{\xi_1, \ldots, \xi_i\}$. Show that

$$\mathbb{E}\left(\sum_{i=1}^{M} \xi_i\right) = \lambda^{-1}\mathbb{E}(M)\,.$$

Let $T_n = \sum_{i=1}^{n} \xi_i$ for $n \in \mathbb{N}$. Let $(N_t)_{t\geqslant 0}$ be the renewal process associated with the interarrival times (ξ_i).

(b) Show that

$$\mathbb{E}(T_{N_t+1}) = \lambda^{-1}(\mathbb{E}(N_t) + 1)\,.$$

(c) Show that $N_t/t \to \lambda$ almost surely as $t \to \infty$.

(d) Assume that ξ_1 is a bounded random variable. Show that $\mathbb{E}(N_t)/t \to \lambda$ as $t \to \infty$. [*Hint: Consider showing* $T_{N_t+1} - \xi_{N_t+1} \leqslant t < T_{N_t+1}$.]

Paper 4, Section II

27K Applied Probability

(a) Consider an $M/M/\infty$ queue with arrival rate $\lambda > 0$ and service rate $\mu > 0$. Let $(X_t)_{t\geqslant 0}$ be the Markov chain representing the number of busy servers at time t. Suppose $X_0 = 0$.

(i) Find the distribution of X_t.

(ii) Find the invariant distribution of $(X_t)_{t\geqslant 0}$.

(iii) Using your expression for the distribution of X_t from part (i), show by direct computation that

$$\lim_{t\to\infty} \mathbb{P}(X_t = k) = \pi_k, \qquad \text{for all } k = 0, 1, 2, \ldots$$

where $\pi = (\pi_0, \pi_1, \ldots)$ is the invariant distribution derived in part (ii).

(b) Let $\lambda : \mathbb{R}^d \to [0, \infty)$ be a non-negative function such that $\int_A \lambda(x)\,dx < \infty$ for every bounded subset $A \subset \mathbb{R}^d$. State what it means for a point process Π on $\mathbb{R}^d$ to be a *Poisson process with intensity function* λ.

(c) Assume that viruses are distributed in $\mathbb{R}^2$ according to a homogeneous spatial Poisson process Π with a constant intensity $\lambda > 0$. A virus can infect any point within a disc of random radius X centred at its location, where X is distributed as Uniform$[0, R]$, for some fixed $R > 0$. The infection distances of the viruses are independent of each other and of Π. Find the distribution of the number of viruses infecting the origin $(0, 0)$.

Paper 2, Section II

32D Asymptotic Methods

(a) Consider the behaviour of the exponential integral function

$$E(x) = \int_x^\infty \frac{e^{-t}}{t}\,dt, \qquad (\dagger)$$

as $x \to \infty$.

(i) Use integration by parts to show that

$$E(x) \sim e^{-x} x^{-1} \sum_{n=0}^{\infty} a_n x^{-n}, \qquad (\star)$$

as $x \to \infty$, where the coefficients a_n are to be determined.

(ii) Verify that Watson's lemma applied to ($\dagger$) also produces ($\star$).

(iii) Determine the number of terms one must retain in ($\star$) so that the asymptotic approximation is optimally truncated. Show that the resulting relative error is exponentially small.

[You may quote, without proof, the result $\Gamma(x+1) \sim \sqrt{2\pi x}(x/e)^x$, as $x \to \infty$.]

(b) Consider

$$J_n = \int_0^\pi \cos\left[n\cos(nt)\right] dt,$$

for n a large positive integer. Show that its leading order asymptotic approximation is $A\cos\alpha$, where A and α are functions of n you need to find. Without calculation, estimate the size of the next term.

[You may quote, without proof, any relevant results concerning the method of stationary phase.]

Paper 3, Section II

30D Asymptotic Methods

Consider the behaviour of the function

$$I(x) = \int_{-1}^{\infty} \exp\left[x\left(t + it - \tfrac{1}{2}t^2\right)\right] dt,$$

as $x \to \infty$.

(a) Transform $I(x)$ into a complex contour integral, so that $t = z \in \mathbb{C}$, and determine the curve of steepest descent through the endpoint $z = -1$.

(b) Find the saddle point and the curve of steepest descent that passes through it.

(c) Deform the integration contour so that it passes through the saddle and draw the contour in the complex plane. Carefully estimate the contribution to $I(x)$ from any 'bridging' curves.

(d) Provide an argument for why the saddle contribution dominates the endpoint contribution, and then show that

$$I(x) \sim \alpha x^{-1/2} e^{ix},$$

as $x \to \infty$, where α is a constant you need to determine. What is the magnitude of the next-largest term in the asymptotic expansion of $I(x)$?

[You may quote, without proof, results concerning Laplace's method.]

Paper 4, Section II

31D Asymptotic Methods

Consider the ordinary differential equation

$$y'' + \lambda^2 q(x) y = 0, \tag{$\dagger$}$$

defined on the interval $[a, b]$, where $q(x) > 0$ is analytic and λ is real.

(a) Derive the asymptotic WKBJ approximation to y on $[a, b]$, in the form:

$$y(x) \sim q^{-1/4} \left(A e^{i\lambda \int_a^x \sqrt{q(t)}dt} + B e^{-i\lambda \int_a^x \sqrt{q(t)}dt} \right),$$

for $\lambda \to \infty$, where A and B are complex constants.

(b) Impose the boundary conditions $y(a) = y(b) = 0$ and regard λ as an eigenvalue of the problem, taking discrete values λ_n.

(i) Show that $\lambda_n \sim E\, n$, as $n \to \infty$, where E is a constant you must determine. Give the form of the associated eigenfunction y_n.

(ii) Show, by direct substitution, that the asymptotic eigenfunctions in part (i) obey the orthogonality condition:

$$\int_a^b y_n(x) y_m(x) q(x) dx = 0,$$

for $n \neq m$.

(iii) Let $q(x) = x^{-2}$ and set $a = 1$ and $b = e$. Solve ($\dagger$) exactly and obtain exact expressions for the eigenvalues λ_n. Compare these with their WKBJ approximations. Show that the magnitude of any discrepancy is consistent with the order of your asymptotic approximation.

Paper 1, Section I

4J Automata and Formal Languages

In this question, we will set $\mathbb{N} = \{0, 1, 2, 3, \ldots\}$.

(a) Define the class of *primitive recursive* functions giving precise definitions of the basic functions and the closure operations that define it.

(b) Show that the addition function $(n, m) \mapsto n + m$ is primitive recursive.

(c) Let $f \colon \mathbb{N}^2 \to \mathbb{N}$ be primitive recursive. Show that its diagonal function $d \colon \mathbb{N} \to \mathbb{N}$, given by $d(n) = f(n, n)$, is primitive recursive.

(d) A function $f \colon \mathbb{N}^2 \to \mathbb{N}$ is called a *complete list* if for each primitive recursive function $g \colon \mathbb{N} \to \mathbb{N}$, there is some $n \in \mathbb{N}$ such that for all $k \in \mathbb{N}$, we have $f(n, k) = g(k)$. Show that a complete list cannot be primitive recursive.

Paper 1, Section I

12I Automata and Formal Languages

(a) Define the notion of a *(variable based) grammar* G and the *language* $\mathcal{L}(G)$ generated by G.

(b) Define *type 2* and *context free* rules, grammars, and languages. State the pumping lemma for context free languages.

In the following, let $\Sigma = \{a, b, c\}$ be a set of terminal symbols and $N = \{S, A, B, C\}$ be a set of non-terminal symbols.

(c) Determine, with justification, whether the following languages over Σ are context free.

(i) The set of words of the form $a^m b^n c^m$ for any $m, n \geqslant 1$.

(ii) The set of words w containing the same number of a's, b's and c's.

(d) Let G be the grammar (N, Σ, P, S) where P is given by

$$S \to abC \mid aSBC, \qquad ABC \to cBa \mid Cab, \qquad CB \to BC, \qquad B \to b, \qquad C \to c.$$

Determine, with justification, whether $\mathcal{L}(G)$ is a type 2 language.

Paper 2, Section I

4I Automata and Formal Languages

(a) For a deterministic finite state automaton (DFA) $D = (Q, \Sigma, \delta, q_0, F)$ with extended transition function $\widehat{\delta}$, define the notion of an *inaccessible state.*

(b) Given a DFA D without inaccessible states, give the definition of the equivalence relation $\sim$, the function δ', and the set F' such that

$$D/\sim \; = (Q/\sim, \Sigma, \delta', [q_0], F')$$

is the minimal DFA for D. Show that δ' is well-defined.

(c) State (without proof) a theorem from the course relating the minimal DFA for D defined in part (b) to the smallest size of a DFA E such that $\mathcal{L}(D) = \mathcal{L}(E)$.

(d) Suppose that $\Sigma = \{a\}$ and fix $n \geqslant 1$. Let $L = \{a^{2m} : m \geqslant 0\}$. Construct an example of a DFA D without inaccessible states with $\mathcal{L}(D) = L$ that has $2n$ states. Prove that $D/\sim$ has two states.

Paper 3, Section I

4I Automata and Formal Languages

(a) Define the notion of a *deterministic finite state automaton* $D = (Q, \Sigma, \delta, q_0, F)$, its *extended transition function* $\widehat{\delta}$, and its *accepted language* $\mathcal{L}(D)$. What is a *regular language*?

(b) State the pumping lemma for regular languages.

(c) Let $\Sigma = \{a, b\}$. Determine whether the following languages are regular. Justify your answer.

(i) The set of words containing no occurrences of 100 successive b's.

(ii) The set $\{w_i : i \geqslant 1\}$ where w_i is recursively defined as follows: $w_1 = ab$ and $w_{i+1} = w_i w_i$.

Paper 3, Section II

12J Automata and Formal Languages

(a) Define the notion of a *register machine*, describe its *computation* on a given input, and how a given register machine is *encoded by a natural number*.

(b) For $m, k \in \mathbb{N} = \{0, 1, 2, 3, \dots\}$, define the *partial function* $f_{m,k}$ and the set W_m.

(c) State the recursion theorem.

(d) Let $\mathbf{Inf} = \{m \in \mathbb{N} : \mathrm{W}_m \text{ is infinite}\}$ and $\mathbf{Fin} = \{m \in \mathbb{N} : \mathrm{W}_m \text{ is finite}\}$. Give explicit numbers i and j such that $i \in \mathbf{Inf}$ and $j \in \mathbf{Fin}$.

(e) Using the numbers i and j from part (d), show that the functions

$$g_0(n) = \begin{cases} j & \text{if } n \in \mathbf{Inf}, \\ i & \text{if } n \in \mathbf{Fin} \end{cases} \qquad \text{and} \qquad g_1(n) = \begin{cases} i & \text{if } n \in \mathbf{Inf}, \\ j & \text{if } n \in \mathbf{Fin} \end{cases}$$

are not total recursive. Is there a total recursive function h such that $n \in \mathbf{Inf}$ if and only if $h(n) \in \mathbf{Fin}$? Justify your answer.

[*You may use theorems from the course without proof, provided that you state them clearly.*]

Paper 4, Section II

4I Automata and Formal Languages

If $f, g \colon \mathbb{N} \to \{0, 1\}$, define $f + g \colon \mathbb{N} \to \{0, 1\}$ to be their pointwise sum modulo 2.

(a) Show that if f and g are computable, then so is $f + g$.

For a function $f \colon \mathbb{N} \to \{0, 1\}$, let $X_f = \{n \in \mathbb{N} : f(n) = 1\}$.

(b) If X_f and X_g are both recursively enumerable, must X_{f+g} be recursively enumerable? Justify your claim.

Let Σ be any finite alphabet and $\beta \colon \Sigma^* \to \mathbb{N}$ be any bijection. We say that a function $f \colon \mathbb{N} \to \{0, 1\}$ is *β-context free* if $\beta^{-1}(X_f)$ is a context free language.

(c) If f and g are both β-context free, must $f + g$ be β-context free? Justify your claim.

[*You may use results and examples from the lectures without proof, provided that you state them clearly.*]

Paper 1, Section I

8A Classical Dynamics

(a) A dynamical system has a single generalized coordinate q, Lagrangian L and action

$$S = \int_{t_1}^{t_2} L(q, \dot{q}, t)\, dt\,.$$

State the principle of least action. Define the *generalized momentum* p and explain how the Hamiltonian H is related to the Lagrangian L. What condition(s) on L are necessary for this Hamiltonian to be well-defined?

(b) The Lagrangian for a particle of charge e and mass m moving in three dimensions in the presence of an electromagnetic field is

$$L = \frac{m}{2}\dot{\mathbf{r}}^2 - e(\phi - \dot{\mathbf{r}} \cdot \mathbf{A})\,,$$

where $\phi(\mathbf{r}, t)$ and $\mathbf{A}(\mathbf{r}, t)$ are the scalar and vector potentials describing the electromagnetic field.

(i) Write down the particle's Hamiltonian.

(ii) Consider the case $\phi = 0$ and $\mathbf{A} = (0, 0, -Bx)$, where B is a constant. Obtain Hamilton's equations for the particle.

Paper 2, Section I

8A Classical Dynamics

A pendulum consists of a light rod of length ℓ with a mass m at the end. It oscillates in the xy-plane making an angle $\theta(t)$ to the downward vertical, along which gravity acts. Assume the constant gravitational acceleration g points in the negative y direction. The pendulum's pivot is attached to a lift that descends with constant acceleration a, so that the position of the pivot as a function of time is $x_{\text{pivot}}(t) = 0$ and $y_{\text{pivot}}(t) = -\frac{1}{2}at^2$.

(a) Derive the Lagrangian for the system, and hence obtain the equation of motion for $\theta(t)$. What is the motion when $a = g$?

(b) Find the equilibrium configurations for arbitrary a. Determine which configuration is stable when $a < g$, and when $a > g$.

Paper 2, Section I

14A Classical Dynamics

A funnel consists of a cylindrical tube of radius $2a$, opening out into a smooth cone whose surface makes an angle $\alpha \in (0, \pi/2)$ with the axis of the cylinder. The funnel is fixed in place with its symmetry axis being vertical and the cylinder at the bottom. A spherical marble of uniform density, mass m, radius a and moment of inertia $(2/5)ma^2$ rolls on the conical surface of the funnel without slipping or twisting (that is, it has no spin about the normal to the surface).

(a) Using appropriate generalised coordinates for unconstrained degrees of freedom, write down the Lagrangian of the system, solving any constraints explicitly.

(b) Identify and interpret two independent conserved quantities.

(c) The marble is launched along the cone horizontally with initial speed v_0 at a height where the cone has radius $8a$ at its point of contact with the marble. Show that the marble never enters the cylinder provided

$$v_0^2 \geqslant v_{0\,\mathrm{min}}^2 = \frac{4}{7} ga \cot\alpha\,,$$

and briefly explain physically why the marble stays out of the cylinder for any $v_0 > v_{0\,\mathrm{min}}$.

(d) In Lagrangian mechanics, what is meant by a *holonomic constraint*? If the constraint is implemented using a Lagrange multiplier, define the *constraint force*. Show that when the constraint has no explicit time dependence, the constraint force does no work during the motion.

(e) The marble is now relaunched horizontally from the same height, but with a speed ensuring it is at the minimum of its effective potential for the radial motion. Calculate the magnitude of the constraint force associated to keeping the particle on the cone's surface.

Paper 3, Section I

8A Classical Dynamics

What is the *Hamiltonian vector field* associated to a smooth function f on phase space? What does it mean for a coordinate transformation on phase space to be *canonical*?

Let D_f and D_g be Hamiltonian vector fields associated to smooth functions f and g on phase space. Show that the commutator $[D_f, D_g] = D_{\{f,g\}}$.

Suppose phase space has generalised coordinates $z^A = (q^a, p_b)$. Show that the coordinate transformations

$$z^A \mapsto Z^A(z,s) = \exp\left(-sD_f\right) z^A$$

define a 1-parameter family of canonical transformations. [*Hint: You may find it helpful to differentiate with respect to s.*]

Paper 4, Section II

8A Classical Dynamics

Consider Hamiltonian dynamics with a time-independent Hamiltonian $H(q^a, p_b)$ on phase space $\mathbb{R}^{2n}$.

(a) Prove that energy is conserved in this system. Show that the Hamiltonian vector field V_H associated to H has vanishing divergence.

(b) Let $\Phi^t : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ denote Hamiltonian flow, so that a system whose initial positions and momenta lie in some region U will be found in the region $\Phi^t(U)$ after time t. Show that the regions $\Phi^t(U)$ have the same phase space volume for all t. [You may assume without proof that the Jacobian matrix $J(t)$ of the coordinate transformation $\Phi^t : (q^a(0), p_b(0)) \mapsto (q^a(t), p_b(t))$ obeys $J^{-1}\partial_t J = \mathrm{div}(V_H) 1_{2n\times 2n}$.]

(c) A classical system is governed by the Hamiltonian $H = \frac{1}{2}(p \cdot p + \cosh(q \cdot q) - 1)$. The system's initial conditions are such that its energy E lies in the range $0 \leqslant E \leqslant 1/2$. Suppose the initial region U is small enough that the system is definitely outside U after some time T. Show that there must exist integers $n > m$ such that

$$\Phi^{nT}(U) \cap \Phi^{mT}(U) \neq \emptyset \, .$$

Explain why this implies that there are initial conditions arbitrarily close to $(q^a(0), p_b(0))$ where the system returns arbitrarily close to its starting point.

Paper 4, Section II

15A Classical Dynamics

A table tennis bat has principal moments of inertia $I_1 < I_2 < I_3$ about the principal axes in the body frame.

(a) Write down Euler's equations describing torque-free rotation of the bat with angular velocity $\boldsymbol{\omega}$. Show that uniform rotation around any of the principal axes is a solution of the equations of motion.

(b) Show that the kinetic energy $T(\boldsymbol{\omega})$ and magnitude of the angular momentum $\mathbf{L}$ are constants of the motion. Show that steady rotations around the principal axes correspond to points where the surfaces $T(\boldsymbol{\omega}) = \text{const}$ and $\mathbf{L}^2 = \text{const}$ are tangent in $\boldsymbol{\omega}$-space.

(c) Let $\boldsymbol{\omega} = (\omega_1, \omega_2, \omega_3)$ in the principal body frame. Eliminate ω_2 to obtain an equation relating ω_1 and ω_3. Sketch the curves described by this equation in the (ω_1, ω_3) plane and relate your sketches to the three-dimensional geometric picture in part (b). Briefly explain what these curves imply for the motion of the bat if it is perturbed slightly from the steady rotations of part (b).

(d) George throws the table tennis bat so that it has initial angular momentum $\boldsymbol{\omega} = (\epsilon u(t), \Omega, \epsilon v(t))$, with $\epsilon \ll 1$. Determine $u(t)$ and $v(t)$. Is your result consistent with your expectation from part (c)?

Paper 1, Section I

3H Coding & Cryptography

Consider the discrete random variable $\mathbf{X}$ taking seven values x_i $(1 \leqslant i \leqslant 7)$ with the following probabilities:

$$\mathbf{X} = \begin{pmatrix} x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & x_7 \\ 0.45 & 0.28 & 0.14 & 0.04 & 0.04 & 0.03 & 0.02 \end{pmatrix}.$$

(a) Find a binary Huffman code for $\mathbf{X}$.

(b) Find the expected codeword length for the encoding.

(c) Find a ternary Huffman code for $\mathbf{X}$ (that is the codewords are strings of elements in $\{0, 1, 2\}$). Show that the expected word length in this case is 1.36.

Paper 1, Section II

11H Coding & Cryptography

(a) Let H be the binary code with the following parity check matrix:

$$\begin{pmatrix} 1 & 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 \end{pmatrix}.$$

Show that $|H| = 16$ and the minimum distance of H is 3.

(b) Let H be as above and let K be the reverse of H (that is $x_1 \dots x_7 \in H$ if and only if $x_7 \dots x_1 \in K$). You may assume that K is a linear code and that the only codewords K and H have in common are 0000000 and 1111111. Let H' be the parity check extension of H and K' the parity check extension of K.

(i) Show that H' and K' have length 8, size 16 and minimum distance 4. Further, show that the only codewords H' and K' have in common are 00000000 and 11111111.

Let the linear code $G \subseteq \mathbb{F}_2^{24}$ be defined as follows:

$$G = \{(a + x, b + x, a + b + x) : a, b \in H', x \in K'\}.$$

(ii) Show that a codeword $(a+x, b+x, a+b+x) \in G$ uniquely determines a, b and x. Hence show that $|G| = 2^{12}$.

(iii) Show that G has minimum distance 8. [You may use without proof that every codeword in G has weight a multiple of 4.]

(iv) Puncture G in the last bit and denote this new code by G^-. Show that G^- is a perfect code.

Paper 2, Section I

3H Coding & Cryptography

Let (X_n) be a source of random variables taking values in some finite alphabet Σ.

(a) What does it mean for the source to be *Bernoulli*?

(b) What does it mean for the source to be *reliably encodable at rate r*?

(c) What is the *information rate* of the source?

(d) What does it mean for the source to satisfy the *asymptotic equipartition property*?

Stating clearly any results you use, show that a Bernoulli source has information rate $H(X)$, where $H(X)$ denotes the entropy of X.

Paper 2, Section II

12H Coding & Cryptography

(a) State and prove Shannon's noiseless coding theorem. [You may use Gibbs' and Kraft's inequalities as long as they are clearly stated].

(b) Consider a random variable with m equiprobable outcomes.

(i) What is the entropy of this information source?

(ii) Describe an optimal binary code for this source when $m = 2^b$.

(iii) For arbitrary $m \geqslant 1$, show that in an optimal binary code the codeword lengths differ by at most 1. Hence, or otherwise, find the expected word length for such an optimal code.

Paper 3, Section I

3H Coding & Cryptography

Let $\mathbb{F}_2$ denote the field of order 2. What does it mean to say $C \subseteq \mathbb{F}_2^n$ is a *cyclic code*?

Show that there is a bijection between the cyclic codes of length n and factors of $X^n - 1$ over the field $\mathbb{F}_2$.

How many cyclic codes are there of length 7?

Paper 4, Section I

3H Coding & Cryptography

Describe the Rabin public encryption scheme. [You should describe both encryption and decryption.]

Suppose Alice chooses to use the Rabin scheme with $N = 2773$. To be sure I am in contact with Alice, I send her the encrypted message $r \equiv 25 \bmod 2773$. On receiving r, Alice uses her knowledge of N and returns $2355 \bmod 2773$. Show that I can now decrypt all messages sent to Alice.

Paper 1, Section I

9D Cosmology

A system of N point particles of equal mass m interacts only through Newtonian gravity. Let $\mathbf{r}_i(t)$ denote the position of the ith particle and $\mathbf{p}_i = m\dot{\mathbf{r}}_i$ its momentum, then the kinetic energy T and potential energy V are given by

$$T = \frac{1}{2}\sum_{i=1}^{N} m\dot{\mathbf{r}}_i^2, \qquad V = -\sum_{i<j}\frac{Gm^2}{|\mathbf{r}_i - \mathbf{r}_j|}.$$

Assume the system is gravitationally bound with particles remaining in a confined region.

(a) The scalar moment $\mathcal{G}$ is defined by

$$\mathcal{G}(t) = \sum_{i=1}^{N} \mathbf{p}_i \cdot \mathbf{r}_i.$$

Show that

$$\frac{d\mathcal{G}}{dt} = 2T + \sum_{i=1}^{N} \mathbf{F}_i \cdot \mathbf{r}_i\,,$$

where the total force on particle i is $\mathbf{F}_i = \sum_{j\neq i}\mathbf{f}_{ij}$, with $\mathbf{f}_{ij}$ the force exerted on particle i by particle j.

(b) Split and rearrange sums over pairwise forces using Newton's third law to express

$$\sum_{i=1}^{N} \mathbf{F}_i \cdot \mathbf{r}_i = \sum_{i}^{N}\sum_{j<i} \mathbf{f}_{ij}\cdot(\mathbf{r}_i - \mathbf{r}_j).$$

Substituting the Newtonian gravitational force described by $\mathbf{f}_{ij}$, show that

$$\sum_{i=1}^{N} \mathbf{F}_i \cdot \mathbf{r}_i = V\,.$$

(c) Deduce the virial theorem by taking a time average over the quantity $d\mathcal{G}/dt$:

$$2\langle T\rangle = -\langle V\rangle.$$

(d) A rich cluster of galaxies can be modeled as a virialized gravitational system. Briefly discuss how redshift measurements of velocity dispersions $\langle v^2\rangle$ of galaxies in a cluster can be combined with the cluster radius $\langle R\rangle$ to estimate the mass of the cluster. How can discrepancies with the estimated mass from luminous matter be resolved?

Paper 1, Section II

15D Cosmology

The Friedmann equation and fluid conservation equation for a homogeneous and isotropic universe are given by

$$\left(\frac{\dot a}{a}\right)^2 + \frac{kc^2}{a^2} \;=\; \frac{8\pi G}{3c^2}\rho + \frac{\Lambda c^2}{3}\,, \qquad\qquad \dot\rho \;=\; -3\frac{\dot a}{a}\,(\rho+P)\;,$$

where a is the scale factor, ρ is the energy density, P is the pressure and k is the curvature.

(a) Consider matter with a linear equation of state $P = w\rho$ where w is a constant. Show that the energy density behaves as $\rho(t) = \rho_0\, a(t)^{-3(1+w)}$, where $\rho_0 \equiv \rho(t_0)$ is the density measured today at $t = t_0$ (assuming the usual normalisation $a(t_0) = 1$). Define the density parameter Ω_i for a given matter component ρ_i using the ratio with the critical energy density of a flat ($k = 0$) universe and expressed using the Hubble parameter H.

(b) The standard concordance cosmology has had great success describing our Universe using just three key constituents: radiation $\rho_{\rm R}$, non-relativistic matter $\rho_{\rm M}$, and a cosmological constant Λ, with the curvature measured to be negligible ($k \approx 0$). Show that the Friedmann equation for this model can be expressed as

$$\left(\frac{\dot a}{a}\right)^2 \;=\; H_0^2\left(\frac{\Omega_{R0}}{a^4} + \frac{\Omega_{M0}}{a^3} + \Omega_{\Lambda 0}\right)\,,$$

where H_0 is the Hubble parameter today and you should define and justify each term.

(c) Recent observations have suggested the concordance model needs an additional ingredient to explain time-varying acceleration. Consider a flat universe dominated by a cosmological constant today with $\Omega_{\Lambda 0} \leqslant 1$ (for brevity $\Omega \equiv \Omega_{\Lambda 0}$) plus a small additional component of 'phantom matter' $\Omega_{P0} = 1 - \Omega \ll 1$, which satisfies a linear equation of state with $w = -4/3$. Show that the Friedmann equation becomes $\dot a = H_0\sqrt{\Omega}\, a\sqrt{1+\beta^2 a}$, where $\beta^2 = (1-\Omega)/\Omega$. Solve this equation by considering $b^2(t) = 1 + \beta^2 a(t)$, that is, you should find the following and then specify the constant κ using $a(t_0) = 1$:

$$\log(b-1) - \log(b+1) = H_0\sqrt{\Omega}\,(t-t_0) + \log\,\kappa\,.$$

Hence, or otherwise, find the late-time concordance solution with Λ and phantom matter:

$$a(t) \;=\; \frac{4\Omega\,\exp\left[H_0\sqrt{\Omega}\,(t-t_0)\right]}{\left((1+\sqrt{\Omega}) \;-\; (1-\sqrt{\Omega})\,\exp\left[H_0\sqrt{\Omega}\,(t-t_0)\right]\right)^2}\,.$$

Verify that the behaviour of $a(t)$ near $t \approx t_0$ for $1 - \Omega \ll 1$ agrees with expectations for a Λ-dominated universe. However, note that even a small component of phantom matter will lead to the catastrophic 'Big Rip' demise of this universe; find the time $t_{\rm BR}$ when the scale factor becomes singular and briefly comment on how quickly this happens. What will be the fate of loosely-bound gravitational objects as $t \to t_{\rm BR}$?

Paper 2, Section I

9D Cosmology

Consider a gas of particles in a cubic box of side length L and volume $V = L^3$. Let $\bar{n}(p)\,dp$ be the mean number of particles in the interval $[p, p+dp]$, where p is the magnitude of the momentum. The total particle number and energy are

$$N = \int_0^\infty \bar{n}(p)\,dp\,, \qquad E = \int_0^\infty \mathcal{E}(p)\,\bar{n}(p)\,dp\,,$$

where the single-particle energy is $\mathcal{E}(p) = \sqrt{p^2c^2 + m^2c^4}$.

(a) Assuming the allowed momentum modes scale as $p \propto L^{-1}$, show that an infinitesimal change in the volume implies

$$\frac{dp}{dV} = -\frac{p}{3V}.$$

Consider a quasistatic change in volume at fixed particle number N and entropy S. Using the induced change in the momenta to compute the corresponding change in the total energy and, identifying it with the thermodynamic relation $dE = -P\,dV$, obtain the pressure

$$P = \frac{1}{3V}\int_0^\infty p\,\frac{d\mathcal{E}}{dp}\,\bar{n}(p)\,dp\,. \qquad (\dagger)$$

(b) Specialize to an ultrarelativistic gas with $\mathcal{E}(p) = pc$ and show that the equation of state is $P = \frac{1}{3}\rho$, where the energy density is $\rho = E/V$. In an expanding universe, assume the gas remains in equilibrium with energy density $\rho \propto T^4$ and number density $n \propto T^3$. Briefly explain why the temperature falls inversely with the scale factor $T \propto a^{-1}$.

(c) In the non-relativistic limit for a gas at temperature T, the typical momentum is given by $\langle p^2/2m\rangle \approx \frac{3}{2}kT \ll mc^2$. Use ($\dagger$) and expand $\mathcal{E}(p)$ in this limit to obtain the equation of state $P = \frac{2}{3}u$, where $u \equiv U/V$ with the internal energy U given by $U = E - N\,mc^2$. Briefly explain how the temperature T is expected to fall with the scale factor a.

Paper 3, Section I

9D Cosmology

(a) The equations governing a period of inflation in a homogenous and isotropic universe are

$$H^2 = \left(\frac{\dot{a}}{a}\right)^2 = \frac{1}{3M_{\rm Pl}^2}\left(\frac{1}{2}\dot{\phi}^2 + V(\phi)\right), \qquad \ddot{\phi} + 3H\dot{\phi} + V_{,\phi} = 0\,,$$

where a is the scale factor, ϕ the inflaton scalar, and $M_{\rm Pl}$ the Planck mass. For the exponential potential, $V(\phi) = V_* e^{-\phi/M_{\rm Pl}}$ there is an exact scaling solution of the form

$$a(t) = a_0\, t^2, \qquad H(t) = \frac{2}{t}\,, \qquad \phi(t) \;=\; \phi_* + 2M_{\rm Pl}\ln(t/t_*)\,.$$

Verify that $a(t)$ and $\phi(t)$ simultaneously satisfy the Friedmann and scalar field equations, provided the initial parameters satisfy a consistency condition:

$$\phi_* = M_{\rm Pl}\ln\left(\frac{V_* t_*^2}{10M_{\rm Pl}^2}\right)\,.$$

Does this power law expansion fulfill the criterion for an inflationary universe? The number of e-folds between time t and t_* is defined by $\mathcal{N} \equiv \ln\left(\frac{a(t)}{a(t_*)}\right)$. How far must the field ϕ traverse from ϕ_* in order for $\mathcal{N} \approx 60$ e-folds?

(b) Briefly explain the horizon problem in a decelerating FLRW universe. Describe how an inflationary phase resolves this problem.

Consider a radiation-dominated universe with scale factor $a(t) \propto t^{1/2}$, evolving from $t \sim 10^{-35}$ s to the present day $t_0 \sim 10^{17}$ s. By extrapolating the present horizon scale backward in time, show that its physical size at $t \sim 10^{-35}$ s was of order one metre.

Assume that for $t < 10^{-35}$ s the universe underwent a de Sitter-like inflationary phase with constant Hubble parameter H, which then reheated and transitioned into a radiation-dominated era at $t \sim 10^{-35}$ s. Approximately estimate the number $\mathcal{N}$ of inflationary e-folds required for the entire observable universe to have originated within a single Hubble volume at the beginning of inflation.

Paper 3, Section II

14D Cosmology

(a) Consider a massive particle at position $\mathbf{r} = a\,\mathbf{x}$ with scale factor a and comoving coordinate $\mathbf{x}$ in an expanding universe. The equation of motion in cosmic time $\ddot{\mathbf{r}} = -\nabla_{\mathbf{r}}\Phi = -\frac{1}{a}\nabla_{\mathbf{x}}\Phi$ is governed by the gravitational potential Φ, satisfying the Poisson equation

$$\nabla_{\mathbf{r}}^2\Phi = \frac{4\pi G}{c^2}\rho_{\mathrm{M}}\,, \qquad (*)$$

where $\rho_{\mathrm{M}}(\mathbf{r},\tau)$ is the dominant matter density. The Friedmann equation in conformal time τ is $\mathcal{H}^2 = \frac{8\pi G}{3c^2}\rho_{\mathrm{M}}a^2$, where $\frac{dt}{d\tau} = a$ and $\mathcal{H} = \frac{1}{a}\frac{da}{d\tau} = \frac{a'}{a}$. We will expand this mildly perturbed universe about a homogeneous background using $\rho_{\mathrm{M}}(\mathbf{r},\tau) \equiv \bar{\rho}_{\mathrm{M}}(\tau)\big(1 + \delta_{\mathrm{M}}(\mathbf{r},\tau)\big)$, where $\bar{\rho}_{\mathrm{M}}$ is the mean density and δ is the density perturbation. Similarly, take $\Phi \equiv \bar{\Phi} + \phi$ where you may assume that the background potential $\bar{\Phi} = -\frac{1}{2}\mathcal{H}'|\mathbf{x}|^2$ satisfies $(*)$ with $\rho_{\mathrm{M}} = \bar{\rho}_{\mathrm{M}}$ and $\Phi = \bar{\Phi}$.

(i) Introduce Lagrangian coordinates $\mathbf{r} = a(\tau)\big(\mathbf{q} + \boldsymbol{\psi}(\mathbf{q},\tau)\big)$ in which $\mathbf{q}$ is the unperturbed comoving position and $\psi(\mathbf{q},\tau)$ is the comoving displacement. Show that the perturbed part of the equation of motion for $\ddot{\mathbf{r}}$ takes the form

$$\mathbf{x}'' + \mathcal{H}\mathbf{x}' \;\simeq\; \boldsymbol{\psi}'' + \mathcal{H}\,\boldsymbol{\psi}' \;\simeq\; -\nabla_{\mathbf{q}}\,\phi\,. \qquad (\dagger)$$

Relate the perturbed potential ϕ to the density perturbation δ_{M} with the Poisson equation.

(ii) Explain why mass conservation implies we can relate matter in a small perturbed comoving volume $d^3\mathbf{r}$ to the unperturbed background $d^3\mathbf{q}$ through $\rho_{\mathrm{M}}(\mathbf{r},\tau)\,d^3\mathbf{r} = \bar{\rho}_{\mathrm{M}}(\tau)\,a(\tau)^3\,d^3\mathbf{q}$. Change coordinates using the Jacobian $|\partial r_i/\partial q_j|^{-1}$ to show that

$$\rho_{\mathrm{M}}(\mathbf{r},\tau) \;\simeq\; \bar{\rho}_{\mathrm{M}}(\tau)\,(1 - \nabla_{\mathbf{q}}\cdot\boldsymbol{\psi} + \ldots)\,.$$

Combine with the divergence of $(\dagger)$ to obtain the density perturbation evolution equation for δ_{M} in a matter-dominated universe:

$$\delta_{\mathrm{M}}''(\mathbf{x},\tau) + \mathcal{H}(\tau)\,\delta_{\mathrm{M}}'(\mathbf{x},\tau) - \frac{4\pi G}{c^2}\,\bar{\rho}_{\mathrm{M}}(\tau)\,a^2(\tau)\,\delta_{\mathrm{M}}(\mathbf{x},\tau) = 0\,.$$

(b) Now consider the evolution of radiation perturbations δ_{R} deep in the radiation-dominated era ($\tau \ll \tau_{\mathrm{eq}}$) for which you are given the Fourier space evolution equation:

$$\delta_{\mathrm{R}}''(\mathbf{k},\tau) + \frac{\alpha\,k}{\mathcal{H}}\,\delta_{\mathrm{R}}'(\mathbf{k},\tau) + \frac{c^2}{3}k^2\,\delta_{\mathrm{R}}(\mathbf{k},\tau) = 0\,, \qquad (k = |\mathbf{k}|)\,,$$

where the last term accounts for relativistic pressure effects and we have neglected the (diminishing) gravitational potential term. Here, the middle term incorporates phenomenological damping due to the flow (or free-streaming) of relativistic particles out of overdense regions (or into underdense regions). For slowly varying damping with $\alpha \ll 1/\tau < k$, seek radiation era solutions of the form $\delta_{\mathrm{R}} = e^{-S}u$ to find the solution

$$\delta_{\mathrm{R}}(\mathbf{k},\tau) \;\simeq\; \exp\big[-S(\mathbf{k},\tau)\big]\Big[A_{\mathbf{k}}\cos(ck\tau/\sqrt{3}) + B_{\mathbf{k}}\sin(ck\tau/\sqrt{3})\Big]\,,$$

where you should specify the $S(\mathbf{k},\tau)$ and state the approximations made. Briefly describe your solution and compare to matter perturbation behaviour during the matter era.

Paper 4, Section I

9D Cosmology

A fermion species X of mass m_X is in thermal equilibrium in the early universe with number density given by

$$n_X = \frac{4\pi}{h^3}\int_0^\infty \frac{p^2\,dp}{\exp[\mathcal{E}(p)/(kT)]+1},$$

where $\mathcal{E}(p) = \sqrt{p^2c^2 + m_X^2c^4}$. Assume the species remains in equilibrium until it becomes non-relativistic.

(a) In the limit $kT \ll m_X\,c^2$, expand $\mathcal{E}(p)$ to leading order, showing that the integral reduces to a Gaussian and obtain the number density,

$$n_X = \left(\frac{2\pi\,m_X\,kT}{h^2}\right)^{3/2}\exp\bigl[-m_X\,c^2/(kT)\bigr]\,.$$

[*Hint: You may assume* $\int_0^\infty e^{-\lambda x^2}dx = \frac{1}{2}\sqrt{\pi/\lambda}$ *for* $\lambda > 0$.]

(b) The fermion X decouples at a temperature $kT_{\rm d} = m_X\,c^2/\alpha$ with $\alpha \gg 1$. Using the photon number density at the same temperature, $n_\gamma = 16\pi\,\zeta(3)(kT/hc)^3$, show that the fermion-to-photon ratio at $T = T_{\rm d}$ is

$$r(\alpha) \equiv \frac{n_X}{n_\gamma} = \frac{\sqrt{2\pi}}{8\,\zeta(3)}\,\alpha^{3/2}\,e^{-\alpha}\,.$$

(c) Assume no significant entropy production after decoupling so that r remains constant. Consider whether the fermion X could be a viable cold dark matter candidate by comparing its mass density ρ_X to the baryon mass density $\rho_B = m_p\,n_B$, where m_p is the proton mass. You may assume that the baryon-to-photon ratio is $\eta = n_B/n_\gamma \simeq 6\times10^{-10}$ and that the measured density parameters today for baryons and dark matter are $\Omega_B \simeq 0.05$ and $\Omega_{DM} \simeq 0.25$ respectively. Hence, or otherwise, determine that for fixed α the required mass for this to be a dark matter fermion is

$$m_X \simeq \frac{\mu}{r(\alpha)}\,m_p\,,$$

where you should specify the numerical constant μ.

Paper 1, Section II

26I Differential Geometry

(a) Let γ be a regular curve. Define the *curvature* and *torsion* of γ, carefully describing any conditions necessary for these to be well defined.

(b) Suppose the image of γ lies on an oriented surface $S \subset \mathbb{R}^3$. Define the *normal* and *geodesic curvature* of γ. What does it mean for γ to be a *geodesic*?

(c) Can a surface $S \subset \mathbb{R}^3$ of positive Gaussian curvature contain a line segment? Can a connected compact surface $S \subset \mathbb{R}^3$ of positive Gaussian curvature contain two non-intersecting closed geodesics?

(d) Consider a regular curve $\gamma : (a,b) \to \mathbb{R}^3$, with $-\infty \leqslant a < b \leqslant \infty$, not necessarily parametrised by arc length. We say that γ is *regularly extendible* if there exists a regular curve $\widetilde{\gamma} : I \to \mathbb{R}^3$, where $I \subset \mathbb{R}$ is a open interval, and an open interval $(c,d) \subset \mathbb{R}$ with $(c,d] \subset I$ or $[c,d) \subset I$ such that $\widetilde{\gamma}|_{(c,d)} : (c,d) \to \mathbb{R}^3$ is a reparametrisation of $\gamma : (a,b) \to \mathbb{R}^3$.

Let $\gamma : (a,b) \to \mathbb{R}^3$ now be a regular curve parametrised by arc length, where we assume $-\infty < a < b < \infty$. Assume that the curvature $k(s)$ satisfies $k(s) \geqslant \epsilon > 0$ for some $\epsilon > 0$ and all $s \in (a,b)$ and let $\tau(s)$ denote the torsion. Show that if

$$\limsup_{s \to a} \left(k(s) + |\tau(s)|\right) = \infty, \qquad \text{and} \qquad \limsup_{s \to b} \left(k(s) + |\tau(s)|\right) = \infty,$$

then γ is not regularly extendible.

Paper 2, Section II

26I Differential Geometry

(a) Define the *exponential map* $\exp_p$ for a surface Σ and point $p \in \Sigma$, describing carefully its domain.

(b) Consider the unit sphere $S^2 = \{(x,y,z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 1\}$. Describe explicitly the exponential map at the point $(0,0,1) \in S^2$, and the associated geodesic polar coordinate system. What is the largest region of S^2 which may be covered by such a coordinate system?

(c) Show that for any two distinct points $p, q \in S^2$ there exists a geodesic $\gamma : [a,b] \to S^2$ such that $\gamma(a) = p$, $\gamma(b) = q$, and such that for any other regular curve $\widetilde{\gamma} : [\tilde{a}, \tilde{b}] \to S^2$ with $\widetilde{\gamma}(\tilde{a}) = p$, $\widetilde{\gamma}(\tilde{b}) = q$ one has $\ell(\widetilde{\gamma}) \geqslant \ell(\gamma)$, where ℓ denotes length.

Show, moreover, that if p and q are not antipodal, then γ is unique when expressed in arc length parametrisation $\gamma : [0,\ell] \to S^2$, and that the above inequality is necessarily strict unless $\widetilde{\gamma}$ is a reparametrisation of γ.

(d) Now consider $\Sigma = S^2 \setminus \{(0,0,1)\}$. Given any two distinct points $p, q \in \Sigma$, does there still necessarily exist some geodesic $\gamma : [a,b] \to \Sigma$ such that $\gamma(a) = p$, $\gamma(b) = q$?

Does there necessarily exist a regular curve $\gamma : [a,b] \to \Sigma$ with $\gamma(a) = p$, $\gamma(b) = q$ which is length minimising in the above sense, i.e. for any other regular curve $\widetilde{\gamma} : [\tilde{a}, \tilde{b}] \to \Sigma$ with $\widetilde{\gamma}(\tilde{a}) = p$, $\widetilde{\gamma}(\tilde{b}) = q$ one has $\ell(\widetilde{\gamma}) \geqslant \ell(\gamma)$? Justify your answers.

Paper 3, Section II

25I Differential Geometry

Let $S \subset \mathbb{R}^3$ denote a surface.

(a) State what it means for a smooth map $\phi : S \to S$ to be an *isometry*.

(b) We call a smooth function $f : S \to \mathbb{R}$ an *isometry invariant* if given any isometry $\phi : S \to S$, we have $f(x) = f(\phi(x))$ for all $x \in S$. Deduce that if f is an isometry invariant and ϕ is an isometry, then ϕ must preserve the level sets of f.

(c) By using the Theorema Egregium, show that $K : S \to \mathbb{R}$ is an isometry invariant, where K denotes the Gaussian curvature of S. If a smooth diffeomorphism $\phi : S \to S$ preserves the level sets of K, does it necessarily follow that ϕ is an isometry?

(d) We say that two isometry invariants $f : S \to \mathbb{R}$, $g : S \to \mathbb{R}$ are *independent at a point* $x \in S$ if $\dim \ker df_x = 1$, $\dim \ker dg_x = 1$ and $\ker df_x + \ker dg_x = T_xS$. Show that if there exist two isometry invariants $f : S \to \mathbb{R}$, $g : S \to \mathbb{R}$ which are independent at some $x \in S$, then there exists a non-empty open neighbourhood $\tilde{S} \subset S$ of x such that any isometry $\phi : \tilde{S} \to \tilde{S}$ must equal the identity.

(e) Suppose the Gaussian curvature $K : S \to \mathbb{R}$ is such that $\dim \ker dK_x = 1$ for all $x \in S$. Show that at each $x \in S$, there exists a unique vector $e_x \in T_xS$ orthogonal to the level set of K through x and such that $dK_x(e_x) > 0$. Defining $g(x) = dK_x(e_x)$, show that g is an isometry invariant of S. [You may assume without proof that g is smooth.]

Paper 4, Section II

25I Differential Geometry

(a) Suppose $X \subset \mathbb{R}^N$ is a k-dimensional manifold. Let us define

$$TX = \bigcup_{p \in X} \{p\} \times T_pX,$$

i.e. the elements of TX are pairs (p, v) where $p \in X$ and $v \in T_pX$. This may be considered as a subset $TX \subset \mathbb{R}^N \times \mathbb{R}^N$.

Show that TX is a $2k$-dimensional manifold.

[*You may wish to consider the following maps:* $(q, w) \mapsto (\phi(q), d\phi|_q(w))$ *where* ϕ *is a local parametrisation of* X*.*]

(b) For $f : X \to Y$ a smooth map of manifolds, define what it means for $y \in Y$ to be a *regular value* of f. State, without proof, the preimage theorem.

(c) Now let $X \subset \mathbb{R}^3$ be a surface and consider the subset

$$SX = \{(p, v) \in TX : I_p(v, v) = 1\}$$

where I_p denotes the first fundamental form of X at p, and TX is defined in part (a). Show that SX is a submanifold of TX of dimension 3.

Paper 1, Section II

32B Dynamical Systems

(a) What are the properties required of a *strict Lyapunov function* on a domain $\mathcal{D}$ for a dynamical system $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$ in $\mathbb{R}^n$ with a fixed point $\mathbf{x}_0 \in \mathcal{D}$? Define the *domain of stability* of $\mathbf{x}_0$.

By considering the function $V = \frac{1}{2}(x^2 + y^2)$ show that the origin is an asymptotically stable fixed point of

$$\dot{x} = -2x + y + x^3 - xy^2,$$
$$\dot{y} = -x - 2y + 6x^2y + 4y^3.$$

Show also that the domain of stability of the origin is contained in $V \leqslant 1$ and includes $V < \frac{1}{4}$.

(b) Find the fixed points of the dynamical system

$$\dot{x} = x(\mu^2 - y),$$
$$\dot{y} = x^2 - y,$$

and determine their type, including the dependence on μ, for $\mu \neq 0$. Find the stable and unstable manifolds of the origin up to 4th order.

Paper 2, Section II

33B Dynamical Systems

Consider the nonlinear oscillator

$$\begin{aligned} \dot{x} &= \ \ 2y + \mu x(1 - ax^2 - y^2), \\ \dot{y} &= -2x, \end{aligned}$$

for some constants $a > 1$ and $\mu > 0$. If μ were zero the system would be Hamiltonian.

(a) Find a forward-invariant region $\mathcal{D}$ that encircles, but does not contain, the origin and hence show that there is at least one periodic orbit in $\mathcal{D}$, referring explicitly to any results that you use.

(b) Define the *Floquet multiplier* λ for a periodic orbit with period T. The *Floquet exponent* σ is defined as $T^{-1} \log \lambda$. If $(x_p(t), y_p(t))$ is a periodic orbit of the above system with period T then show, clearly explaining any general results that you use, that the Floquet exponent σ is given by

$$\sigma = \frac{\mu}{T} \int_0^T dt \, (1 - 3ax_p(t)^2 - y_p(t)^2).$$

(c) Describe the *energy balance method.* Use it to show that for $\mu \ll 1$ there is a single periodic orbit and find its distance from the origin. Confirm that your answer is indeed in $\mathcal{D}$.

(d) Calculate the Floquet exponent at leading order in μ to confirm that the orbit is stable when $0 < \mu \ll 1$.

(e) Show that the Floquet exponent can be deduced directly from the expression for the change in the Hamiltonian over one orbit as derived by the energy balance method. Verify that the value obtained by this approach is equal to that obtained from part (d).

Paper 3, Section II

31B Dynamical Systems

Consider the system

$$\ddot{x} = a + bx - \dot{x} - x^2 - \dot{x}^2, \qquad (*)$$

where a and b are real parameters.

(a) Write $(*)$ as a dynamical system in $\mathbb{R}^2$.

(b) Find all the fixed points as a function of a and b, and show the locations of any bifurcations in the (a, b)-plane.

(c) For the case $a = 0$, find the leading-order approximation to the extended centre manifold of the bifurcation of the origin as b varies. Find also the evolution equation on the extended centre manifold to leading order. Deduce the type of bifurcation and sketch the bifurcation diagram in the (b, x)-plane.

(d) Now repeat part (c) for the case $b = 1$ and letting a vary. Sketch the bifurcation diagram in the (a, x)-plane.

(e) Are there any periodic orbits for the case $a = -1$, $b = 1$? Hence deduce the existence or absence of a bounding function $V(x, y)$ in $\mathbb{R}^2$. In either case be careful to explain any theoretical results that you use.

Paper 4, Section II

32B Dynamical Systems

(a) Let $F : \mathbb{R} \to \mathbb{R}$ be a continuous one-dimensional map of the real line to itself. Define what it means for

(i) F to have a *horseshoe*,

(ii) F to be *chaotic* according to Glendinning's definition.

Prove that if F has a 3-cycle, $x_1 \to x_2 \to x_3$ with $x_1 < x_2 < x_3$, then F is chaotic.

(b) Now consider the map F defined by

$$x_{n+1} = F(x_n; \mu) := x_n(px_n^2 + qx_n - 1 - \mu),$$

where $p > 0$ and $q > 0$ are constants and μ is a parameter.

(i) One fixed point of F is at $x = 0$. Determine its linear stability. Find the other fixed points of F. Hence show that there are bifurcations at $\mu = -2$, $\mu = 0$, and a third value $\mu = \mu(q, p)$ to be determined. Determine the stability of the non-zero fixed points, explaining your reasoning, and sketch the (μ, x) bifurcation diagram showing the locus and stability of all the fixed points. State the type of each bifurcation present.

(ii) Now set $q = 0$ and consider the bifurcation at $\mu = 0$ of $x = 0$. For $0 < \mu \ll 1$, by considering x_{n+2} as a function of x_n, assuming that $x_n = O(\mu^{1/2})$ and retaining terms up to $O(\mu^{3/2})$ only, show that there is a stable period-2 orbit with

$$|x_n| = \sqrt{\frac{\mu}{p}}$$

to leading order in μ.

Paper 1, Section II

37D Electrodynamics

(a) A point particle of mass m and charge q moves in a background electromagnetic field with field-strength tensor $F_{\mu\nu}$. The velocity 4-vector $u^\mu = dx^\mu/d\tau$, with τ the particle's proper time, satisfies

$$m\frac{du^\mu}{d\tau} = qF^\mu{}_\nu u^\nu \,, \qquad (*)$$

where $F^\mu{}_\nu = \eta^{\mu\rho}F_{\rho\nu}$ and $\eta_{\mu\nu} = \mathrm{diag}(-1,+1,+1,+1)$ the Minkowski metric. In some inertial frame, let the particle's velocity be $\mathbf{v}$, with Lorentz factor $\gamma = (1-|\mathbf{v}|^2/c^2)^{-1/2}$, and the electric and magnetic fields $\mathbf{E}$ and $\mathbf{B}$, respectively. Show that the components of $(*)$ reduce to

$$\frac{d\mathbf{p}}{dt} = q(\mathbf{E} + \mathbf{v}\times\mathbf{B})\,,$$
$$mc^2\frac{d\gamma}{dt} = q\mathbf{E}\cdot\mathbf{v}\,,$$

where the momentum $\mathbf{p} = \gamma m\mathbf{v}$. Give a physical interpretation of the second equation.

[*You may assume the following relations between the components of* $F_{\mu\nu}$ *and those of the electric and magnetic fields,* $\mathbf{E} = (E_1, E_2, E_3)$ *and* $\mathbf{B} = (B_1, B_2, B_3)$*:*

$$F_{i0} = -F_{0i} = E_i/c \qquad \textit{and} \qquad F_{ij} = \varepsilon_{ijk}B_k\,,$$

for $i, j = 1, 2, 3$.]

(b) Maxwell's equations in vacuum admit wavelike solutions with propagation speed c. A circularly polarized wave propagating along the x-axis of some inertial frame has electric field

$$\mathbf{E} = E_0(0, \cos\omega\xi, \sin\omega\xi)\,,$$

where E_0 and ω are constants and $\xi = t - x/c$. Determine the corresponding magnetic field $\mathbf{B}(\xi)$. [You should neglect any constant term.]

The particle in part (a) moves under the influence of this electromagnetic wave. Show that circular motion in the plane $x = 0$ with constant speed is possible, and that the Lorentz factor satisfies

$$\gamma^2 = 1 + \left(\frac{qE_0}{m\omega c}\right)^2 .$$

Give the radius of the circle in terms of q, E_0, m, ω and γ.

Paper 3, Section II

36D Electrodynamics

In the Lorenz gauge, $\partial_\mu A^\mu = 0$, the 4-vector potential A^μ due to a localised charge/current distribution with conserved current density 4-vector J^μ is

$$A^\mu(t,\mathbf{x}) = \frac{\mu_0}{4\pi}\int \frac{J^\mu(t-|\mathbf{x}-\mathbf{x}'|/c,\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|}\, d^3\mathbf{x}' \,.$$

For charges in non-relativistic periodic motion localised near the origin $\mathbf{x}=\mathbf{0}$, use the dipole approximation to show that at sufficiently large distances $r=|\mathbf{x}|$ the magnetic field is given approximately by

$$\mathbf{B}(t,\mathbf{x}) \approx \frac{\mu_0}{4\pi r c}\ddot{\mathbf{p}}(t-r/c)\times\hat{\mathbf{x}}\,,$$

where $\mathbf{p}$ is the electric dipole moment, which you should define, and $\hat{\mathbf{x}}$ is a unit vector in the radial direction. State clearly the conditions under which this approximate result holds.

Given that the electric field at such large distances is $\mathbf{E}(t,\mathbf{x}) \approx c\mathbf{B}(t,\mathbf{x})\times\hat{\mathbf{x}}$, determine the Poynting vector and show that the instantaneous power P radiated per solid angle is

$$\frac{dP}{d\Omega} = \frac{\mu_0}{(4\pi)^2 c}\left[|\ddot{\mathbf{p}}|^2 - (\hat{\mathbf{x}}\cdot\ddot{\mathbf{p}})^2\right] \,.$$

A point charge q moves non-relativistically in a circle of radius $\mathcal{R}$ with angular frequency ω in the plane $z=0$, so that $(x,y)=\mathcal{R}(\cos\omega t,\sin\omega t)$. Find the total radiated power in the dipole approximation and its time-averaged angular distribution.

Show further that the electric field along the z-axis is circularly polarized with

$$\mathbf{E}(t,\mathbf{x}) \approx \frac{\mu_0 q\omega^2\mathcal{R}}{4\pi r}\left(\cos\omega\tau,\sin\omega\tau,0\right) \,,$$

where $\tau = t-r/c$, and along the x-axis is linearly polarized with

$$\mathbf{E}(t,\mathbf{x}) \approx \frac{\mu_0 q\omega^2\mathcal{R}}{4\pi r}\left(0,\sin\omega\tau,0\right) \,.$$

If the angular frequency is increased so that the charge moves relativistically, determine the total radiated power P by considering the acceleration in the instantaneous rest frame, or otherwise.

Paper 4, Section II

36D Electrodynamics

(a) By considering the force per unit volume, $\mathbf{f} = \rho\mathbf{E} + \mathbf{J} \times \mathbf{B}$, on a charge density ρ and current density $\mathbf{J}$ due to an electric field $\mathbf{E}$ and magnetic field $\mathbf{B}$, show from Maxwell's equations that

$$\frac{\partial g_i}{\partial t} + \frac{\partial \sigma_{ij}}{\partial x_j} = -f_i \,,$$

where $\mathbf{g} = \epsilon_0 \mathbf{E} \times \mathbf{B}$ and

$$\sigma_{ij} = -\epsilon_0 \left(E_i E_j - \frac{1}{2}|\mathbf{E}|^2 \delta_{ij} \right) - \frac{1}{\mu_0} \left(B_i B_j - \frac{1}{2}|\mathbf{B}|^2 \delta_{ij} \right) .$$

Give a physical interpretation of $\mathbf{g}$ and σ_{ij}.

(b) A homogeneous, linear dielectric with relative permittivity ϵ_r fills the region $z > 0$, while the region $z < 0$ is vacuum. State the junction conditions for $\mathbf{E}$ and the electric displacement $\mathbf{D}$ at the plane $z = 0$ assuming there is no free surface charge density or current.

A point charge q is held at rest at position $(0, 0, -d)$. Let $R_1 = \sqrt{x^2 + y^2 + (z + d)^2}$ be the distance of the point (x, y, z) from the charge q, and $R_2 = \sqrt{x^2 + y^2 + (z - d)^2}$ be the distance from the point $(0, 0, d)$. Using the method-of-images ansatz $\mathbf{E} = -\boldsymbol{\nabla}\Phi$, where

$$\Phi = \begin{cases} \dfrac{1}{4\pi\epsilon_0} \left(\dfrac{q}{R_1} + \dfrac{q'}{R_2} \right) & \text{for } z < 0 \,, \\[2ex] \dfrac{q''}{4\pi\epsilon_0 \epsilon_r R_1} & \text{for } z > 0 \,, \end{cases}$$

determine q'/q and q''/q in terms of ϵ_r.

Find the components of $\mathbf{E}$ in the plane $z = 0^-$ just outside the dielectric.

(c) For the set-up in part (b), integrate the relevant components of σ_{ij} from part (a) over the plane $z = 0^-$ to determine the force on the charge q due to the dielectric. Comment on your result.

[*You may assume the integrals*

$$2\pi \int_0^\infty \frac{r dr}{(r^2 + d^2)^3} = \frac{\pi}{2d^4} \quad \textit{and} \quad 2\pi \int_0^\infty \frac{r^3 dr}{(r^2 + d^2)^3} = \frac{\pi}{2d^2} \,. \qquad]$$

Paper 1, Section II

39C Fluid Dynamics II

The two-dimensional steady boundary-layer equations can be written as

$$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=U\frac{dU}{dx}+\nu\frac{\partial^2 u}{\partial y^2}, \qquad \frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0,$$

where U is the velocity far from the layer, and the other symbols take their usual meanings.

State the approximations that have been made to the two-dimensional Navier–Stokes equations to arrive at the above equations.

Calculate the components in plane polar coordinates of the potential flow $\mathbf{u}=\boldsymbol{\nabla}\phi$ for $\phi=-Ak^{-1}r^k\cos(k\theta)$, where $1<k<2$ and $A>0$ are constants. Explain why this could represent an inviscid flow past an infinite wedge with rigid boundaries at $\theta=\pm\pi/k$, but would require boundary layers if the viscosity was nonzero but small.

Consider the boundary layer on $\theta=\pi/k$, and let x denote radial distance along the boundary. Write down $U(x)$ and use a scaling argument to derive the estimate

$$\delta(x)=(\nu x^{2-k}/A)^{1/2},$$

for the thickness of the boundary layer.

Show that the boundary-layer equations are satisfied by a self-similar stream function of the form

$$\psi(x,y)=U(x)\delta(x)f(\eta), \quad \text{where} \quad \eta=\frac{y}{\delta(x)},$$

provided that the dimensionless function $f(\eta)$ satisfies an equation of the form

$$f'''=\alpha f'^2+\beta f f''+\gamma\,,$$

where the numerical coefficients α, β and γ are to be determined in terms of k. What are the boundary conditions satisfied by $f(\eta)$?

Show that the boundary-layer assumption is invalid sufficiently close to the tip of the wedge, but is self-consistent for $x\gg(\nu/A)^a$, where a is to be found.

Paper 2, Section II

39C Fluid Dynamics II

(a) A uniform rigid cylinder of radius a and finite length falls through unbounded and very viscous fluid. The axis of the cylinder is tilted at some angle to the vertical. Write down the Stokes equations (with no body force) and explain what is meant by the *reversibility* of Stokes flow. Making use of suitable diagrams, use reversibility and symmetries to show that the orientation of the cylinder's axis does not change in any direction as it falls.

(b) Suppose now that the same cylinder falls with its axis vertical through viscous fluid towards a rigid horizontal plane. The gap width $h(t)$ between the closest end of the cylinder and the plane is uniform and satisfies $h(t) \ll a$.

For a prescribed value of $V \equiv dh/dt$, use lubrication theory to find the radial velocity and fluid pressure in the gap. [Assume that the fluid surrounding the gap has constant pressure p_0 and that any end effects near the circular edge of the gap are negligible.]

Deduce that

$$\frac{dh}{dt} = -\frac{2Wh^3}{3\pi\mu a^4},$$

where W is the weight of the cylinder, adjusted for buoyancy. How does h vary asymptotically at large time?

Paper 3, Section II

38C Fluid Dynamics

Viscous incompressible fluid is contained in a very long cylindrical pipe of radius a. The fluid in the central region $0 < r < c < a$ has viscosity μ_c, while that in the surrounding annular region $c < r < a$ has viscosity μ_a. The fluid moves along the pipe due to a fixed pressure difference applied to the ends of the pipe and you may assume that the flow is unidirectional with $\mathbf{u} = u(r)\mathbf{e}_z$ and that the pressure gradient $G \equiv -\partial p/\partial z$ is uniform.

Starting from the steady Navier–Stokes equations, derive the equations satisfied by $u(r)$ in each region, and state all the boundary conditions. Hence determine the flow profiles $u_c(r)$ and $u_a(r)$ in the central and annular regions respectively. Why does $u_a(r)$ not depend on μ_c?

The mechanical energy equation for viscous flow in a fixed volume V with boundary ∂V is given (in standard notation) by

$$\frac{d}{dt}\int_V \tfrac{1}{2}\rho u^2 \, dV + \int_{\partial V} \tfrac{1}{2}\rho u^2 \, \mathbf{u}\cdot\mathbf{n}\, dS = \int_{\partial V} \mathbf{u}\cdot\boldsymbol{\sigma}\cdot\mathbf{n}\, dS - 2\mu \int_V e_{ij}e_{ij}\, dV\,.$$

Verify this equation holds for the annular volume $V = \{c < r < a,\ 0 < z < L\}$ by calculating the various terms.

[*Hint: In cylindrical polar coordinates* (r,θ,z),

$$e_{rz} = \frac{1}{2}\Big(\frac{\partial u_r}{\partial z} + \frac{\partial u_z}{\partial r}\Big) \qquad \textit{and} \qquad \nabla^2 f(r) = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial f}{\partial r}\right),$$

for a function $f = f(r)$.]

Paper 4, Section II

38C Fluid Dynamics

Starting from the Navier–Stokes equations for incompressible viscous flow with conservative forces, obtain the vorticity equation (in standard notation)

$$\frac{\partial \boldsymbol{\omega}}{\partial t} + (\mathbf{u} \cdot \boldsymbol{\nabla})\boldsymbol{\omega} = (\boldsymbol{\omega} \cdot \boldsymbol{\nabla})\mathbf{u} + \nu \nabla^2 \boldsymbol{\omega}.$$

Consider a vorticity field $\boldsymbol{\omega} = \big(0, 0, \omega(r,t)\big)$ in cylindrical polar coordinates (r, θ, z), where ω is a smooth function and is exponentially small as $r \to \infty$. If the corresponding velocity field is $\big(0, v(r,t), 0\big)$ and v is finite as $r \to 0$, find $v(r,t)$ in terms of $\omega(r,t)$.

Suppose now that the velocity field is $\big(-\alpha r,\, v(r,t),\, 2\alpha z\big)$, where α is a positive constant. Show that the vorticity is still $\big(0, 0, \omega(r,t)\big)$. Find the scalar partial differential equation satisfied by $\omega(r,t)$ and deduce that the total circulation,

$$\Gamma = 2\pi \int_0^\infty r\omega(r,t)\, dr\,,$$

is constant in time.

Obtain the steady solution $\omega_s(r)$ for a given value of Γ.

$\Big[$ *Hint: In cylindrical polar coordinates* (r, θ, z)*, the Laplacian of a function* $f = f(r)$ *is given by*

$$\nabla^2 f(r) = \frac{1}{r}\frac{\partial}{\partial r}\left(r \frac{\partial f}{\partial r}\right)\,. \Big]$$

Paper 1, Section I

7D Further Complex Methods

Define the *Cauchy principal value* $\mathcal{P}\int_{-\infty}^{\infty} f(x)\,dx$ of an improper integral.

Let $\omega \in \mathbb{R} \setminus \{0\}$. Consider

$$\int_{-\infty}^{\infty} \frac{e^{i\omega x}}{x^2 - x - 6}\,dx.$$

Explain why this integral does not exist in the usual sense.

Show that the Cauchy principal value

$$I(\omega) = \mathcal{P}\int_{-\infty}^{\infty} \frac{e^{i\omega x}}{x^2 - x - 6}\,dx$$

exists, and evaluate it by considering an appropriate contour integral. State clearly any standard results involving contour integrals that you use.

Hence deduce the values of

$$\mathcal{P}\int_{-\infty}^{\infty} \frac{\cos(\omega x)}{x^2 - x - 6}\,dx \qquad \text{and} \qquad \mathcal{P}\int_{-\infty}^{\infty} \frac{\sin(\omega x)}{x^2 - x - 6}\,dx.$$

Paper 1, Section II

14D Further Complex Methods

Let $I(z) = \int_0^\infty t^{z-1} e^{-t}\, dt$ be Euler's integral. You may assume it converges and is analytic for $\mathrm{Re}\,(z) > 0$.

Show how the Gamma function $\Gamma(z)$ can be defined in terms of $I(z)$ such that $\Gamma(n) = (n-1)!$.

Prove the reflection formula $\Gamma(z)\Gamma(1-z) = \dfrac{\pi}{\sin(\pi z)}$.

The expressions $\Gamma(z)\Gamma\left(z+\frac{1}{2}\right)$ and $\Gamma(2z)$ define two meromorphic functions with the same poles, which are simple. Hence there exists an analytic entire function $g : \mathbb{C} \to \mathbb{C}$ with

$$\Gamma(z)\Gamma\left(z+\frac{1}{2}\right) = e^{g(z)}\Gamma(2z).$$

Using the reflection formula, show that $g(z) = -2\,z\,\log 2 + h(z)$ with $h(z+1) = h(z)$. Why is $h(z)$ entire? The entire 1-periodic function $h(z)$ has the expansion $h(z) = \sum_{n\in\mathbb{Z}} a_n e^{2\pi i n z}$ and has at most exponential growth in any vertical strip $\{s \leqslant \mathrm{Re}\, z \leqslant t\}$, i.e. $|h(z)| \leqslant Ce^{\alpha|\mathrm{Im}\, z|}$ for suitable constants $C, \alpha > 0$. Deduce that $h(z) = b$ is constant and by considering a suitable z, determine the value of b. Deduce

$$\Gamma(z)\Gamma\left(z+\frac{1}{2}\right) = 2^{1-2z}\sqrt{\pi}\,\Gamma(2z).$$

Find $\Gamma(\frac{1}{6})$ in terms of $\Gamma(\frac{1}{3})$.

Paper 2, Section I

7D Further Complex Methods

Consider a function $f(t)$ such that $f(t) = 0$ for $t < 0$ and $f(t) \neq 0$ for $t \geqslant 0$.

Define the *Laplace transform* $\hat{f}(p)$ and the *inverse Laplace transform.* Explain what a *Bromwich contour* is. Explain how the Bromwich inversion integral is evaluated by closing the contour for $t > 0$ and for $t < 0$.

Let $u(x,t)$ be the concentration of a chemical compound on a half-line $x > 0$ for $t > 0$. It obeys the partial differential equation

$$u_t = D\, u_{xx} + e^{-t}, \qquad x > 0,\ t > 0,$$

for $D > 0$ with initial condition $u(x,0) = 0$ for $x > 0$ and boundary condition $u(0,t) = e^{-t}$ for $t > 0$.

Write down an integral representation for $u(x,t)$ using the Bromwich contour, identifying clearly any singularities and explaining how one would choose a contour around them.

Paper 2, Section II

13D Further Complex Methods

Consider a multivalued function $f(z) : \mathbb{C} \to \mathbb{C}$. Define a *branch point*, a *branch cut*, and a *branch* of $f(z)$.

Explain how the function $\arcsin(z)$ can be defined as a complex integral and why it is a multivalued function. How can a single-valued function $\mathrm{Arcsin}(z)$ be constructed such that $\mathrm{Arcsin}(0) = 0$?

Let

$$\arccos(z) = \int_z^1 \frac{dt}{\sqrt{1-t^2}}.$$

Show, using branch cut integration around a suitably chosen branch cut, that for a suitably chosen path $\gamma : 0+ \to 1$

$$\arccos(0+) = \frac{\pi}{2}$$

defining carefully the point $z = 0+$.

Define a single-valued branch of $\mathrm{Arccos}(z)$ such that

$$\mathrm{Arcsin}(z) + \mathrm{Arccos}(z) = \frac{\pi}{2}$$

on its domain. State the domain on which your definition is intended to hold.

Consider now

$$\arctan(z) := \int_0^z \frac{dt}{1+t^2}.$$

By calculating values of $\arctan(1)$ show that it is a multi-valued function. How can it be made to a single-valued function $\mathrm{Arctan}(z)$?

For z in the domain of $\mathrm{Arctan}(z)$, show that

$$\mathrm{Arctan}(z) = \mathrm{Arcsin}\left(\frac{z}{\sqrt{1+z^2}}\right).$$

Paper 3, Section I

7D Further Complex Methods

Consider the Fuchsian ordinary differential equation

$$w'' + p(z)w' + q(z)w = 0$$

with three regular singular points.

Provide a definition of *regular singular points* and explain what the *Papperitz symbol*

$$P\left\{\begin{array}{cccc} a & b & c & \\ \alpha & \beta & \gamma & ; z \\ \alpha' & \beta' & \gamma' & \end{array}\right\}$$

represents.

For

$$z(1-z)w'' + (2-3z)w' - w = 0\,, \qquad (\dagger)$$

show that there are two regular singular points a and b in $\mathbb{C}$. Where is the third regular singular point c located?

Find α, α', β and β' by considering corresponding indicial equations.

Write $(\dagger)$ as a hypergeometric equation, hence determine γ and γ' and deduce the Papperitz symbol.

Paper 4, Section I

7D Further Complex Methods

State the identity theorem for analytic functions.

Define what is meant by the *analytic continuation* of a function.

Consider

$$F(z) = \int_{-\infty}^{\infty} \frac{e^{-5it}}{t^2 - tz + ti - iz}dt,$$

defined for $\operatorname{Im} z < 0$. Why is $F(z)$ analytic for $\operatorname{Im} z < 0$? Construct an analytic continuation of $F(z)$ to $\operatorname{Im} z > 0$ and show that

$$\widetilde{F}(z) = \int_{-\infty}^{\infty} \frac{e^{-5it}}{t^2 - tz + ti - iz}dt, \quad \operatorname{Im} z \neq 0$$

cannot be such an analytic continuation.

Let

$$D = \{z \in \mathbb{C} : \operatorname{Re} z > 0\}$$

and define a sequence $z_n = 1/n$.

Suppose G is analytic on D and satisfies $G(z_n) = 0$ for all $n \in \mathbb{N}$.

Does this information determine G uniquely on D? Justify your answer.

Paper 1, Section II
18F Galois Theory

Let $f \in K[X]$ be a separable polynomial of degree n. Define the *Galois group* $\mathrm{Gal}(f/K)$ of f, and show that it is a transitive subgroup of S_n if and only if f is irreducible. Give an example of two polynomials $f, g \in \mathbb{Q}[X]$ of the same degree n which have the same splitting field, but such that $\mathrm{Gal}(f/\mathbb{Q})$ is not conjugate to $\mathrm{Gal}(g/\mathbb{Q})$ in S_n.

Let $f \in \mathbb{F}_p[X]$ be a separable polynomial whose irreducible factors have degrees $n_1, \ldots, n_r$. Show that $\mathrm{Gal}(f/\mathbb{F}_p)$ is generated by an element of cycle type $(n_1, \ldots, n_r)$. [Standard facts about the Galois theory of finite fields may be used without proof.]

Find the Galois group of the polynomial $X^4 + X^3 + 2X^2 + 1$ (i) over $\mathbb{F}_2$, (ii) over $\mathbb{Q}$.

Paper 2, Section II
18F Galois Theory

(a) Define the nth *cyclotomic polynomial* $\Phi_n(X) \in \mathbb{C}[X]$. Prove that $\Phi_n(X) \in \mathbb{Z}[X]$. For which integers $n \geqslant 1$ is $\Phi_n(X)$ irreducible over $\mathbb{Q}$? Prove your assertion when $n = p$ is a prime number.

(b) Show that if n is odd, then $\Phi_{2n}(X) = \Phi_n(-X)$.

(c) Let $L = \mathbb{Q}(\zeta_{15})$ where ζ_{15} is a primitive 15th root of unity. Show that $G = \mathrm{Gal}(L/\mathbb{Q})$ can be generated by two elements, say σ and τ. Write down the lattice of subgroups of G, and the corresponding lattice of subfields of L. [You should write each intermediate field F explicitly in terms of generators, for example $F = \mathbb{Q}(\sqrt{-3}, \sqrt{5})$.] Which of these fields are contained in $\mathbb{R}$?

Paper 3, Section II
18F Galois Theory

Let L/K be an extension of fields.

(a) What does it mean to say that L/K is *finite*? What does it mean to say that L/K is *algebraic*? Show that any finite extension is algebraic, and give an example of an extension L/K which is algebraic but not finite.

(b) Now assume L/K is algebraic. Let $x \in L$. What does it mean to say that x is *separable* over K? Show that if $\mathrm{char}\, K = 0$ then x is separable over K, but if instead $\mathrm{char}\, K = p > 0$ then there exists an integer $n \geqslant 0$ such that x^{p^n} is separable over K.

(c) Let $L = \mathbb{Q}(\sqrt{2}, \sqrt[3]{2}, \zeta_3)$ where ζ_3 is a primitive cube root of unity. Compute the degree $[L : \mathbb{Q}]$ and decide whether $L/\mathbb{Q}$ is Galois. For which integers $n \geqslant 1$ does there exist $x \in L$ with $x^n = 2$? Justify your answers.

Paper 4, Section II

18F Galois Theory

(a) State Artin's theorem on the invariants of a field L under the action of a finite subgroup of $\mathrm{Aut}(L)$. Deduce that any finite group G arises as the Galois group of some extension of fields.

(b) Define the *elementary symmetric polynomials* $s_1, \ldots, s_n \in \mathbb{Z}[X_1, \ldots, X_n]$. Let $K = \mathbb{C}(s_1, \ldots, s_n) \subset L = \mathbb{C}(X_1, \ldots, X_n)$. Compute the degree $[L : K]$ using Artin's theorem. Show that there exists an element $f \in L$ such that $f^2 \in K$ but $f \notin K$.

(c) Let $M = \mathbb{C}(X, Y)$ and let σ, $\tau \in \mathrm{Aut}(M/\mathbb{C})$ be the automorphisms given by

$$\sigma(X) = \zeta_n X, \qquad \sigma(Y) = \zeta_n^{-1} Y,$$

$$\tau(X) = Y, \qquad \tau(Y) = X,$$

where ζ_n is a primitive nth root of unity. Show that σ and τ generate a group G isomorphic to the dihedral group D_{2n}, and that $M^G = \mathbb{C}(X^n + Y^n, XY)$.

Paper 1, Section II
38A General Relativity

The Schwarzschild metric in units with $c = 1$ is given by

$$ds^2 = -\Big(1 - \frac{r_s}{r}\Big) dt^2 + \Big(1 - \frac{r_s}{r}\Big)^{-1} dr^2 + r^2(d\theta^2 + \sin^2\theta \, d\phi^2) \ ,$$

where r_s is a positive constant, the Schwarzschild radius.

(a) For a photon moving in the Schwarzschild metric (with $r > 0$) explain why the trajectory can be assumed to lie in the equatorial plane $\theta = \pi/2$. With this assumption, show that the photon trajectory satisfies

$$\Big(\frac{dr}{d\lambda}\Big)^2 = F(r) \, ,$$

where λ is an affine parameter and $F(r)$ is a function of r that is to be determined.

Setting $u = 1/r$, show that

$$\Big(\frac{du}{d\phi}\Big)^2 = \frac{1}{b^2} - u^2 + r_s u^3 \, ,$$

where b is a constant, to be defined.

(b) Now consider solutions of the form

$$u(\phi) = \frac{1}{b}(v_0(\phi) + \varepsilon \, v_1(\phi) + O(\varepsilon^2)) \, ,$$

where $\varepsilon = r_s/b \ll 1$ and where the following symmetry conditions are imposed:

$$v_0(\pi - \phi) = v_0(\phi) \quad \text{and} \quad v_1(\pi - \phi) = v_1(\phi) \, .$$

Show that to zeroth order in ε the trajectories are straight lines and explain, with reference to a sketch, the geometrical significance of the constant b.

Show that there is a solution for the first order correction of the form

$$v_1(\phi) = A + B \cos 2\phi \, ,$$

where A and B are constants, to be determined. Hence deduce an approximate expression for the total angle through which the photon is deflected. Comment briefly on how this result can be tested experimentally.

Paper 2, Section II
38A General Relativity

(a) Define the *Einstein tensor* $G_{\mu\nu}$ in terms of the Ricci tensor $R_{\mu\nu}$. State, without proof, the property of the Einstein tensor that is required for consistency of Einstein's equations

$$G_{\mu\nu} \;=\; 8\pi T_{\mu\nu}\,,$$

where $T_{\mu\nu}$ is the energy-momentum tensor. Units with $G = c = 1$ are used throughout.

(b) Consider a FLRW metric in coordinates $x^0 = t$, x^i $(i = 1, 2, 3)$:

$$ds^2 \;=\; -dt^2 \;+\; a(t)^2 h_{ij}\, dx^i dx^j\,,$$

where h_{ij} is a time-independent spatially homogeneous metric depending on a discrete parameter $k = 1, 0$ or -1. The components of the Ricci tensor for this metric are:

$$R_{00} \;=\; -3\frac{\ddot{a}}{a}\,, \qquad R_{ij} \;=\; (\, a\ddot{a} \;+\; 2\dot{a}^2 \;+\; 2k\,)\, h_{ij}\,, \qquad R_{0i} = R_{i0} = 0\,,$$

where dots denote derivatives with respect to t. Given that the energy-momentum tensor takes the form

$$T_{00} = \rho(t)\,, \qquad T_{ij} \;=\; p(t)\, a(t)^2 h_{ij}\,, \qquad T_{0i} = T_{i0} = 0\,,$$

where $\rho(t)$ and $p(t)$ are independent of x^i, show that

$$\dot{a}^2 \;=\; \alpha\, a^2 + \beta\,, \qquad \ddot{a} = \gamma\, a\,, \qquad (*)$$

where α, β, γ are coefficients depending on ρ, p, k that you should determine.

(c) In the Einstein static universe the energy-momentum tensor has contributions from a cosmological constant $\Lambda > 0$ and dust with constant density ρ_m:

$$\rho = \rho_0 := \frac{\Lambda}{8\pi} + \rho_m\,, \qquad p = p_0 := -\frac{\Lambda}{8\pi}\,.$$

Show that there is a static solution to $(*)$ with $a = a_0$, a constant, for a certain choice of the discrete parameter k, and find the values of a_0 and ρ_m in terms of Λ.

(d) Now consider a time-dependent perturbation of the static solution in part (c) corresponding to the addition of a small amount of radiation:

$$a = a_0 + \delta a(t)\,, \quad \rho = \rho_0 + \delta\rho(t)\,, \quad p = p_0 + \delta p(t)\,, \quad \text{with} \quad \delta p(t) \;=\; \frac{1}{3}\delta\rho(t)\,,$$

where $\delta a(t)$, $\delta\rho(t)$ are small and independent of x^i. Determine whether the Einstein static universe is stable or unstable under such a perturbation.

Paper 3, Section II

37A General Relativity

(a) Consider a manifold with metric tensor $g_{\mu\nu}$ and metric connection $\Gamma_{\alpha\ \beta}^{\ \gamma} = \Gamma_{\beta\ \alpha}^{\ \gamma}$. From the Ricci identity

$$\nabla_\alpha \nabla_\beta V^\mu \, - \, \nabla_\beta \nabla_\alpha V^\mu \, = \, R^\mu{}_{\nu\alpha\beta} V^\nu$$

for any vector field V^μ, and using the definition of ∇_α, find an explicit expression for the Riemann curvature tensor $R^\mu{}_{\nu\alpha\beta}$ in terms of the connection.

(b) State clearly the symmetries of the Riemann tensor $R_{\mu\nu\alpha\beta}$. Explain briefly, without carrying out calculations, how your expression for the Riemann curvature tensor allows all symmetries to be easily checked.

(c) Given vector fields V^μ and W^μ, show that the commutator defined by

$$[\,V,\,W\,]^\mu \, = \, V^\alpha \partial_\alpha W^\mu \, - \, W^\alpha \partial_\alpha V^\mu$$

is also a vector field.

(d) A vector field V^μ is defined to be a Killing vector field if

$$\nabla_\alpha V_\beta \, + \, \nabla_\beta V_\alpha \, = \, 0\,.$$

Use this property to obtain an expression for $\nabla_\alpha \nabla_\beta V^\mu$ in terms of the Riemann tensor contracted with the Killing vector field.

(e) Show that if V^μ and W^μ are Killing vector fields, then the commutator $[V, W]^\mu$ is also a Killing vector field.

Paper 4, Section II
37A General Relativity

(a) The metric for de Sitter spacetime can be written

$$ds^2 = -\left(1 - r^2/b^2\right) dt^2 + \left(1 - r^2/b^2\right)^{-1} dr^2 + r^2\left(d\theta^2 + \sin^2\theta \, d\phi^2\right),$$

using static, spherically symmetric coordinates (t, r, θ, ϕ), where b is a positive constant and units with $c = 1$ are chosen. Defining

$$u = t - b \tanh^{-1}(r/b) \quad \text{for} \quad 0 \leqslant r < b\,,$$

find the components of the vector field $U_\mu = \partial_\mu u$ in the (t, r, θ, ϕ) coordinate system and verify that U_μ is null.

Find the form of the metric in coordinates (u, r, θ, ϕ) and hence show that the metric is non-singular on the surface $r = b$.

(b) Show that null geodesics which are radial ($d\theta = d\phi = 0$) must obey either

$$\frac{du}{dr} = 0 \qquad \text{or} \qquad \frac{du}{dr} = -\frac{2}{1 - r^2/b^2}\,.$$

Determine which of these geodesics are outgoing, $dr/dt > 0$, and which are ingoing, $dr/dt < 0$, for $0 \leqslant r < b$. Sketch members of both families in the r-t_* plane, where $t_* = r + u$, for $0 \leqslant r < b$.

Is it possible for a light signal travelling radially inwards in the region $r < b$ to have been sent from a point in the region $r > b$?

(c) Consider an observer travelling radially outward from $r = 0$ on a timelike trajectory $r(\tau)$, where τ is proper time. By finding a suitable trajectory explicitly, show that it is possible for the observer to reach the surface $r = b$ after finite proper time has elapsed.

Paper 1, Section II

17J Graph Theory

(a) Let G be a bipartite graph with vertex classes X and Y. What does it mean to say that G *contains a matching from* X *to* Y? State and prove Hall's marriage theorem.

(b) Let $G = (V, E)$ be a bipartite graph with vertex classes X and Y. Suppose that every $x \in X$ has degree $d(x) \geqslant 1$, and that if $x \in X$ and $y \in Y$ with $xy \in E$, then $d(x) \geqslant d(y)$. Show that G contains a matching from X to Y. [*Hint: If not, consider a minimal subset of* X *that fails Hall's condition.*]

(c) Let $n \geqslant 1$ and $k \geqslant 1$ be integers. Assume that we are given a sequence $A_1, A_2, \ldots, A_k$ of subsets of $\{1, 2, \ldots, n\}$ that is an *antichain*, which means that $A_i \not\subseteq A_j$ for all $i \neq j$. Suppose further that $|A_1 \cup \cdots \cup A_k| > k$. Show that there exist i and $x \in A_i$ such that if we replace A_i in the sequence by $A_i \setminus \{x\}$, then the resulting sequence is still an antichain. [*Hint: Argue by contradiction.*]

Paper 2, Section II

17J Graph Theory

(a) State and prove Ramsey's theorem for $R(s, t)$ for $s, t \geqslant 2$. Deduce that the Ramsey number $R(t)$ satisfies $R(t) \leqslant 2^{2t}$.

(b) Let the positive integers be 2-coloured. Show that, for any integer $n \geqslant 3$, there exists a monochromatic set $\{x_1, x_2, \ldots, x_n\}$, where $x_1, \ldots, x_n$ are not necessarily distinct, such that

$$x_1 + x_2 + \ldots + x_{n-1} = x_n.$$

(c) Show that we can 2-colour $\mathbb{R}^2$ such that there is no equilateral triangle with sidelength 1 whose vertices have the same colour. [*Hint: Consider colouring* $\mathbb{R}^2$ *in vertical strips of a certain width.*]

Show that if we 3-colour $\mathbb{R}^2$, we can always find two points of the same colour at distance 1 apart.

Paper 3, Section II

17J Graph Theory

(a) Define the *adjacency matrix* A of a graph G. What are the *eigenvalues* of G?

(i) Show that if λ is an eigenvalue of G, then $|\lambda| \leqslant \Delta$, where Δ is the maximum degree of G.

(ii) Show that if G is connected, then Δ is an eigenvalue of G if and only if G is regular. In this case show that Δ has multiplicity 1 and eigenvector $(1, 1, \ldots, 1)^T$.

(b) Let G have adjacency matrix A. For each $r \in \mathbb{N}$, give a characterisation of the rth power A^r of A in terms of walks in G. Assume now that G is d-regular and that for all $r \in \mathbb{N}$, the number of circuits in G of length r is

$$5^r + 3 \cdot 5^{r/2} + 3 \cdot (-1)^r \cdot 5^{r/2} + 5 \cdot (-1)^r.$$

Find the order and size of G. Compute d.

(c) What is meant by a graph G of order n being *strongly regular with parameters* (d, a, b)? If such a graph G exists and $b > 0$ and G is not complete, write down the rationality condition.

Let G be a graph of order at least 2 containing no triangles, in which every pair of non-adjacent vertices has exactly three common neighbours. Show that G must be d-regular of order $1 + d(d+2)/3$ for some $d \in \{1, 3, 21\}$. Show that such a graph exists for $d = 3$.

Paper 4, Section II

17J Graph Theory

In this question all graphs have at least 3 vertices.

(a) Let G be a graph of order n with minimum degree δ. What does it mean to say that G is *Hamiltonian*? Show that if $\delta \geqslant \frac{n}{2}$, then G is Hamiltonian. For each $n \geqslant 3$, give examples to show that this result does not remain true if we weaken the condition to $\delta \geqslant \frac{n}{2} - 1$ (for n even) or $\delta \geqslant \frac{n-1}{2}$ (for n odd).

(b) Let G be a Hamiltonian graph. Show that G has the property that for all subsets V_0 of $V(G)$, the number of connected components after removing V_0 is at most $|V_0|$.

Is it true that every graph with this property must be Hamiltonian? Give a proof or a counterexample.

(c) Let G be a graph and let $\chi(G)$ be its chromatic number. A *Hamilton path* is a path that visits each vertex of G exactly once. Suppose that G has the property that for every pair of vertices there exists a Hamilton path between the two vertices. Show that $\chi(G) \geqslant 3$.

Paper 1, Section II

33B Integrable Systems

Let M be a six–dimensional phase–space with canonical coordinates (x, y, z, p_x, p_y, p_z), with the Hamiltonian

$$H = \frac{1}{2}({p_x}^2 + {p_y}^2 + {p_z}^2) + V(x, y, z). \tag{†}$$

(a) Write down Hamilton's equations for (M, H), define a first integral of the system and state what it means to say that the system is *integrable*.

(b) State the Arnold–Liouville theorem.

In the remainder of the question restrict attention to the case where $V(x, y, z) = U(r)$, with $r^2 = x^2 + y^2 + z^2$.

(c) Show that $m_z = xp_y - yp_x$ and $m_y = zp_x - xp_z$ Poisson commute with H. Consider $m_x \equiv \{m_y, m_z\}$ and find a non–constant function $G = G(m_x, m_y, m_z)$ that Poisson commutes with m_x. Deduce that Hamilton's equations, with the Hamiltonian given by (†) with $V(x, y, z) = U(r)$, are integrable. [You do not need to show that (H, G, m_x) are independent everywhere on M, but you should show that (H, G, m_x) are independent in a neighbourhood of each point $\mathbf{x} = (1, 0, 0), \mathbf{p} = (p_x, p_y, p_z)$ with $p_y \neq 0, p_z \neq 0$.]

Paper 2, Section II

34B Integrable Systems

Let $L = -{\partial_x}^2 + u(x, t)$ and let A be a differential operator involving ∂_x such that

$$\partial_t L = [L, A]. \tag{†}$$

Show that eigenvalues of L corresponding to normalisable eigenfunctions do not depend on t.

Let $A = {\partial_x}^3 + a(x, t)\partial_x + b(x, t)$. By constructing an eigenfunction f of L such that $\partial_t f + Af = 0$, find 2×2 matrices M_A, M_L such that the system of PDEs (†) for u, a, b admits a zero–curvature representation

$$\partial_t M_L - \partial_x M_A - [M_A, M_L] = 0.$$

[You may assume that the eigenvalues of L are non–degenerate.]

Now assume that the functions u, a and b do not depend on x. Show that $\mathrm{Tr}\{(M_L)^k\}$ does not depend on t for any positive integer k.

Paper 3, Section II

32B Integrable Systems

Let $u : \mathbb{R}^n \to \mathbb{R}$. Find a one-parameter group of transformations G of $\mathbb{R}^{n+1}$ generated by a vector field

$$W = \phi u \frac{\partial}{\partial u} + \sum_{i=1}^{n} \alpha^i x^i \frac{\partial}{\partial x^i} \qquad (\dagger)$$

where $\phi, \alpha^1, \ldots, \alpha^n$ are constants not all equal to zero. State what is meant by an *invariant* of G, and show that $f : \mathbb{R}^{n+1} \to \mathbb{R}$ is an invariant if and only if $W(f) = 0$.

Find two independent vector fields W_1 and W_2 of the form ($\dagger$) generating Lie point symmetries of a PDE

$$u_{yy} - u_{tx} + (uu_x)_x = 0. \qquad (\ddagger)$$

Find three functionally independent invariants of W_1 and do the same for W_2. Find two independent non-constant functions $f(x, y, t, u)$ and $g(x, y, t, u)$ which are invariant for both W_1 and W_2.

Show that, if solutions of ($\ddagger$) are to be invariant under a group generated by

$$W = u\partial_u + x\partial_x + \frac{1}{2} y\partial_y, \quad W_3 = \partial_y,$$

they must be of the form $u = xF(t)$. Construct the corresponding group-invariant solutions.

Paper 1, Section II

22I Linear Analysis

(a) State and prove the open mapping lemma. State the open mapping theorem.

(b) Let $T\colon X \to Y$ be a linear map between normed spaces. We say that T is *bounded below* if there exists $\delta > 0$ such that $\|Tx\| \geqslant \delta\|x\|$ for all $x \in X$. Assume now that X and Y are Banach spaces and that T is bounded and injective. Show that T is bounded below if and only if the image of T is closed in Y.

(c) Let X and Y be Banach spaces. Show that each of the following properties of a bounded linear map $T \in \mathcal{B}(X, Y)$ are stable under perturbation, *i.e.* that for some $\varepsilon > 0$, any $S \in \mathcal{B}(X, Y)$ with $\|S - T\| < \varepsilon$ has the same property.

(i) T is bounded below.

(ii) T is surjective.

(iii) T is an isomorphism.

Paper 2, Section II

22I Linear Analysis

(a) State and prove the Riesz representation theorem.

(b) Let H be a complex Hilbert space. A *sesquilinear form on* H is a map $\theta\colon H \times H \to \mathbb{C}$ such that for each $y \in H$ the map $H \to \mathbb{C}$ given by $x \mapsto \theta(x, y)$ is linear, and for each $x \in H$ the map $H \to \mathbb{C}$ given by $y \mapsto \theta(x, y)$ is conjugate-linear. We say θ is *bounded* if

$$\|\theta\| = \sup\{|\theta(x, y)| : x, y \in H, \|x\| \leqslant 1, \|y\| \leqslant 1\} < \infty \ .$$

Show that if θ is a bounded sesquilinear form on H, then there is a unique map $S\colon H \to H$ such that $\theta(x, y) = \langle x, Sy\rangle$ for all $x, y \in H$. Show further that $S \in \mathcal{B}(H)$ and $\|S\| = \|\theta\|$.

(c) Define the *adjoint* of an operator $T \in \mathcal{B}(H)$. Using part (b), or otherwise, prove the existence and uniqueness of the adjoint of T, and compute its norm.

(d) Define *hermitian, unitary* and *normal* operators. Let $T \in \mathcal{B}(H)$ be hermitian. Show that every approximate eigenvalue of T is real. Deduce that every element of the spectrum of T is a real approximate eigenvalue.

Paper 3, Section II

21I Linear Analysis

(a) Let X be a non-zero, complex Banach space and $T \in \mathcal{B}(X)$. Define the *spectrum* $\sigma(T)$ and the *spectral radius* $r(T)$ of T. Briefly explain why $r(T)$ is well-defined. Show that $r(T) \leqslant \inf_{n\in\mathbb{N}}\|T^n\|^{1/n}$.

(b) State the complex version of the Stone–Weierstrass theorem and deduce it from the real version. Let K and L be compact Hausdorff spaces. Show that finite sums of functions on the product space $K \times L$ of the form $(x, y) \mapsto f(x)g(y)$ ($f \in C(K)$, $g \in C(L)$) are dense in $C(K \times L)$.

(c) Let $h \in C([0,1]^2)$. For $f \in C[0,1]$ define

$$T_h f(x) = \int_0^1 h(x,y)f(y)\,\mathrm{d}y, \quad x \in [0,1]\,.$$

Show that $T_h \in \mathcal{B}(C[0,1])$. [You do not need to show that $T_h f$ is continuous.] Show that the map $T\colon h \mapsto T_h$ from $C([0,1]^2) \to \mathcal{B}(C[0,1])$ is a bounded linear map.

(d) Now fix $h \in C([0,1]^2)$. Using part (b), or otherwise, show that T_h is the limit of a sequence of finite-rank operators. Explain briefly why this means that T_h is compact. By considering T_h^n, or otherwise, show that if $h(x,y) = 0$ whenever $0 \leqslant x \leqslant y \leqslant 1$, then $\sigma(T_h) = \{0\}$.

Paper 4, Section II

22I Linear Analysis

(a) Define the *dual space* X^* of a real normed space X. Define the *dual operator* T^* of a bounded linear map $T\colon X \to Y$ between normed spaces X and Y, and show that T^* is a bounded linear map.

(b) Let X and Y be normed spaces. Show that if $X \sim Y$, *i.e.* X and Y are isomorphic, then $X^* \sim Y^*$. Show also, that if $X \cong Y$, *i.e.* X and Y are isometrically isomorphic, then $X^* \cong Y^*$.

(c) Let X be a Banach space such that $X^* \cong \ell_2$. Show that $X \cong \ell_2$. [You may use the fact that the canonical embedding $x \mapsto \hat{x} : X \to X^{**}$ is isometric, where $\hat{x}(f) = f(x)$.]

(d) Let K be an infinite compact Hausdorff space. Fix pairwise distinct elements $k_1, k_2, \ldots$ in K and, for fixed $y = (y_n) \in \ell_1$, consider the map $\varphi_y : C(K) \to \mathbb{R}$ given by

$$\varphi_y(f) = \sum_n f(k_n)y_n\,.$$

Using this map, or otherwise, show that ℓ_1 is isometrically isomorphic to a subspace of $C(K)^*$.

(e) Is it true that $C[0,1]^* \cong \ell_1$? [*Hint: You may wish to consider the following elements of* $C[0,1]^*$*: for fixed* $x \in [0,1]$*, the map* $\delta_x(f) = f(x)$*.*]

Paper 1, Section II

16J Logic & Set Theory

(a) State the axiom of choice and Zermelo's wellordering theorem, providing definitions of concepts used in the statements.

(b) Prove Zermelo's wellordering theorem assuming all axioms of Zermelo-Fraenkel set theory with the axiom of choice.

(c) State a version of Hessenberg's theorem that can be proved without the axiom of choice. [You do not need to argue that it can be proved without the axiom of choice.]

(d) Work in Zermelo-Fraenkel set theory without the axiom of choice and prove that the axiom of choice is equivalent to the statement "for every infinite set X, there is a bijection between X and $X \times X$". [*Hint: For one of the implications, consider sets of the form $X \cup \alpha$ where α is an ordinal disjoint from the set X.*]

[*Throughout this question, you may use that the following can all be proved in Zermelo-Fraenkel set theory without the axiom of choice: the recursion theorem, the representation theorem for wellorders, Hartogs's theorem, and the fact that Zermelo's wellordering theorem implies the axiom of choice.*]

Comment.

Parts (a), (b), and (c) are bookwork and should be doable by all students who properly revised. Part (d), $\Rightarrow$ is new, but straightforward, especially given that all of the components are explicitly mentioned in the question elsewhere.

Part (d), $\Leftarrow$ could be challenging: this is known as Tarski's theorem. If it is considered to be too hard by the examiners, it could be made easier by giving the following hint: "*Hint.* For the direction from right to left, let α be the Hartogs aleph of X and consider the set $Y := X \times \alpha$."

Paper 2, Section II

16J Logic & Set Theory

Let $n \geqslant 2$ be a natural number and let $H_n = \{0\} \cup \{1/k;\ 1 \leqslant k < n\}$. For $a, b \in H_n$, let

$$a \to b = \begin{cases} 1 & \text{if } a \leqslant b, \text{ and} \\ b & \text{if } a > b. \end{cases}$$

Let $\mathbf{H}_n = (H_n, \to, 0)$. We say that a map v from the set of implicative formulas (i.e., formulas that do not contain the symbols $\wedge$, $\vee$, $\neg$, or $\top$) to H_n is an *n-valuation* if it is a homomorphism into $\mathbf{H}_n$. An implicative formula φ is called *n-valid* if for all n-valuations v, we have that $v(\varphi) = 1$.

(a) Let $a, b \in H_n$ and $P(a, b) = ((a \to b) \to a) \to a$. Show that

$$P(a, b) = \begin{cases} 1 & \text{if } a \leqslant b, \text{ and} \\ a & \text{if } a > b. \end{cases}$$

(b) Let $a, b, c \in H_n$ and $T(a, b, c) = (b \to c) \to a$. Show that

$$T(a, b, c) = \begin{cases} 1 & \text{if } b > c \text{ and } c \leqslant a, \text{ and} \\ a & \text{otherwise.} \end{cases}$$

(c) State the axioms of types 1, 2, and 3 used in proofs in propositional logic.

(d) Show that all axioms of type 1 and 2 are n-valid for all $n \geqslant 2$ and that, for any $n \geqslant 3$, there are axioms of type 3 that are not n-valid.

(e) Using parts (a) and (b), or otherwise, find a formula that is 3-valid, but not 4-valid. Justify your claim.

Comments. Parts (a) and (b) are very simple algebraic transformations and should be doable for any student who reads the question (they do not even require any knowledge from the lecture course, just engagement with the question). Part (c) is bookwork. Part (d) consists of very simple calculations, very similar to the proof of soundness for propositional calculus. Students who understood the basics of propositional logic should be able to to parts (a) to (d) without any problems and get to 14 marks. Part (e) is new material and could be challenging.

Note that in some years, there is a example on Example Sheet #1 whose solution is equivalent to "the formula corresponding to $P(a, b)$ is valid, but not 3-valid". This example will not be on the sheets this year.

Paper 3, Section II

16J Logic & Set Theory

Work in the language of ordered fields with binary function symbols $+$ and $\cdot$, a binary relation symbol $<$, and two constant symbols 0 and 1. Let T be the axioms of ordered fields (i.e., the usual field axioms, the usual axioms for strict total orders, and the additional axioms $\forall x \forall y \forall z (x < y \Rightarrow x + z < y + z)$ and $\forall x \forall y (x > 0 \wedge y > 0 \Rightarrow x \cdot y > 0)$). Let $\mathbf{F} = (F, +, \cdot, 0, 1, <) \models T$. Let t_n, $n \in \mathbb{N}$, be terms defined recursively by $t_0 = 0$, $t_1 = 1$, and $t_{n+1} = t_n + 1$. We say that $\mathbf{F}$ is *Archimedean* if for all $x \in F$ there is an n such that $x < t_n^{\mathbf{F}}$.

(a) State what it means that a class of ordered fields is *axiomatisable.*

(b) State the compactness theorem for predicate logic.

(c) Show that the class of Archimedean ordered fields is not axiomatisable.

For $x, y \in F$ (we are allowing the case $x = y$), sets of the form (x, y), $[x, y)$, $(x, y]$, $[x, y]$, $(-\infty, x)$, $(-\infty, x]$, (x, ∞), or $[x, \infty)$ are called *intervals.* Finite unions of intervals are called *basic sets.*

(d) Let $\varphi(x)$ be a quantifier-free formula with one free variable. Show that $\{x : \mathbf{F} \models \varphi(x)\}$ is a basic set. [You may use without proof the fact that for a polynomial function p, the sets $\{x : p(x) = 0\}$ and $\{x : p(x) < 0\}$ are basic sets.]

(e) Assume that an ordered field $\mathbf{F}$ has the following property. For each formula $\varphi(x)$ with one free variable, there is a quantifier-free formula $\psi(x)$ such that for all $x \in F$, we have that $\mathbf{F} \models \varphi(x)$ if and only if $\mathbf{F} \models \psi(x)$.

Show that every non-negative element of F is a square, i.e., for each $x > 0$, there is a y such that $y^2 = x$. [You may use without proof the following statements that hold in all ordered fields: if $x > 0$ and $y > 1$, then $x \cdot y > x$, and if $x > 0$ and $y < 1$, then $x \cdot y < x$.]

Comment.

Part (a) and (b) are bookwork and part (c) is very similar to examples seen in the lectures. Part (d) is a typical easy induction proof on formula complexity. All well-prepared students should find it easy to get 12 marks.

Part (e) should not be too hard, but requires some idea what needs to be done. It is (a slightly weaker form of) the converse of the model theoretic result that real-closed fields satisfy quantifier elimination.

Paper 4, Section II

16J Logic & Set Theory

Work in ZFC in this question.

(a) Define the following:

(i) a *transitive set*,

(ii) the *transitive closure* $\mathrm{tcl}(x)$ *of a set* x,

(iii) the levels $\mathbf{V}_\alpha$ of the *von Neumann hierarchy*, and

(iv) the *Mirimanoff rank* ϱ.

(b) Prove that for every set x and $\gamma < \varrho(x)$, there is some $y \in x$ such that $\gamma \leqslant \varrho(y)$.

(c) For any set x, show that $\varrho(x) = \varrho(\mathrm{tcl}(x))$.

(d) By induction or otherwise, prove that for every transitive set x and $\gamma < \varrho(x)$, there is some $y \in x$ such that $\gamma = \varrho(y)$.

We call a set x *hereditarily countable* if $\mathrm{tcl}(x)$ is countable and write $\mathbf{HC}$ for the collection of hereditarily countable sets.

(e) Show that $\mathbf{HC} \subseteq \mathbf{V}_{\omega_1}$.

(f) Show that $|\mathbf{HC}| = 2^{\aleph_0}$.

Comment. (a) is just definitions. (b) is an easier version of what is usually called the "rank formula", i.e., $\varrho(x) = \bigcup\{\varrho(y)\,;\, y \in x\}$ which is lectured. (c) is an easy argument that any student who understands the definitions should be able to do.
(d) is a typical induction argument and should not be hard. However, there is a little twist (noted in the comment after the model solution). Marks should be subtracted if the student gets this wrong. Students should easily be able to get 12 marks, possibly minus the marks subtracted for getting the twist wrong.
(e) and (f) are new, but (e) follows quite easily from the previous statements (c) and (d), so should not be too challenging.

Paper 1, Section I

6C Mathematical Biology

A certain sexually transmitted disease is only passed on by heterosexual sex. The populations of susceptible heterosexual males and females are denoted by S and $\widehat{S}$, respectively. The populations of infected males and females are denoted by I and $\widehat{I}$, respectively. The disease is modelled by the equations

$$\frac{d\widehat{S}}{dt} = -\widehat{\beta}\widehat{S}I + \widehat{\nu}\widehat{I} \quad \text{and} \quad \frac{d\widehat{I}}{dt} = \widehat{\beta}\widehat{S}I - \widehat{\nu}\widehat{I},$$
$$\frac{dS}{dt} = -\beta S\widehat{I} + \nu I \quad \text{and} \quad \frac{dI}{dt} = \beta S\widehat{I} - \nu I,$$

with β, $\widehat{\beta}$, ν, and $\widehat{\nu}$ constant.

(a) Give a biological interpretation of each term in the equations.

(b) Define $N = S + I$ and $\widehat{N} = \widehat{S} + \widehat{I}$ and assume that

$$N\widehat{N}\beta\widehat{\beta} > \nu\widehat{\nu} \ .$$

By writing equations involving only I and $\widehat{I}$, show that there are fixed points at $I = \widehat{I} = 0$ and at

$$I = \frac{\widehat{N}N - \widehat{\nu}\nu/(\widehat{\beta}\beta)}{\nu/\beta + \widehat{N}} \quad \text{and} \quad \widehat{I} = \frac{\widehat{N}N - \widehat{\nu}\nu/(\widehat{\beta}\beta)}{\widehat{\nu}/\widehat{\beta} + N} \ .$$

(c) Show that the fixed point at the origin is unstable. Why must the other fixed point be stable?

Paper 2, Section I

6C Mathematical Biology

Spruce budworms have a population density $n(x,t)$ and are restricted to lie in an interval $x \in [0, L]$. The population is described by the diffusion equation

$$\frac{\partial n}{\partial t} = D\frac{\partial^2 n}{\partial x^2} \ ,$$

where D is a constant.

(a) What boundary conditions at $x = 0$ and $x = L$ would ensure that the total population is conserved?

(b) The population grows at a rate α per capita. What change should we make to the diffusion equation to capture this?

(c) Suppose the surface of the domain is hostile, so that we now apply the boundary conditions $n(0,t) = n(L,t) = 0$. Derive a condition on the parameters D, α, and L that is necessary if the population is not to die out.

Paper 3, Section I

6C Mathematical Biology

Bacteria have a population density $n(x,t)$ obeying the reaction-diffusion equation

$$\frac{\partial n}{\partial t} = \frac{\partial^2 n}{\partial x^2} - n(n-A)(n-B)\,,$$

with $0 < A < B$.

(a) Find the stable, homogeneous fixed points.

(b) Suppose $n(x,t) \to 0$ as $x \to -\infty$ and $n(x,t) \to C$ as $x \to +\infty$, with C a non-zero stable homogeneous fixed point.

Show that the system supports a solution of the form $n(x,t) = f(x-ct)$, for c a constant, and f some suitable function you need not determine.

What further condition on A and B ensures that the population does not die out in the future?

Paper 3, Section II

13E Mathematical Biology

At any time $t \in \mathbb{R}$, a certain species has a population size $n \in \mathbb{Z}^+$. The probability rate for the species to evolve from size n to size $n+r$, with $r \in \mathbb{Z}$, is given by $W(n,r)$.

(a) Write down the master equation that governs the probability distribution $P(n,t)$.

(b) In what circumstance can the discrete population size n be approximated by a continuous variable $x \in \mathbb{R}^+$? By making appropriate assumptions, derive the Fokker–Planck equation,

$$\frac{\partial P}{\partial t} = -\frac{\partial (uP)}{\partial x} + \frac{\partial^2 (DP)}{\partial x^2}\,, \tag{$\dagger$}$$

for a probability distribution $P(x,t)$, where $u(x)$ and $D(x)$ are functions that you should define in terms of $W(x,r)$. Explain how the Fokker–Planck equation ($\dagger$) ensures that probability is conserved and give a physical interpretation of $u(x)$ and $D(x)$. Give an example of probability rates $W(x,r)$ for which ($\dagger$) reduces to the diffusion equation.

(c) Compute the time derivatives $d\langle x\rangle/dt$ and $d(\mathrm{var}(x))/dt$, where $\langle\cdot\rangle$ denotes an expectation value and $\mathrm{var}(x)$ is the variance of x.

(d) Consider the rates $W(x,1) = e^{-\lambda x}$ and $W(x,-1) = e^{\lambda x}$, with $W(x,r) = 0$ for $r \neq \pm 1$. Use the Fokker–Planck equation ($\dagger$) to derive the steady-state probability distribution $P(x,t)$. [You need not compute the overall normalisation constant of this distribution.]

Sketch the distribution.

(e) Now take the rates to be $W(x,1) = a$ and $W(x,-1) = b$, where a and b are unequal, positive constants. As before, $W(x,r) = 0$ for $r \neq \pm 1$.

(i) Change variables from x to $z = x - ut$, and from ($\dagger$) deduce the partial differential equation for $\tilde{P}(z,t) = P(x,t)$.

(ii) Given $\tilde{P}(z,0) \geqslant 0$ for all z, argue that, if there exists any time $T > 0$ at which $\tilde{P}(z_*,T) < 0$ for some z_*, then the equation deduced in part (e)(i) is contradictory. [*Hint: Consider the earliest such time T.*]

(iii) Let us add a term to ($\dagger$), obtaining

$$\frac{\partial P}{\partial t} = -\frac{\partial (uP)}{\partial x} + \frac{\partial^2 (DP)}{\partial x^2} - \frac{\partial^3 (FP)}{\partial x^3}\,. \tag{$\star$}$$

Amending your answer to part (b), express $F(x)$ in terms of a and b. Explain why the argument of part (e)(ii) does not hold for ($\star$).

Paper 4, Section I

6C Mathematical Biology

A delay model for a population N_t is given by

$$N_{t+1} = N_t \exp[r(1 - N_{t-1}/K)]\,,$$

where t is discrete time and $r, K > 0$.

Show that there is a stable, non-zero fixed point for $0 < r < 1$.

Show that at $r = 1$ the steady state bifurcates to a periodic solution of period 6.

Paper 4, Section II

14C Mathematical Biology

In a forest the number of trees of height s, at time t, is denoted by $n(s,t)$. Trees grow at a rate $g(s) > 0$. The production rate of seedlings per tree is $b(s)$, and the death rate per tree is $\mu(s)$. The evolution of the forest is described by the equation

$$\frac{\partial n}{\partial t} + \frac{\partial (g\,n)}{\partial s} = -\mu\, n\ .$$

(a) Explain why this equation is subject to the boundary condition

$$g(0)n(0,t) = \int_0^\infty ds\ b(s)\, n(s,t)\,.$$

(b) A separable solution takes the form $n(s,t) = \widetilde{n}(s)e^{rt}$ with r constant.

(i) Derive an expression for $\widetilde{n}(s)$ and show how the boundary condition yields an integral expression for r.

(ii) Suppose that $g(s) = g$ and $\mu(s) = \mu$, both constant, while $b(s) = B$ for $s > s_0$ and zero otherwise, for constants B and s_0. What is the mean number of seedlings per tree? What value of B ensures that the tree population is independent of time?

(iii) Suppose now that g and μ are constant but that trees only produce seedlings when they are a particular height $s^\star$, so that $b(s) = B\,\delta(s - s^\star)$, where δ is the Dirac delta function. What is the value of r?

Paper 1, Section II

31K Mathematics of Machine Learning

(a) Suppose $\mathcal{H}$ is a class of functions $h : \mathbb{R}^p \to \{-1,1\}$ with $|\mathcal{H}| \geqslant 2$. Define the *shattering coefficient* $s(\mathcal{H},n)$ and the *VC dimension* $\mathrm{VC}(\mathcal{H})$ of $\mathcal{H}$.

Let $\mathcal{A}$ be the set of all partitions $\mathcal{I}$ of $\mathbb{R}$ of the form

$$\mathcal{I} := \{(-\infty,a_1),[a_1,a_2),[a_2,a_3),\ldots,[a_{m-2},a_{m-1}),[a_{m-1},\infty)\}$$

for some $a_1 < \cdots < a_{m-1}$. We denote $(-\infty,a_1)$ by I_1, $[a_{k-1},a_k)$ by I_k for $k = 2,\ldots,m-1$, and $[a_{m-1},\infty)$ by I_m. For a given input set $\mathcal{X}$ and a class $\mathcal{F}$ of functions $f : \mathcal{X} \to \mathbb{R}$, we write $\mathcal{H}_{\mathcal{F}}$ to denote the class of functions $\{\operatorname{sgn} \circ f : f \in \mathcal{F}\}$.

(b) Given fixed $\beta_1,\ldots,\beta_m \in \mathbb{R}$, let $\mathcal{F}_1$ be the class of functions from $\mathbb{R}$ to $\mathbb{R}$ given by

$$\mathcal{F}_1 := \left\{ x \mapsto \sum_{k=1}^{m} \beta_k \mathbb{1}_{I_k}(x) : \mathcal{I} = \{I_1,\ldots,I_m\}, \mathcal{I} \in \mathcal{A} \right\}.$$

Show that $s(\mathcal{H}_{\mathcal{F}_1},n) \leqslant (n+1)^{m-1}$. [*Hint: Given* $x_1 < x_2 < \cdots < x_n$*, consider the partition* $(-\infty,x_1],(x_1,x_2],(x_2,x_3],\ldots,(x_{n-1},x_n],(x_n,\infty)$.]

(c) Given fixed $\mathcal{I}_1,\ldots,\mathcal{I}_p \in \mathcal{A}$, let $\mathcal{F}_2$ be a class of functions from $\mathbb{R}^p \to \mathbb{R}$ given by

$$\mathcal{F}_2 = \mathcal{F}_2(\mathcal{I}_1,\ldots,\mathcal{I}_p) := \left\{ x \mapsto \sum_{j=1}^{p} \sum_{I \in \mathcal{I}_j} \beta_{j,I} \mathbb{1}_I(x_j) \ : \beta_{j,I} \in \mathbb{R} \text{ for all } j, I \right\}.$$

Show that $\mathrm{VC}(\mathcal{H}_{\mathcal{F}_2}) \leqslant 1 + p(m-1)$.

(d) Now let

$$\mathcal{F}_3 := \left\{ x \mapsto \sum_{j=1}^{p} \sum_{I \in \mathcal{I}_j} \beta_{j,I} \mathbb{1}_I(x_j) \ : \beta_{j,I} \in \mathbb{R} \text{ for all } j, I \text{ and } \mathcal{I}_1,\ldots,\mathcal{I}_p \in \mathcal{A} \right\},$$

so the difference compared to $\mathcal{F}_2$ is that the partitions are not fixed. Show that

$$s(\mathcal{H}_{\mathcal{F}_3},n) \leqslant (n+1)^{1+2p(m-1)}.$$

Paper 2, Section II

31K Mathematics of Machine Learning

Consider a classification setting with i.i.d. input–output pairs $(X_1, Y_1), \ldots, (X_n, Y_n) \in [-1,1]^p \times \{-1,1\}$.

(a) What is meant by the *empirical ϕ-risk* $\hat{R}_\phi(h)$ of hypothesis $h : \mathbb{R}^p \to \mathbb{R}$ given convex surrogate loss ϕ?

(b) Given a class of hypotheses $\mathcal{H}$, what is the *Rademacher complexity* $\mathcal{R}_n(\mathcal{H})$?

(c) Show that the empirical ϕ-risk minimiser $\hat{h}$ and the minimiser h^* of the ϕ-risk R_ϕ satisfy

$$\mathbb{E}R_\phi(\hat{h}) - R_\phi(h^*) \leqslant \mathbb{E}\sup_{h \in \mathcal{H}}\{R_\phi(h) - \hat{R}_\phi(h)\}.$$

Suppose ϕ is the hinge loss. Quoting or naming any results that you need, argue that

$$\mathbb{E}R_\phi(\hat{h}) - R_\phi(h^*) \leqslant 2\mathcal{R}_n(\mathcal{H}).$$

(d) Let $\varepsilon_1, \ldots, \varepsilon_n$ be i.i.d. Rademacher random variables, independent of the data. Show that

$$\mathbb{E}\left(\max_{j=1,\ldots,p}\left|\frac{1}{n}\sum_{i=1}^n \varepsilon_i X_{ij}\right|\right) \leqslant \sqrt{2\log(2p)/n}.$$

[Standard properties of sub-Gaussian random variables may be used without proof.]

Suppose now that the hypothesis class $\mathcal{H}$ is formed from single hidden layer neural networks with no bias terms of the form

$$x \mapsto h(x) = \sum_{k=1}^m \alpha_k \psi(\beta_k^\top x).$$

Here ψ is the reLU activation function and $\alpha \in \mathbb{R}^m$ and $\beta_k \in \mathbb{R}^p$ for all k. Suppose moreover that we have the additional constraints $\|\beta_k\|_1 \leqslant 1$ for all k and $\|\alpha\|_1 \leqslant \lambda$ for some $\lambda > 0$.

(e) Show that

$$\mathcal{R}_n(\mathcal{H}) \leqslant \lambda\mathbb{E}\left(\sup_{b:\|b\|_1 \leqslant 1}\left|\frac{1}{n}\sum_{i=1}^n \psi(b^\top X_i)\varepsilon_i\right|\right).$$

(f) Finally show that

$$\mathbb{E}R_\phi(\hat{h}) - R_\phi(h^*) \leqslant \frac{4\lambda\sqrt{2\log(2p)}}{\sqrt{n}}.$$

Paper 4, Section II

30K Mathematics of Machine Learning

Suppose $(x_1, y_1), \ldots, (x_n, y_n) \in \mathbb{R}^p \times \{-1, 1\}$ are i.i.d. input–output pairs. Describe the procedure of *gradient boosting* using a base regression procedure $\hat{h}$ and using the logistic loss, computing any derivatives involved.

Consider now the base regression procedure given by the empirical risk minimiser with squared error loss over the hypothesis class

$$\mathcal{H} := \{x_0 \mapsto \beta \operatorname{sgn}(x_{0j} - \alpha) : j = 1, \ldots, p,\ \alpha, \beta \in \mathbb{R}\}.$$

Explain how an empirical risk minimiser may be computed in $O(np)$ operations. [You may assume for simplicity that for each j, the n numbers $x_{1j}, x_{2j}, \ldots, x_{nj}$ are all distinct, and moreover, that we have already computed the permutations π_j such that $x_{\pi_j(1)j} < x_{\pi_j(2)j} < \cdots < x_{\pi_j(n)j}$, for each j.]

Paper 1, Section II
20G Number Fields

(a) Let $d \neq 0, 1$ be a square-free integer. Compute an integral basis for $K = \mathbb{Q}(\sqrt{d})$ and compute the discriminant D_K.

(b) Let $m \neq 0, \pm 1$ be an odd square-free integer. Let $L = \mathbb{Q}(i, \sqrt{m})$. Show that if $\theta \in \mathcal{O}_L$ then $\theta = \frac{1}{2}(a + bi + c\sqrt{m} + di\sqrt{m})$ where $a, b, c, d \in \mathbb{Z}$ satisfy

$$\begin{aligned} a^2 - b^2 &\equiv m(c^2 - d^2) \pmod 4, \\ 2ab &\equiv 2mcd \pmod 4. \end{aligned}$$

Compute an integral basis for L and compute the discriminant D_L.

[*You should split into cases according to the values of d or m mod* 4.]

Paper 2, Section II
20G Number Fields

Let $K = \mathbb{Q}(\alpha)$ where α is an algebraic integer. Let R be a ring with $\mathbb{Z}[\alpha] \subset R \subset \mathcal{O}_K$. Let $y_1, \ldots, y_n$ be a $\mathbb{Q}$-basis for K with $\mathrm{Tr}_{K/\mathbb{Q}}(\alpha^{i-1} y_j) = \delta_{ij}$ for all $1 \leqslant i, j \leqslant n$.

(a) Show that $\mathcal{O}_K \subset \mathbb{Z}y_1 + \ldots + \mathbb{Z}y_n$. Deduce that R is Noetherian.

(b) Let I be an R-submodule of K. Show that the following are equivalent.

(i) There exists $0 \neq c \in K$ such that $cI \subset R$.

(ii) I is finitely generated as an R-module.

If these conditions hold then we say that I is a *fractional ideal.* It is *invertible* if there exists a fractional ideal J with $IJ = R$.

(c) Show that the following are equivalent.

(i) For every non-zero ideal $I \subset R$ there exists a non-zero ideal $J \subset R$ with IJ principal.

(ii) Every non-zero fractional ideal is invertible.

(d) Suppose that the conditions in part (c) hold. Show that if $\beta \in \mathcal{O}_K$ then $I = R[\beta]$ is a fractional ideal. By considering I^2, or otherwise, show that $R = \mathcal{O}_K$.

(e) Find explicit non-zero ideals I, J_1, J_2 in $R = \mathbb{Z}[\sqrt{5}]$ with $IJ_1 = IJ_2$, yet $J_1 \neq J_2$.

Paper 4, Section II

20G Number Fields

(a) Show that there are no integers x, y, z, t (not all zero) with

$$x^3 + 7y^3 + 49z^3 - 21xyz = 3t^3.$$

Deduce that if $K = \mathbb{Q}(\sqrt[3]{7})$ then there does not exist $\theta \in K$ with $N_{K/\mathbb{Q}}(\theta) = 3$.

(b) State Dedekind's criterion. Assuming that $\mathcal{O}_K = \mathbb{Z}[\sqrt[3]{7}]$, factor 2, 3, 5 and 7 as products of prime ideals in $\mathcal{O}_K$, and compute the class group of K.

Paper 1, Section I

1G Number Theory

Let $N \geqslant 1$ be an integer. Define what it means for a positive integer a to be a *primitive root modulo* N. Let p be an odd prime. Define what it means for an integer a with $a \not\equiv 0 \mod p$ to be a *quadratic residue modulo* p.

Show that when p is an odd prime, no primitive root is a quadratic residue mod p.

Find all primitive roots mod 11. What is the smallest primitive root mod 121?

[*Any results from lectures that you use should be stated clearly.*]

Paper 2, Section I

1G Number Theory

Define the *Möbius function* μ and the *Riemann zeta function* ζ.

Show that

$$\mu(n)^2 = \sum_{d^2 | n} \mu(d).$$

[You may assume, without proof, properties of the Möbius function covered in lectures.]

Deduce that for $s \in \mathbb{C}$ with $\Re(s) > 1$,

$$\sum_{n=1}^{\infty} \frac{\mu(n)^2}{n^s} = \frac{\zeta(s)}{\zeta(2s)}.$$

[*Hint: You may wish to use the fact that*

$$\zeta(s) \cdot \sum_{n=1}^{\infty} \frac{\mu(n)}{n^s} = 1$$

for $s \in \mathbb{C}$ *with* $\Re(s) > 1$.]

Paper 3, Section I

1G Number Theory

Define the *Jacobi symbol* $\left(\frac{a}{n}\right)$ for $a \in \mathbb{Z}$ and $n \geqslant 1$ an odd integer. State and prove the reciprocity law for the Jacobi symbol. [You may assume the reciprocity law for the Legendre symbol.]

Compute $\left(\frac{5}{959}\right)$. Determine whether the congruence $x^2 \equiv 5 \mod 959$ has a solution.

Paper 3, Section II

11G Number Theory

(a) Define the *convergents* $(p_n/q_n)_{n\geqslant 0}$ of the continued fraction expansion $[a_0, a_1, a_2, ...]$.

Show that for all $n \geqslant 1$, $p_n q_{n-1} - p_{n-1} q_n = (-1)^{n+1}$. Show further that if $n \geqslant 1$, $\beta > 0$ and $\alpha = [a_0, a_1, \ldots, a_n, \beta]$, then

$$\alpha = \frac{p_n \beta + p_{n-1}}{q_n \beta + q_{n-1}}$$

and α lies strictly between p_n/q_n and p_{n-1}/q_{n-1}.

(b) Given $\theta \in \mathbb{R} \setminus \mathbb{Q}$, let $[a_0, a_1, a_2, ...]$ be the continued fraction expansion of θ with convergents $(p_n/q_n)_{n\geqslant 0}$. Let $b_n = q_n^2 |\theta - p_n/q_n|$.

By considering $b_n/q_n^2 + b_{n+1}/q_{n+1}^2$, show that for every $n \geqslant 1$,

$$\frac{q_{n+1}}{q_n} = \frac{1 \pm \sqrt{1 - 4b_n b_{n+1}}}{2b_n}.$$

Deduce that

$$2c_n a_{n+1} \leqslant 2b_n a_{n+1} \leqslant \sqrt{1 - 4b_n b_{n+1}} + \sqrt{1 - 4b_{n-1} b_n} \leqslant 2\sqrt{1 - 4c_n^2},$$

where $c_n = \min\{b_{n-1}, b_n, b_{n+1}\}$.

(c) Deduce from part (b) that if $\theta \in \mathbb{R} \setminus \mathbb{Q}$, then there are infinitely many rationals p/q such that

$$\left|\theta - \frac{p}{q}\right| \leqslant \frac{1}{\sqrt{5} q^2}.$$

Paper 4, Section I

1G Number Theory

Define what we mean by a *Carmichael number*.

Show that if N is a Carmichael number and p is an odd prime such that $p \mid N$, then $p - 1 \mid N - 1$.

Show that if N is a Carmichael number, then N is square-free.

Deduce that if N is a Carmichael number and p is an odd prime such that $p \mid N$, then $p \leqslant \sqrt{N}$.

Paper 4, Section II

11G Number Theory

Define what it means for a binary quadratic form to be *positive definite*, and what it means for two binary quadratic forms $f = (a, b, c)$ and $g = (a', b', c')$ to be *equivalent*. Define what it means for a positive definite binary quadratic form to be *reduced*.

Show that every positive definite binary quadratic form is equivalent to a reduced form. Show further that if $f = (a, b, c)$ is a reduced positive definite binary quadratic form of discriminant d, then $|b| \leqslant a \leqslant \sqrt{|d|/3}$ and $b \equiv d \mod 2$.

Find congruence conditions for a prime p to be represented by the form $x^2 + 7y^2$. [You should state carefully any further results from lectures that you require.]

Paper 1, Section II

41B Numerical Analysis

(a) Describe the *power method* for computing the largest eigenvalue (in modulus) of a square matrix.

(b) Let $A \in \mathbb{R}^{n\times n}$ be symmetric. Show that all eigenvectors of A are real and that eigenvectors corresponding to different eigenvalues are orthogonal. Then use Householder reflections to show that there exists a real orthogonal matrix $Q \in \mathbb{R}^{n\times n}$ such that

$$Q^T AQ = \Lambda = \operatorname{diag}(\lambda_1, \ldots, \lambda_n),$$

and that, for $k = 1, \ldots, n$, the kth column of Q is an eigenvector, i.e.

$$A\,Qe_k = \lambda_k\,Qe_k,$$

where e_k is the kth canonical basis vector of $\mathbb{R}^n$.

(c) Suppose $A \in \mathbb{R}^{n\times n}$ is symmetric and that

$$Q^T AQ = \operatorname{diag}(\lambda_1, \ldots, \lambda_n),$$

where $Q = \big[\,q_1 \mid \cdots \mid q_n\,\big]$ is orthogonal and $|\lambda_1| > |\lambda_2| \geqslant \cdots \geqslant |\lambda_n|$. Let the vectors $q^{(k)}$ be the k-th iteration vectors of the power method, for $k = 0, 1, 2, \ldots$, and the scalars $\lambda^{(k)}$ be defined by $\lambda^{(k)} = \big(q^{(k)}\big)^T A\,q^{(k)}$. Define $\theta_k \in [0, \pi/2]$ by

$$\cos\theta_k = \big|q_1^T q^{(k)}\big|.$$

Show that if $\cos\theta_0 \neq 0$, then for $k = 0, 1, \ldots$ we have

$$|\sin\theta_k| \;\leqslant\; \left|\frac{\lambda_2}{\lambda_1}\right|^k \tan\theta_0,$$

$$\big|\lambda^{(k)} - \lambda_1\big| \;\leqslant\; \max_{2\leqslant i\leqslant n} |\lambda_1 - \lambda_i| \left|\frac{\lambda_2}{\lambda_1}\right|^{2k} \tan^2\theta_0.$$

Paper 2, Section II

41B Numerical Analysis

Consider the Chebyshev polynomials $\{T_n\}_{n=0}^{\infty}$ defined on the interval $[-1,1]$, given by the three-term recurrence relation

$$\begin{aligned} T_0(x) = 1, \quad T_1(x) = x, \\ T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x), \quad n \geqslant 1. \end{aligned}$$

In your answers to the following questions clearly state any properties of the Chebyshev polynomials that you assume.

(a) Consider the inner product $\langle f, g\rangle_w := \int_{-1}^{1} f(x)g(x)w(x)dx$ for functions $f, g : [-1,1] \to \mathbb{R}$, and let $\mathcal{H}$ denote the set of functions $f : [-1,1] \to \mathbb{R}$ such that $\|f\|_w := \langle f, f\rangle_w < \infty$, where $w(x) = (1-x^2)^{-1/2}$. Show that the Chebyshev polynomials are orthogonal with respect to this inner product. Define, using the convention specified in lectures, the *Chebyshev coefficients* $\{c_k\}_{k=0}^{\infty}$ of a function $f \in \mathcal{H}$, and determine the Chebyshev coefficients of $f(x) = \sqrt{1-x^2}$.

(b) Prove that Chebyshev series approximate functions in $\mathcal{H}$ with convergence in the $\|\cdot\|_w$ norm. [You may use the fact that Fourier series converge in the L^2-norm for $L^2[-\pi,\pi]$ functions.]

(c) Derive the equality

$$\frac{\pi}{2}\sum_{k=1}^{\infty}|c_k|^2 + \pi|c_0|^2 = \|f\|_w,$$

where the function $f \in \mathcal{H}$ and $\{c_k\}_{k=0}^{\infty}$ are the Chebyshev coefficients of f.

(d) Prove that Chebyshev series approximations to functions in $\mathcal{H} \cap C^{\infty}[-1,1]$ yield spectral convergence, i.e. the N-term approximation converges faster in the $\|\cdot\|_w$ norm than $\mathcal{O}(N^{-p})$ for any $p = 1, 2, \ldots$.

Paper 3, Section II

40B Numerical Analysis

The Poisson equation $\nabla^2 u = f$, where f is sufficiently smooth, in the unit square $\Omega = [0,1] \times [0,1]$, is equipped with zero boundary conditions on $\partial\Omega$. It is discretised with the 9-point formula:

$$\begin{aligned}\Gamma_9[u_{m,n}] := &-\frac{10}{3}u_{m,n} + \frac{2}{3}\big(u_{m+1,n} + u_{m-1,n} + u_{m,n+1} + u_{m,n-1}\big)\\ &+ \frac{1}{6}\big(u_{m+1,n+1} + u_{m+1,n-1} + u_{m-1,n+1} + u_{m-1,n-1}\big) = h^2 f_{m,n},\end{aligned}$$

where $1 \leqslant m,n \leqslant M$, $h = 1/(M+1)$ is the step size, (mh, nh) are grid points, $f_{m,n} = f(mh, nh)$ and $u_{m,n} \approx u(mh, nh)$.

(a) Find the local error of approximation of the 9-point formula.

(b) Let $\boldsymbol{e} \in \mathbb{R}^{M\times M}$ be the error, that is $e_{m,n} = u_{m,n} - u(mh, nh)$. Determine the global error $\|\boldsymbol{e}\|_2$ as a function of h. [You may assume that the 9-point formula induces a matrix A_h (similar to the 5-point formula discussed in lectures) for which $\|A_h^{-1}\|_2 \leqslant C/h^2$, for some constant $C > 0$, independent of the ordering of the grid.]

(c) Determine the local and global error of the 9-point formula if f satisfies the Laplace equation $\nabla^2 f = 0$.

(d) Show that the modified 9-point scheme

$$\begin{aligned}\Gamma_9[u_{m,n}] &= h^2 f_{m,n} + \frac{1}{12}h^2\Gamma_5[f_{m,n}]\\ &:= h^2 f_{m,n} + \frac{1}{12}h^2\big(f_{m+1,n} + f_{m-1,n} + f_{m,n+1} + f_{m,n-1} - 4f_{m,n}\big)\end{aligned}$$

has the same local error as that obtained in part (c).

Paper 4, Section II

40B Numerical Analysis

A nonnegative matrix is a matrix whose entries are nonnegative ($\geqslant 0$). Let A, M, N be three real $n \times n$ matrices satisfying $A = M - N$. The pair of matrices M, N is a *regular splitting* of A, if M is nonsingular and M^{-1} and N are nonnegative.

With a regular splitting, we associate the iteration

$$x_{k+1} = M^{-1}Nx_k + M^{-1}b, \qquad x_0, b \in \mathbb{R}^n, \quad k = 0, 1, 2, \ldots. \tag{$\dagger$}$$

The questions below address the conditions for such an iteration to converge for all x_0.

Each result must be proved directly rather than quoting from lectures, except that you may assume the existence of Jordan block structure.

(a) Show that the sequence A^k, $k = 0, 1, \ldots$, converges to zero if and only if $\rho(A) < 1$, where $\rho(A)$ denotes the spectral radius of A.

(b) Show that the series $\sum_{k=0}^{\infty} A^k$ converges if and only if $\rho(A) < 1$, and that, under this condition, $I - A$ is nonsingular and the limit of the series is equal to $(I - A)^{-1}$.

(c) Let $B \in \mathbb{R}^{n \times n}$ be a nonnegative matrix. Show that if $\rho(B) < 1$, then $(I - B)^{-1}$ is nonnegative.

(d) Let M, N be a regular splitting of a matrix A. Then, show that $\rho(M^{-1}N) < 1$ if and only if A^{-1} exists and is non-negative.

Deduce that if $Ax^* = b$ then ($\dagger$) converges to x^* for all x_0 if and only if A^{-1} exists and is non-negative.

[You may use the fact that if G is a nonnegative matrix, there is a nonnegative eigenvector x with eigenvalue $\rho(G)$.]

Paper 1, Section II

34A Principles of Quantum Mechanics

(a) Define the *uncertainty* of a Hermitian operator Q in the normalised state $|\psi\rangle$. Then state the *generalised uncertainty principle* for two Hermitian operators A and B.

(b) A one-dimensional quantum harmonic oscillator with Hamiltonian

$$H = \frac{1}{2}(P^2 + \omega^2 X^2)$$

is in the coherent state $|\alpha\rangle = e^{\alpha A^\dagger - \bar{\alpha} A}|0\rangle$, where α is a complex number, $\bar{\alpha}$ its complex conjugate, and A is the annihilation operator. Compute the norm of $|\alpha\rangle$ and of $A|\alpha\rangle$.

(c) Compute the expectation value and uncertainty of the position X and the Hamiltonian H in the state $|\alpha\rangle$.

(d) When is the generalised uncertainty principle for these two operators saturated?

(e) Compute the time evolution $|\alpha(t)\rangle$ given that $|\alpha(0)\rangle = |\alpha\rangle$. When is the generalised uncertainty principle for X and H in the state $|\alpha(t)\rangle$ saturated?

Paper 2, Section II

35A Principles of Quantum Mechanics

Consider two qubits A and B and the following Bell states, written in the basis $\{|+\rangle, |-\rangle\}$ for each qubit:

$$\begin{aligned}
|\mathcal{B}_1\rangle &= \frac{1}{\sqrt{2}}|+\rangle_A|+\rangle_B + \frac{1}{\sqrt{2}}|-\rangle_A|-\rangle_B, \\
|\mathcal{B}_2\rangle &= \frac{1}{\sqrt{2}}|+\rangle_A|+\rangle_B - \frac{1}{\sqrt{2}}|-\rangle_A|-\rangle_B, \\
|\mathcal{B}_3\rangle &= \frac{1}{\sqrt{2}}|+\rangle_A|-\rangle_B + \frac{1}{\sqrt{2}}|-\rangle_A|+\rangle_B, \\
|\mathcal{B}_4\rangle &= \frac{1}{\sqrt{2}}|+\rangle_A|-\rangle_B - \frac{1}{\sqrt{2}}|-\rangle_A|+\rangle_B.
\end{aligned}$$

Here $|+\rangle$ and $|-\rangle$ are eigenstates of the spin σ_z with eigenvalues $+1$ and -1 respectively.

(a) Compute $\langle\mathcal{B}_1|\mathcal{B}_i\rangle$ for $i = 1, 2, 3, 4$.

(b) Compute the reduced density matrix $\rho_A = \mathrm{Tr}_B(|\mathcal{B}_1\rangle\langle\mathcal{B}_1|)$ of the qubit A in the state $|\mathcal{B}_1\rangle$. Compute the von Neumann entropy of ρ_A and determine whether it is pure or mixed.

(c) Can $|\mathcal{B}_1\rangle$ be written as a product state $|\psi\rangle_A \otimes |\chi\rangle_B$? Briefly justify.

(d) Compute $\langle\mathcal{B}_i|\, \sigma_z^{(A)}\sigma_z^{(B)}\, |\mathcal{B}_i\rangle$ for $i = 1, \ldots, 4$. Discuss the implication of this result for measurements of the spin of the two qubits.

(e) Show that the Bell states are eigenstates of

$$T^{AB} := \sigma_x^{(A)}\sigma_x^{(B)} + \sigma_y^{(A)}\sigma_y^{(B)} + \sigma_z^{(A)}\sigma_z^{(B)},$$

and find the corresponding eigenvalues. Here σ_x, σ_y and σ_z are Pauli matrices and the superscript indicates which qubit they act on.

(f) Consider local unitary transformations of the form $U = U_A \otimes U_B$, with $U_{A,B}$ acting on qubit A or B. Show that all four Bell states are locally equivalent, that is, for any i, j there exist U_A, U_B such that

$$|\mathcal{B}_j\rangle = (U_A \otimes U_B)\, |\mathcal{B}_i\rangle.$$

[*Hint: Consider* $U = 1_A \otimes \sigma_i^{(B)}$ *acting on* $|\mathcal{B}_1\rangle$*.*]

Paper 3, Section II

33A Principles of Quantum Mechanics

A quantum system has Hamiltonian $H(t) = H_0 + \lambda V(t)$, where H_0 is time-independent with discrete, non-degenerate spectrum $H_0 |n\rangle = E_n |n\rangle$, and $|\lambda| \ll 1$. The system is prepared at $t = 0$ in a state that is a linear superposition of two normalised eigenstates $|i_1\rangle \neq |i_2\rangle$ of H_0,

$$|\psi(0)\rangle = \alpha_1 \; |i_1\rangle + \alpha_2 \; |i_2\rangle \, , \qquad |\alpha_1|^2 + |\alpha_2|^2 = 1, \qquad H_0 \, |i_k\rangle = E_{i_k} \, |i_k\rangle \;\; (k = 1, 2).$$

Consider the perturbation

$$V(t) = W \, e^{-\gamma t} \cos(\omega t) \, \Theta(t) \, ,$$

where W is a time-independent Hermitian operator, ω and γ are positive constants, and Θ is the Heaviside function. For a fixed final eigenstate $|f\rangle$ of H_0 (with $f \neq i_1, i_2$), you are given the matrix elements

$$W_k \equiv \langle f| \, W \, |i_k\rangle \qquad (k = 1, 2) \, .$$

(a) Working in the interaction picture, compute the first-order transition amplitude $a_f^{(1)}(t)$ for $|\psi\rangle \to |f\rangle$.

(b) Take the limit $t \to \infty$ and express $a_f^{(1)}(\infty)$ in terms of α_k, W_k and the complex response factor

$$F_k \equiv \frac{1}{2} \left[\frac{1}{\gamma - i(\omega_{fi_k} + b_1 \omega)} + \frac{b_2}{\gamma - i(\omega_{fi_k} + b_3 \omega)} \right]$$

where b_1, b_2, b_3 and ω_{fi_k} are coefficients you should determine.

(c) Hence compute the first-order transition probability $P_{\psi \to f}(\infty)$ and briefly discuss the result.

(d) Now consider the more general initial state $|\phi_0\rangle = \sum_k \alpha_k \, |i_k\rangle$ for $k = 1, \ldots, N$ with $\sum_k |\alpha_k|^2 = 1$. Assume that W_k and F_k can be approximated by k-independent constants $\overline{W}$ and $\overline{F}$. For fixed λ, $\overline{F}$ and $\overline{W}$, determine the largest value of $P_{\phi \to f}(\infty)$ as function of N that can be obtained by optimally choosing α_k.

Paper 4, Section II

33A Principles of Quantum Mechanics

Consider a particle of mass m and spin $s = 1/2$ moving in a spherically symmetric potential $V(|\mathbf{X}|)$ with Hamiltonian

$$H_0 = \frac{\mathbf{P}^2}{2m} + V(|\mathbf{X}|)\,.$$

(a) Let $\mathbf{J}$, $\mathbf{L}$, and $\mathbf{S}$ be respectively the total angular momentum, the orbital angular momentum and the spin operators.

(i) State all the commutation relations satisfied by $\mathbf{L}$, $\mathbf{S}$ and $\mathbf{J}$. Briefly explain why we can choose a basis $\{|n, \ell, s, m, \sigma\rangle\}$ for the Hilbert space, labelled by the quantum numbers of the operators $(H_0, L^2, S^2, L_z, S_z)$ respectively.

(ii) Show that the energy E_n of $|n, \ell, s, m, \sigma\rangle$ is independent of m and σ. Hence compute the degeneracy of the energy eigenstates for a given fixed ℓ and $s = 1/2$.

(iii) Briefly explain why we may also choose a basis $\{|n, \ell, s, j, j_z\rangle\}$ for the Hilbert space, labelled by the quantum numbers of the operators $(H_0, L^2, S^2, J^2, J_z)$ respectively.

(iv) For fixed $\ell \geqslant 1$ and $s = 1/2$, list the allowed values of j. You may use that addition of spin and orbital angular momentum works in the same way as addition of angular momentum of two subsystems. Hence, in the basis $\{|n, \ell, s, j, j_z\rangle\}$ compute the degeneracy of the energy eigenstates for a given fixed ℓ and $s = 1/2$ and compare it to your previous result. [Note that in this question the eigenvalue of $J_z/\hbar$ is called j_z to distinguish it from that of $L_z/\hbar$, which is called m.]

(b) Now the same particle evolves according to the new Hamiltonian

$$H = H_0 + H_{\mathrm{SO}}, \qquad H_{\mathrm{SO}} = \lambda\, \mathbf{L} \cdot \mathbf{S},$$

where $\lambda > 0$ is a real constant and H_{SO} is the spin-orbit interaction.

(i) By recalling the relation among $\mathbf{J}$, $\mathbf{L}$, and $\mathbf{S}$ or otherwise, show that $\mathbf{L} \cdot \mathbf{S}$ can be written in terms of J^2, L^2 and S^2. Use this fact to briefly explain why one of the two bases considered in part (a) will be more convenient for studying the spectrum of the new Hamiltonian.

(ii) In the basis $\{|n, \ell, s, j, j_z\rangle\}$ compute the energy eigenvalues of H and their degeneracy for all states with fixed n, for $\ell \geqslant 1$ and $s = 1/2$. Discuss how the spin-orbit coupling lifts part of the degeneracy encountered in part (a). [The quantum number n still labels the eigenvalues E_n of H_0.]

(iii) The eigenvalues of $\lambda\, \mathbf{L} \cdot \mathbf{S}$ are the spin-orbit energy shifts. Show that the degeneracy-weighted average of these eigenvalues vanishes.

Paper 1, Section II

29L Principles of Statistics

Let $f(x;\theta)$ be a regular statistical model for a real-valued parameter θ.

(a) Explain what is meant by the *maximum likelihood estimator* $\hat{\theta}_n$ based on a sample $X_1,\ldots,X_n$.

(b) State the standard asymptotic result on the distribution of the maximum likelihood estimator in terms of the Fisher information $I(\theta)$. [You are not expected to give precise conditions for the validity of this result.]

Consider the model defined by the density

$$f(x;\theta) = e^{-(x-\theta)}\mathbf{1}_{\{x\geqslant\theta\}}, \quad x\in\mathbb{R}.$$

(c) Find the maximum likelihood estimator $\hat{\theta}_n$ in this case and explain briefly why the standard asymptotics for the distribution of the maximum likelihood estimator may fail to apply.

(d) Show that there exists a constant $\alpha>0$ such that the sequence of random variables $n^{\alpha}(\hat{\theta}_n-\theta)$ converges in distribution, with limit distribution of non-zero variance, where α and the limiting distribution are to be determined.

(e) Consider the alternative estimator $\tilde{\theta}_n = \bar{X}_n - 1$, where $\bar{X}_n$ is the sample mean. Compare the efficiency of $\hat{\theta}_n$ relative to $\tilde{\theta}_n$ by calculating the limit of the ratio of their mean squared errors as $n\to\infty$.

Paper 2, Section II

29L Principles of Statistics

Let $X_1, \ldots, X_n$ be an independent random sample from the uniform distribution on $[0, \theta]$, where $\theta > 0$ is an unknown parameter. Throughout this question, we consider the scale-invariant loss function

$$L(\delta, \theta) = \theta^{-2}(\delta - \theta)^2.$$

(a) Find the maximum likelihood estimator $\hat{\theta}_n$ and calculate its risk. [*Hint: The first and second moments of a Beta*$(n, 1)$ *random variable* Y *are given by*

$$\mathbb{E}(Y) = \frac{n}{n+1}, \quad \mathbb{E}(Y^2) = \frac{n}{n+2}. \quad]$$

(b) Is the maximum likelihood estimator admissible? Justify your answer. [*Hint: Consider estimators of the form* $c\hat{\theta}_n$.]

(c) State the meaning of the following notions in decision theory:

(i) the *Bayes risk* $r(\pi, \delta)$ of an estimator δ with respect to a prior π,

(ii) a *Bayes estimator.*

For $\alpha, \ell \in (0, \infty)$, the Pareto$(\alpha, \ell)$ density $\pi_{\alpha,\ell}$ is given by

$$\pi_{\alpha,\ell}(\theta) = \frac{\alpha \ell^\alpha}{\theta^{\alpha+1}} \mathbf{1}_{\{\theta \geqslant \ell\}}.$$

(d) Suppose we take $\pi_{\alpha,\ell}$ as prior for θ. Find the Bayes estimator $\hat{\theta}_{\alpha,\ell}$ for the scale-invariant loss function $L(\delta, \theta)$. [*Hint: The negative moments of a Pareto*(α, ℓ) *random variable* T *are given by*

$$\mathbb{E}(T^{-k}) = \frac{\alpha \ell^{-k}}{\alpha + k}, \quad k \geqslant 1. \quad]$$

Paper 3, Section II

28L Principles of Statistics

Let $X_1, \ldots, X_n$ be an independent random sample from an unknown distribution. It is desired to estimate a scalar parameter $\theta = g(\mathbb{E}(X_1))$. Denote the sample mean by $\bar{X}_n$ and set

$$\hat{\theta}_n = g(\bar{X}_n), \quad G(t) = \mathbb{P}(\sqrt{n}(\hat{\theta}_n - \theta) \leqslant t).$$

Assume here that G is strictly increasing and continuous and does not depend on n or θ.

(a) In the case where G is a known function, describe the construction of a $(1-\alpha)$-confidence interval C_n for θ.

(b) Explain what is meant by a *bootstrap sample* $X_1^*, \ldots, X_n^*$ constructed from $X_1, \ldots, X_n$.

Denote the sample mean of the bootstrap sample by $\bar{X}_n^*$ and set

$$\hat{\theta}_n^* = g(\bar{X}_n^*), \quad \hat{G}_n(t) = \mathbb{P}\left(\sqrt{n}(\hat{\theta}_n^* - \hat{\theta}_n) \leqslant t \,\middle|\, X_1, \ldots, X_n\right).$$

Assume that $\sup_{t \in \mathbb{R}} |\hat{G}_n(t) - G(t)| \to 0$ in probability.

(c) Describe the construction of an approximate $(1-\alpha)$-confidence interval $\hat{C}_n$ for θ using $\hat{G}_n$ and show that indeed

$$\mathbb{P}(\theta \in \hat{C}_n) \to 1 - \alpha.$$

(d) Why might you want to simulate a number of independent bootstrap samples $X_1^{*j}, \ldots, X_n^{*j}$ for $j = 1, \ldots, B$ in order to implement a similar procedure in practice? Explain how you would use the B samples. [You are not expected to justify convergence of the procedure you propose.]

Paper 4, Section II

28L Principles of Statistics

Let $\{f(x;\theta) : \theta \in \mathbb{R}\}$ be a regular statistical model and let X have density $f(.;\theta)$. Let $\delta = \delta(X)$ be an estimator for θ with finite variance and having a differentiable bias function

$$b(\theta) = \mathbb{E}_\theta[\delta] - \theta.$$

(a) Define the *Fisher information* $I(\theta)$ for the model and state the Cramér–Rao lower bound for the variance of δ. [You are not expected to set out regularity conditions for its validity.]

(b) State what is meant by the *quadratic risk* $R(\delta,\theta)$.

(c) Suppose that quadratic risk satisfies $R(\delta,\theta) \leqslant \gamma I(\theta)^{-1}$ for some constant $0 < \gamma < 1$. Use the Cramér–Rao lower bound to show that the derivative of the bias must satisfy

$$b'(\theta) \leqslant -\left(1 - \sqrt{\gamma - I(\theta)b(\theta)^2}\right).$$

Consider from now on the model $X \sim N(\theta, 1)$. The maximum likelihood estimator $\delta_*(X) = X$ is known to have a constant risk 1. We consider a "super-efficient" estimator $\delta(X)$ that achieves a risk $R(\delta,\theta) \leqslant \gamma < 1$ for all θ in some interval $[\theta_1, \theta_2]$.

(d) Using the result from part (c), show that the change in bias over this interval satisfies

$$b(\theta_2) - b(\theta_1) \leqslant -(1 - \sqrt{\gamma})(\theta_2 - \theta_1).$$

Deduce that the length of the interval $L = \theta_2 - \theta_1$ satisfies

$$L \leqslant \frac{2\sqrt{\gamma}}{1 - \sqrt{\gamma}}.$$

(e) Consider, for the same model, the linear shrinkage estimator $\delta_c(X) = cX$ for a fixed constant $0 \leqslant c < 1$. Fix $\gamma \in (c^2, 1)$ and set

$$I_\gamma = \{\theta : R(\delta_c, \theta) \leqslant \gamma\}.$$

Show that I_γ is an interval of length of L_γ, where

$$L_\gamma = \frac{2\sqrt{\gamma - c^2}}{1 - c}.$$

Verify that this length satisfies the bound derived in part (d).

Paper 1, Section II

27L Probability and Measure

(a) State and prove the first and second Borel–Cantelli lemmas.

Let $(X_n)_{n\geqslant 1}$ be a sequence of independent identically distributed real random variables with density p and characteristic function Φ given by

$$p(x) = \frac{1}{\pi(1+x^2)}, \quad \Phi(\xi) = e^{-|\xi|}.$$

Set $S_n = X_1 + \cdots + X_n$.

(b) Show that the sequence of random variables S_n/n converges in law.

(c) By considering the convergence of the series $\sum_{n\geqslant 1} \mathbb{P}(X_n/n > 1)$ or otherwise, show that almost surely S_n/n does not converge to a finite limit.

Paper 2, Section II

27L Probability and Measure

Let E be a set and let $\mathcal{A}$ be a σ-algebra on E.

(a) What does it mean to say that μ is a *measure* on $(E, \mathcal{A})$?

Let λ denote Lebesgue measure on $\mathbb{R}^d$.

(b) Show that λ has the following dilatation property: for all $r > 0$ and all $A \in \mathcal{B}(\mathbb{R}^d)$, for $rA = \{rx : x \in A\}$, we have

$$\lambda(rA) = r^d \lambda(A).$$

[*Hint: You may find it helpful to consider a characterization of λ in terms of its values on sets of a simple form.*]

Given a Borel measurable function $f : \mathbb{R}^d \to [0, \infty)$, consider for $k \in \mathbb{Z}$ the set

$$E_k = \{x \in \mathbb{R}^d : 2^k < f(x) \leqslant 2^{k+1}\}$$

and let

$$F_n(x) = \sum_{k=-n}^{n} 2^k \mathbf{1}_{E_k}(x).$$

(c) Show that $F_n(x)$ converges for all x, with limit $F(x)$ satisfying

$$F(x) \leqslant f(x) \leqslant 2F(x).$$

(d) Show that f is Lebesgue integrable if and only if the following condition holds

$$\sum_{k=-\infty}^{\infty} 2^k \lambda(E_k) < \infty.$$

(e) Use part (d) to determine for which $b \in (0, \infty)$ the following function f on $\mathbb{R}^d$ is Lebesgue integrable

$$f(x) = \mathbf{1}_{\{|x|<1\}} |x|^{-b}.$$

Paper 3, Section II

26L Probability and Measure

(a) State Fatou's lemma and deduce Lebesgue's dominated convergence theorem.

Let $(X_n)_{n\geqslant 1}$ be a sequence of real-valued random variables.

(b) Show that $X_n \to X$ almost surely implies $X_n \to X$ in probability.

For the rest of the question, assume that $X_n \to 0$ almost surely and $\mathbb{E}[X_n^2] \leqslant 1$ for all n.

(c) Does the sequence $(X_n)_{n\geqslant 1}$ converge in L^1? Justify your answer.

(d) Give an example to show that $(X_n)_{n\geqslant 1}$ does not necessarily satisfy the assumptions of the dominated convergence theorem.

Paper 4, Section II

26L Probability and Measure

(a) What does it mean to say that a sequence of real-valued random variables $(X_n : n \geqslant 1)$ *converges in law*?

(b) State a characterization of convergence in law in terms of the sequence of characteristic functions $(\Phi_{X_n} : n \geqslant 1)$.

Let $(X_n : n \geqslant 1)$ be a sequence of independent random variables such that

$$\mathbb{P}(X_n = 1) = \mathbb{P}(X_n = -1) = 1/2, \quad n \geqslant 1.$$

Set

$$S_n = \sum_{j=1}^{n} \frac{X_j}{2^j}, \quad T_n = \sum_{j=1}^{n} \frac{X_j + 1}{3^j}.$$

(c) Show that $S_n \to S$ almost surely for some random variable S and determine the law of S.

[You may use the following identity

$$\sin\left(\frac{\xi}{2^n}\right) \prod_{j=1}^{n} \cos\left(\frac{\xi}{2^j}\right) = \frac{\sin \xi}{2^n}. \quad]$$

(d) Show that $T_n \to T$ almost surely for some random variable T and that T is neither a discrete random variable nor does it have a density with respect to Lebesgue measure.

[*Hint: You may find it helpful to work out the distributions of* T_1, T_2, T_3 *explicitly.*]

Paper 1, Section I

10E Quantum Information and Computation

(a) State the no-signalling principle for quantum states of a bipartite system AB.

Given the orthonormal basis $\mathcal{B}(\theta) = \{|\mu(\theta)\rangle, |\nu(\theta)\rangle\}$, with

$$|\mu(\theta)\rangle = \cos\theta\,|0\rangle + \sin\theta\,|1\rangle \quad \text{and} \quad |\nu(\theta)\rangle = \sin\theta\,|0\rangle - \cos\theta\,|1\rangle\,,$$

and the 2-qubit state $|\phi^+\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$, show that

$$\frac{1}{\sqrt{2}}\Big(\,|\mu(\theta)\rangle\,|\mu(\theta)\rangle + |\nu(\theta)\rangle\,|\nu(\theta)\rangle\,\Big) = \big|\phi^+\big\rangle \quad \text{for all } \theta.$$

Alice and Bob share the state $|\phi^+\rangle_{AB}$. Alice measures her qubit in the basis $\mathcal{B}(\theta)$, and then Bob measures his qubit in the computational basis. Compute Bob's measurement outcome probabilities in terms of θ and state how your result relates to the no-signalling principle.

(b) State the Helstrom–Holevo theorem on distinguishing two quantum states, $|\alpha_0\rangle$ and $|\alpha_1\rangle$.

Suppose that for any states $|\alpha_0\rangle$ and $|\alpha_1\rangle$ with $|\langle\alpha_0|\alpha_1\rangle| = \cos\theta$, there is a quantum measurement process that will correctly identify the state with probability $p(\theta) = \frac{1}{2}(1+\sin\theta)$. Show that if Alice could clone states from the set $\{|0\rangle, |+\rangle\}$, then the Helstrom–Holevo theorem could be violated.

Paper 2, Section I

10E Quantum Information and Computation

Consider a quantum database whose $N = 2^n$ items are labelled by the computational basis states $\{|x\rangle : x \in \{0,1\}^n\}$. Let $|\psi_0\rangle$ denotes the uniform superposition state.

(a) If the database has a single marked item, say $|x_0\rangle$, then Grover's search algorithm uses the Grover iteration operator $Q := -I_{|\psi_0\rangle} I_{|x_0\rangle}$. For any $|\psi\rangle \in (\mathbb{C}^2)^{\otimes n}$, give the definition of the reflection operator $I_{|\psi\rangle}$, and specify the action of $I_{|x_0\rangle}$ on any computational basis state.

(b) Next, suppose the database has $1 < M < N$ marked items, which form the set

$$G = \{|g_1\rangle, |g_2\rangle, \ldots, |g_M\rangle \ : \ g_i \in \{0,1\}^n \text{ for } i = 1, 2, \ldots, M\}.$$

The Grover iteration operator used to search for a marked item is now $Q_M := -I_{|\psi_0\rangle} I_G$, for a suitable reflection operator I_G.

(i) Write the reflection operator I_G in terms of the states in G.

(ii) Show that Q_M preserves the two-dimensional subspace $\mathcal{S}$ spanned by $|\psi_0\rangle$ and the state $|\psi_G\rangle$, the equal superposition of all marked items.

(iii) Construct an orthonormal basis for $\mathcal{S}$ and express Q_M as a 2×2 matrix in this basis.

(iv) Show that Q_M can be interpreted as a rotation in the subspace $\mathcal{S}$. Defining the angle α through $\sin^2 \alpha := M/N$, determine the angle that a vector in $\mathcal{S}$ is rotated after one application of Q_M.

(v) Assuming that $M \ll N$ and starting from $|\psi_0\rangle$, find the number of repeated applications of Q_M needed to find a marked item with high probability.

Paper 2, Section II

15E Quantum Information and Computation

This question concerns quantum key distribution using two different protocols, both in the absence and presence of noise, as described below.

(a) As in the standard BB84 quantum key distribution protocol, Alice and Bob are spatially separated and can communicate through both classical and quantum channels. An eavesdropper, Eve, performs an intercept–resend attack by measuring each qubit (sent by Alice) in the *Breidbart basis* $\{|b_0\rangle, |b_1\rangle\}$, where

$$|b_0\rangle = \cos\left(\frac{\pi}{8}\right)|0\rangle + \sin\left(\frac{\pi}{8}\right)|1\rangle, \quad |b_1\rangle = \sin\left(\frac{\pi}{8}\right)|0\rangle - \cos\left(\frac{\pi}{8}\right)|1\rangle.$$

and subsequently sending the post-measurement state on to Bob. Assume there is no noise in transmission.

Briefly describe the procedure Alice and Bob take to generate a private encryption key and to test whether eavesdropping has occurred.

Given that Bob chooses his basis randomly, calculate the probability (for a single sent qubit) that Alice and Bob can detect the eavesdropping. [*Hint: First determine the probability that Bob gets a different outcome from Alice's bit when he measures in the same basis as Alice.*]

(b) Now consider a six-state quantum key distribution protocol which is a generalisation of BB84 in the following way. Alice and Bob use the following *three* bases to respectively send and measure a qubit: the computational basis, the Hadamard basis $\{|+\rangle, |-\rangle\}$, and the circular basis $\{|R\rangle, |L\rangle\}$, where $|R\rangle = \frac{1}{\sqrt{2}}(|0\rangle + i|1\rangle)$, and $|L\rangle = \frac{1}{\sqrt{2}}(|0\rangle - i|1\rangle)$. Alice and Bob choose their bases uniformly at random for each qubit. Eavesdropper Eve performs an intercept–resend attack, but she can *only* measure in the computational, Hadamard, and circular bases. [You may use without proof that these bases are *mutually unbiased*: that is, for any pair of distinct bases $\{|e_0\rangle, |e_1\rangle\}$ and $\{|e_0'\rangle, |e_1'\rangle\}$, one has $|\langle e_i|e_j'\rangle|^2 = \frac{1}{2}$ for $i, j \in \{0, 1\}$. Again assume that there is no noise in transmission.]

If Bob measures in the same basis that Alice transmits, calculate the probability that Bob gets a measurement outcome that is different from Alice's encoded bit, and hence determine the probability that, with a single transmitted qubit, Alice and Bob can detect Eve's attack.

(c) Now consider the possibility that on the way from Alice to Bob, transmitted qubits are affected by noise which acts in the following way. Let $p \in [0, 1]$. With probability $p/4$ an X gate gets applied to the qubit, with probability $p/4$ a Y gate gets applied to the qubit, with probability $p/4$ a Z gate gets applied to the qubit, and with probability $1 - 3p/4$ the qubit gets sent through unchanged.

Suppose that Alice transmits state $|\psi\rangle$ and that Bob measures in the computational basis. Assuming that there is no eavesdropper (no Eve), what is the probability that Bob's measurement outcome is 0? [You may use without proof that, for any pair of qubit states $|\phi\rangle$ and $|\psi\rangle$, $|\langle\phi|X\psi\rangle|^2 + |\langle\phi|Y\psi\rangle|^2 + |\langle\phi|Z\psi\rangle|^2 = 2 - |\langle\phi|\psi\rangle|^2$.]

Now consider parts (a) and (b) in the presence of noise. In each case, determine the range of p for which Alice and Bob can detect Eve's eavesdropping. State any additional assumptions you must make.

Paper 3, Section I

10E Quantum Information and Computation

Let $f : \{0,1\}^n \to \{0,1\}$ be a Boolean function. We say that f is *k-balanced* if $f(x) = 1$ for exactly k inputs and $f(x) = 0$ for $2^n - k$ inputs. Thus f is constant if $k = 0$ or $k = 2^n$, and f is perfectly balanced if $k = 2^{n-1}$. In the following, assume that we are given a quantum oracle U_f for the function f.

Starting with initial state $|0\rangle^{\otimes n}|-\rangle$ and applying the standard Deutsch–Jozsa algorithm, express the probability of obtaining the n-bit string $000\ldots0$ in the final measurement, in terms of k and n. Explain qualitatively how this probability behaves in the following cases

(i) f is k-balanced with $k \notin \{0, 2^{n-1}, 2^n\}$,

(ii) f is constant,

(iii) f is perfectly balanced.

Discuss why the Deutsch–Jozsa algorithm no longer perfectly distinguishes between the two cases, constant or k-balanced for $k \notin \{0, 2^{n-1}, 2^n\}$. Given $k \notin \{0, 2^{n-1}, 2^n\}$ and $\varepsilon \in (0,1)$, state an algorithm that distinguishes between the two cases in such a way that the following two conditions hold.

- If f is constant, the algorithm always correctly identifies f as constant.
- If f is k-balanced, the algorithm correctly identifies f as k-balanced with probability at least $1 - \varepsilon$.

For this algorithm, state how the number of uses of the oracle U_f depends on ε and k.

Paper 3, Section II

15E Quantum Information and Computation

Alice (A), Bob (B), and Charlie (C) are separated in space. They can perform quantum operations on systems held locally in their possession, and they can communicate classically. They are unable to directly transmit quantum systems between each other. Each also has a supply of local qubit ancillas all in state $|0\rangle$. This scenario is referred to as 'local operations and classical communication,' or LOCC.

(a) Consider the 3-qubit state

$$|\varphi\rangle = \alpha\,|000\rangle + \beta\,|111\rangle + \gamma\,|100\rangle\,, \text{ with } |\alpha|^2 + |\beta|^2 + |\gamma|^2 = 1\,.$$

Show that $|\varphi\rangle$ is entangled (that is, not a product state of three qubits) for all $\alpha, \beta \neq 0$ and any γ.

(b) Suppose that Alice, Bob, and Charlie share the state

$$|\Omega\rangle_{ABC} = \frac{1}{\sqrt{2}}\Big(\,|000\rangle_{ABC} + |111\rangle_{ABC}\Big)\,.$$

Alice also possesses a 1-qubit state $|\xi\rangle$ and wishes to transfer it to Bob.

(i) Show how this may be achieved by a suitable LOCC protocol, stating clearly the sequence of individual, local operations and classical communications used. [*Hint: It may be useful to consider Charlie first performing a suitable measurement.*]

(ii) Can the transfer be done with an LOCC protocol that does not involve any classical communication between parties A, B, and C? Give a reason for your answer.

(c) Now instead suppose that Alice and Bob possess two copies of the state

$$|\psi\rangle_{AB} = \alpha\,|00\rangle_{AB} + \beta\,|11\rangle_{AB}\,, \text{ with } \alpha, \beta \neq 0\,;$$

that is, they have the 4-qubit state $|\psi\rangle_{A_1B_1} \otimes |\psi\rangle_{A_2B_2}$. They wish to obtain the state

$$|\phi^+\rangle_{AB} = \frac{1}{\sqrt{2}}\Big(\,|00\rangle_{AB} + |11\rangle_{AB}\Big)\,.$$

Due to soaring energy costs, for this task Alice's and Bob's local quantum capability is restricted to performing only CX and Pauli gates, and to making (possibly incomplete) measurements in the computational basis. Show how they may obtain the state $|\phi^+\rangle_{AB}$ with a nonzero probability of success by a LOCC protocol, and give the success probability of your protocol. [*Hint: It may be useful to consider the expression for the 4-qubit state* $|\psi\rangle_{A_1B_1} \otimes |\psi\rangle_{A_2B_2}$ *with qubit labels reordered as* A_1, A_2, B_1, B_2.]

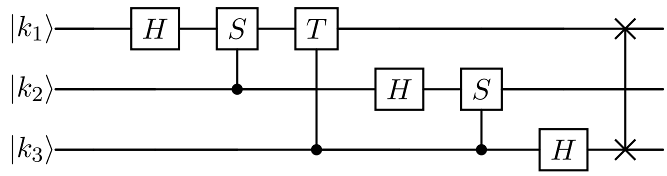
(d) Alice and Bob now possess one copy of $|\phi^+\rangle_{AB}$ and wish to convert it back to $|\psi\rangle_{AB}$, with these states defined as in part (c). Describe an LOCC protocol, now with general local quantum actions allowed, that will achieve this task with certainty.

Paper 4, Section I

10E Quantum Information and Computation

(a) Let QFT_N denote the quantum Fourier transform over $\mathbb{Z}_N = \{0, 1, \ldots, N-1\}$. How does QFT_N act on a state $|k\rangle$, where $k \in \{0, \ldots, N-1\}$?

(b) Consider the following 3-qubit circuit



where

$$H = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}, \qquad S = \begin{bmatrix} 1 & 0 \\ 0 & e^{i\pi/2} \end{bmatrix}, \qquad T = \begin{bmatrix} 1 & 0 \\ 0 & e^{i\pi/4} \end{bmatrix},$$

and $\times\!\!-\!\!\!-\!\!\times$ in the final step of the circuit denotes the SWAP gate, defined through $\mathrm{SWAP}\,|a\rangle\,|b\rangle = |b\rangle\,|a\rangle$ for all single-qubit states $|a\rangle$ and $|b\rangle$. Evaluate the actions of the circuit on the following states (i) $|000\rangle$ and (ii) $|011\rangle$, and show that these are identical to the actions of QFT_8 on the states $\{|k\rangle\}$, where $|k\rangle = |k_1k_2k_3\rangle$.

(c) Let $|\phi\rangle = \frac{1}{\sqrt{A}}\sum_{j=0}^{A-1}|x_0 + jr\rangle$ be a 3-qubit state, with $x_0 \in \mathbb{Z}_8$ and $A = 8/r$ an integer for some $r \in \mathbb{Z}_8$. Show that the amplitudes of $\mathrm{QFT}_8\,|\phi\rangle$ are concentrated on multiples of A. If one measures this state in the basis $\{|0\rangle, |1\rangle, \ldots, |7\rangle\}$, do the measurement outcomes and their probabilities depend on x_0? Justify your answer.

Paper 1, Section II

19F Representation Theory

(a) Compute the character table of the alternating group A_5.

[You may use without proof that there are 5 conjugacy classes, and they have sizes $(1, 15, 20, 12, 12)$ and consist of elements of order $(1, 2, 3, 5, 5)$, respectively, and any other structure of the group and its subgroups that you need. You may also use without proof that there is an irreducible representation of dimension 3, with character values $(3, -1, 0, \frac{1+\sqrt{5}}{2}, \frac{1-\sqrt{5}}{2})$, respectively.]

If V is any irreducible complex representation of A_5 of dimension 3, show that V is not the restriction of a representation of the symmetric group S_5.

(b) Let G be a finite group acting on a finite set X.

Define the *permutation representation* $\mathbb{C}[X]$, and compute its character. Compute $\dim \mathrm{Hom}_G(\mathbf{1}, \mathbb{C}[X])$ in terms of the orbits for G acting on X.

Let $d = |X|$. Let $\Gamma_n = \{\Sigma \subseteq X : |\Sigma| = n\}$ be the set of all n-element subsets of X. Show that $\mathbb{C}[\Gamma_n]$ is isomorphic to $\mathbb{C}[\Gamma_{d-n}]$ for all $0 \leqslant n \leqslant d$.

Now take $G = A_5$ acting on $X = \{1, \ldots, 5\}$ in the natural way. Is it true that $\mathbb{C}[\Gamma_n]$ is isomorphic to $\wedge^n \mathbb{C}[X]$ for all $0 \leqslant n \leqslant 5$? Give a proof or counterexample.

Paper 2, Section II

19F Representation Theory

Let G be a finite group and let Z be its centre.

(a) Let V be a representation of G. Define the *kernel* of V, and show that it is a normal subgroup. The derived subgroup G' is the subgroup of G generated by the elements $ghg^{-1}h^{-1}$, for $g, h \in G$. Show that the 1-dimensional representations of G are precisely the irreducible representations of G with G' in their kernel.

(b) Let $\Theta : Z \to \mathbb{C}^*$ be a 1-dimensional representation, and V an irreducible summand of $\operatorname{Ind}_Z^G \Theta$. Describe how Z acts on V, and determine the number of times V occurs in the decomposition of $\operatorname{Ind}_Z^G \Theta$ into irreducibles.

(c) Let p be a prime number and let

$$G = \left\{ \begin{pmatrix} 1 & x & c \\ & 1 & y \\ & & 1 \end{pmatrix} \middle| \, x, y, c \in \mathbb{F}_p \right\}.$$

Compute $|G|$ and deduce that G has no irreducible representation V with $1 < \dim V < p$. Compute Z and G'.

Show that if $\Theta : Z \to \mathbb{C}^*$ is a 1-dimensional representation, then $\operatorname{Ind}_Z^G \Theta$ is either a direct sum of 1-dimensional representations, or of the form $V_\Theta \oplus \ldots \oplus V_\Theta$ where V_Θ is an irreducible representation of dimension p.

Hence, or otherwise, list all the irreducible representations of G. You need not determine the characters, but you must prove that your list is complete.

[You may use any theorems from lectures, provided you state them clearly.]

Paper 3, Section II

19F Representation Theory

(a) Define the topological group SU_2, and show that it is homeomorphic to S^3. Write down a subgroup T of SU_2 that is isomorphic to the circle group S^1.

Let $\mathcal{O}$ be a conjugacy class in SU_2, and suppose $|\mathcal{O}| > 1$. Show $\mathcal{O}$ is homeomorphic to the 2-sphere S^2, and $\mathcal{O}$ meets T at exactly 2 points. Parameterise all the conjugacy classes in SU_2.

Let $\rho : SU_2 \to \mathrm{GL}(V)$ be an n-dimensional complex representation of SU_2. Show that $\det \rho(g) = 1$ for all $g \in SU_2$.

(b) Let $G = Q_8$ be the quaternion group of order 8. Construct the 2-dimensional irreducible complex representation V of G, and use this to show that $G \leqslant SU_2$.

Decompose $V \otimes V$ and $V \otimes V \otimes V$ as representations of SU_2, carefully stating any results from lectures that you use. For each irreducible summand W of $V \otimes V$ as a representation of SU_2, decompose $\operatorname{Res}_G^{SU_2} W$ as a representation of G.

Paper 4, Section II
19F Representation Theory

(a) Let H, K be subgroups of a finite group G, V a complex representation of H, and W a complex representation of K.

Suppose $|H|$ and $|K|$ are coprime. Compute the inner product $\langle \operatorname{Ind}_H^G V, \operatorname{Ind}_K^G W \rangle_G$.

(b) Let G be a finite group of odd order. Let $\chi_1, \ldots, \chi_r$ be the characters of the irreducible complex representations of G.

Show that the map $g \mapsto g^2$ is a bijection on G. Let χ be the character of an irreducible complex representation of G, and write $\widetilde{\chi}$ for the function $g \mapsto \chi(g^2)$. Show that

(i) $\langle \widetilde{\chi}, \widetilde{\chi} \rangle = 1$,

(ii) $\widetilde{\chi} = \sum_{i=1}^{r} n_i \chi_i$, for some $n_i \in \mathbb{Z}$.

Deduce that $\widetilde{\chi}_1, \ldots, \widetilde{\chi}_r$ are some permutation of $\chi_1, \ldots, \chi_r$.

Paper 1, Section II

24G Riemann Surfaces

(a) Let $\pi\colon X \to Y$ be a continuous map of path-connected Hausdorff topological spaces. What does it mean to say that π is a *covering map*? What does it mean to say that π is a *regular covering map*? Give an example of a covering map between Riemann surfaces which is surjective and not regular.

(b) Let (f, U) and (g, V) be function elements on domains U, $V \subset \mathbb{C}$. What does it mean to say that (g, V) is an *analytic continuation* of (f, U)? What is a *complete analytic function*?

Let $\mathcal{F}$ be a complete analytic function. Explain how to construct the Riemann surface R of $\mathcal{F}$ along with its canonical covering map to $\mathbb{C}$, and describe how $\mathcal{F}$ gives rise to an analytic function $R \to \mathbb{C}$. [Proofs are not required.]

(c) Let (f, U) be a function element on a domain $U \subset \mathbb{C}$ and $F\colon U \to \mathbb{C}$ an analytic function with $F' = f$.

(i) Let (g, V) be an analytic continuation of (f, U), with $U \cap V$ connected and V simply-connected. Show that there exists an analytic function $G\colon V \to \mathbb{C}$ with $G' = g$ such that (G, V) is an analytic continuation of (F, U).

(ii) Let $\mathcal{F}$ and $\widetilde{\mathcal{F}}$ be the complete analytic functions determined by (f, U) and (F, U), respectively. Show that the Riemann surface of $\widetilde{\mathcal{F}}$ is a surjective regular cover of the Riemann surface of $\mathcal{F}$.

Paper 2, Section II

24G Riemann Surfaces

(a) State and prove the Riemann–Hurwitz formula. [You may assume the valency theorem.]

(b) Let $g(z) \in \mathbb{C}[z]$ be a monic polynomial of degree n with no repeated roots, and $d > 1$ an integer dividing n. Let X be the Riemann surface (with branch points) of the function $\sqrt[d]{g}$ and $f\colon X \to \mathbb{C}$ the analytic map given by z. Show that there exists a compact Riemann surface $\widehat{X}$, containing X as a dense open subset, and an analytic map $\hat{f}\colon \widehat{X} \to \mathbb{C}_\infty$ extending f.

What is the cardinality of $\widehat{X} \setminus X$? Compute the genus of $\widehat{X}$ in terms of n and d.

Paper 3, Section II

23G Riemann Surfaces

Let $\Lambda = \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2 \subset \mathbb{C}$ be a lattice, where $\omega_1, \omega_2 \in \mathbb{C} \setminus \{0\}$ and $\mathrm{Im}(\omega_2/\omega_1) > 0$.

What is an *elliptic function* with respect to Λ? Show that if f is an elliptic function with respect to Λ, and $\mathcal{P}$ is a period parallelogram, then f has as many poles in $\mathcal{P}$ as it has zeroes (counted with multiplicity).

Define the *Weierstrass* $\wp$*-function* attached to Λ and show that it is an elliptic function. Write $\omega_3 = -\omega_1 - \omega_2$ and $e_j = \wp(\omega_j/2)$ $(1 \leqslant j \leqslant 3)$. Show that the e_j are distinct and that $\wp(z)$ satisfies the differential equation

$$\wp'(z)^2 = 4(\wp(z) - e_1)(\wp(z) - e_2)(\wp(z) - e_3).$$

Prove that

$$\frac{\wp'(z + \omega_1/2)}{\wp'(z)} = -\left(\frac{\wp(\omega_1/4) - e_1}{\wp(z) - e_1}\right)^2.$$

[*You may assume that the series*

$$\sum_{0 \neq \omega \in \Lambda} \frac{1}{|\omega|^\alpha}$$

converges for every real $\alpha > 2$.]

Paper 1, Section II

5K Statistical Modelling

Define the *normal linear model* with response $Y \in \mathbb{R}^n$, $X \in \mathbb{R}^{n \times p}$, coefficient vector $\beta \in \mathbb{R}^p$, and noise variance $\sigma^2 > 0$. What is the maximum likelihood estimator of β? [You do not need to derive it and can assume X has full column rank.]

Define the *leverage* of the ith data point for $1 \leqslant i \leqslant n$, then express the variance of the residual of the ith data point (treating X as fixed) in terms of its leverage and σ^2. Explain why we pay attention to high-leverage points in regression analysis.

Paper 1, Section II

13K Statistical Modelling

Let Poisson(λ) denote the Poisson distribution with rate parameter λ.

(a) Show that Poisson(λ), $\lambda > 0$ forms an exponential family with natural parameter $\theta = \log \lambda$ by identifying its sufficient statistic and cumulant function.

(b) Let $Y_i \sim \text{Poisson}(\lambda_i), i = 1, \ldots, n$ be independent and $M = \sum_{i=1}^n Y_i$. Show that the conditional distribution of $(Y_1, \ldots, Y_n)$ given $M = m$ is

$$(Y_1, \ldots, Y_n) \mid M = m \sim \text{Multinomial}(m, (\pi_1, \ldots, \pi_n)),$$

where $\pi_i = \lambda_i / \lambda_+, i = 1, \ldots, n$ for $\lambda_+ = \lambda_1 + \cdots + \lambda_n$.

(c) The `lizards` dataset records the species (`Sagrei` or `Distichus`), diameter of the branch they were perched on (`narrow` or `wide`), and the height of that same branch (`high` or `low`) for a sample of 409 lizards.

```
> (data <- as.data.frame(table(lizards)))
    Species Diameter Height Freq
1    Sagrei   narrow   high   86
2 Distichus   narrow   high   73
3    Sagrei     wide   high   35
4 Distichus     wide   high   70
5    Sagrei   narrow    low   32
6 Distichus   narrow    low   61
7    Sagrei     wide    low   11
8 Distichus     wide    low   41
```

Consider the two models fitted by the R code below. Why does the statistical model in `fit1` imply that the variables `Species`, `Diameter`, and `Height` are independent? What independence relations between these three variables are implied by the model in `fit2`?

```
> fit1 <- glm(Freq ~ Species + Diameter + Height, family = poisson, data)
> fit2 <- glm(Freq ~ Species * Diameter + Height, family = poisson, data)
```

What can you conclude from the analysis of deviance below? How would you test the independence relations implied by the model in `fit2`?

```
> anova(fit1, fit2, test = "LR")
Analysis of Deviance Table

Model 1: Freq ~ Species + Diameter + Height
Model 2: Freq ~ Species * Diameter + Height
  Resid. Df Resid. Dev Df Deviance  Pr(>Chi)
1         4     25.037
2         3     12.431  1   12.606 0.0003845 ***
---
Signif. codes:  0 ‘***’ 0.001 ‘**’ 0.01 ‘*’ 0.05 ‘.’ 0.1 ‘ ’ 1
```

Paper 2, Section I

5K Statistical Modelling

Write down the *logistic regression* model for $(X_i, Y_i) \in \mathbb{R}^p \times \{0,1\}$, $i = 1, \ldots, n$. Then show that the logit function

$$\operatorname{logit}(\pi) = \log \frac{\pi}{1-\pi}, \; 0 < \pi < 1,$$

is the canonical link function for the binomial generalised linear model. Name an algorithm that you can use for fitting logistic regression with a dataset. [You do not need to further describe the algorithm.]

Recall that linear discriminant analysis assumes

$$X_i \mid Y_i = 0 \sim \mathrm{N}(\mu_0, \Sigma), \quad X_i \mid Y_i = 1 \sim \mathrm{N}(\mu_1, \Sigma),$$

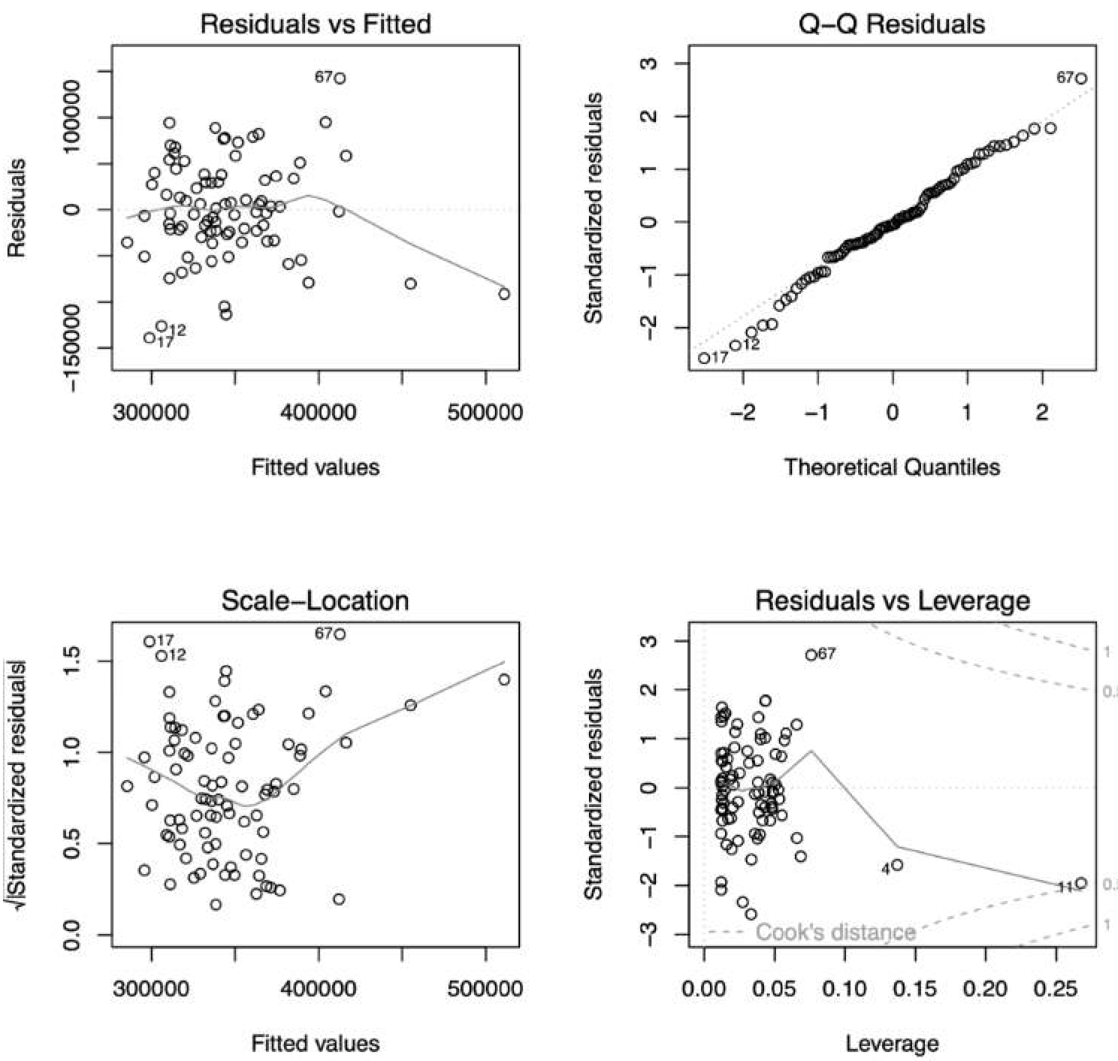
where $\mu_0, \mu_1 \in \mathbb{R}^p$ and $\Sigma \in \mathbb{R}^{p \times p}$ is positive definite. Show that this is a submodel of logistic regression with an intercept term (that is, all assumptions of the logistic regression are satisfied).

Paper 3, Section I

5K Statistical Modelling

Consider the dataset `HousePrices` which includes the house sale prices along with various factors that might be relevant for predicting the sale price. Consider the `R` code and output below.

```
> head(HousePrices, 3)
  Bedrooms Bathrooms Living.area Lot.size Year.built Property.tax Sale.price
1        4         1        1380     6000       1948         8360     350000
2        4         2        1761     7400       1951         5754     360000
3        4         2        1564     6000       1948         8982     350000
> par(mfrow = c(2,2))
> plot(lm(Sale.price ~ Bedrooms + Living.area, data = HousePrices))
```



Describe your observations in each plot above. Does the model appear to be adequate in light of these plots? If not, what might you do to improve the model?

Paper 4, Section I

5K Statistical Modelling

Let $\{f(y;\theta) \mid \theta \in \Theta \subseteq \mathbb{R}\}$ be an exponential family with natural parameter θ. Define the corresponding generalised linear model for the data $(X_i, Y_i) \in \mathbb{R}^p \times \mathbb{R}$, $i = 1, \ldots, n$ with parameter vector $\beta = (\beta_1, \ldots, \beta_p) \in \mathbb{R}^p$.

Find the log-likelihood function and the normal equations that the maximum likelihood estimator $\hat{\beta} \in \mathbb{R}^p$ must satisfy. What is the geometric interpretation of the normal equations?

Describe a two-sided hypothesis test for $H_0 : \beta_1 = 0$ versus $H_1 : \beta_1 \neq 0$.

Paper 4, Section II

13K Statistical Modelling

Let $(X_1, Y_1, Z_1), \ldots, (X_n, Y_n, Z_n) \in \mathbb{R}^3$ be n independent and identically distributed samples generated by the R code below. Suppose the R object `X` stores $X_1, \ldots, X_n$, object `Y` stores $Y_1, \ldots, Y_n$, and object `Z` stores $Z_1, \ldots, Z_n$.

(a) The `R` function `cov(X,Y)` returns the sample covariance between `X` and `Y` defined as

$$S_{XY} = \frac{1}{n-1} \sum_{i=1}^{n} (X_i - \bar{X})(Y_i - \bar{Y}),$$

where $\bar{X}$ is the sample mean of $X_1, \ldots, X_n$ and $\bar{Y}$ is similarly defined. Show that this is an unbiased estimator of $\mathsf{Cov}(X_1, Y_1)$.

(b) Consider the following R code and output.

```
> n <- 1000
> Z <- rnorm(n)
> U <- rnorm(n)
> X <- Z + U + rnorm(n)
> Y <- X + U + rnorm(n)
> coef(lm(Y ~ X))
(Intercept)           X
-0.01579827  1.36753704
> cov(X, Y) / cov(X, X)
[1] 1.367537
```

Why is the coefficient of `X` in the linear model identical to the ratio of covariances?

How do you expect the coefficient of `X` in this linear model to behave when n increases? Find the limiting value of this coefficient when $n \to \infty$ in the example above.

(c) Define the *two-stage least squares* estimator of the causal effect of X on Y using Z as the instrumental variable, and show that in the setting above, it can be calculated by `cov(Z, Y) / cov(Z, X)`. What do you expect the two-stage least squares estimator to converge to when $n \to \infty$ in this example? Justify your answer.

Paper 1, Section II

36E Statistical Physics

Throughout this problem you may assume that all gasses are ideal; that air is a combination of 80% nitrogen gas (N_2 molecules composed of the N-14 isotope) and 20% oxygen gas (O_2 molecules composed of the O-16 isotope); that helium gas is He composed of the He-4 isotope; that nuclear binding energies and electrons may be neglected so that atomic masses (in atomic units) are given by their isotope numbers.

(a) Consider a spherical, rubber party balloon, in a room at room temperature and standard pressure, which is filled with a fixed number N of helium atoms at room temperature T. Neglecting in this part the additional pressure on the internal gas caused by the rubber tension, calculate

(i) the mass density ρ of the He gas relative to the surrounding air,

(ii) the specific heat capacity c_p (at fixed pressure) of the helium, relative to the surrounding air, i.e. $c_p^{\mathrm{He}}/c_p^{\mathrm{air}}$.

(b) In this part, you will take the tension of the rubber into account. Suppose that the rubber has a free energy $F = tA$, where A is its area, and t is the (assumed to be constant) tension per unit length. (Since rubber forces are mostly entropic, it is necessary to add this to the free energy F rather than to the energy E.)

(i) Write down an expression for the free energy F of the balloon, including both the rubber and the internal gas.

(ii) Write down an equation for the equilibrium of the system with respect to changes of the balloon radius r, taking into account the surrounding air pressure.

(iii) At room temperature T, calculate the effect of tension on the radius r to $O(t)$ in a small-t expansion.

Paper 2, Section II

37E Statistical Physics

(a) Consider the grand Gibbs canonical potential, defined as

$$\Gamma := \Phi + pV \ = \ E - TS + pV - \mu N,$$

for any substance made of a single type of molecule (with only short-range interactions). Write down a formula for the variation $d\Gamma$. Show that

$$\Gamma = 0\,,$$

regardless of the nature of the substance.

(b) A large number N of atoms are trapped in a two-dimensional plane, with no ability to move or vibrate in the third dimension. At low temperatures, these atoms form a planar crystal, arranged in a square grid with area A. Let the speed of sound be given by c_s. (Although it is not realistic, assume that c_s is the same for both transverse and longitudinal modes.)

(i) Calculate the density of states $g(\omega)$, where $g(\omega)d\omega$ is the density of one-phonon states available in a small frequency window $[\omega, \omega + d\omega)$.

(ii) Estimate the Debye frequncy ω_D for the system.

(iii) Show that the partition function of the gas of phonons at low temperature T is given by

$$Z = \exp\left(\frac{XAk_B^2T^2}{c_s^2\hbar^2}\right),$$

where X is a numerical coefficient you should express as an infinite series. For what temperatures is your answer valid?

(iv) For the same low temperature regime, calculate the entropy $S(T)$.

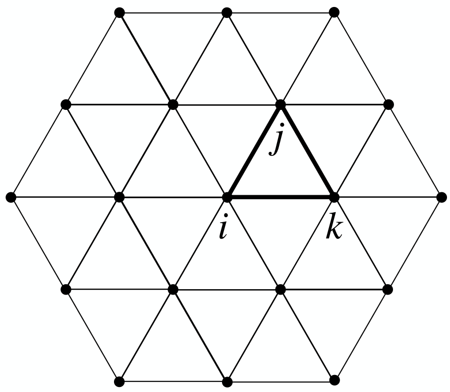
Paper 3, Section II

35E Statistical Physics

Consider N spins, $s_i \in \{-1, 1\}$, placed on the vertices $i, j, k \ldots$ of a triangular lattice with hexagonal symmetry (see figure) in $d = 2$ spatial dimensions, subject to the following Hamiltonian

$$H = -B \sum_i s_i - J \sum_{\langle ij \rangle} s_i s_j - K \sum_{\langle ijk \rangle} s_i s_j s_k \,,$$

where $\langle ij \rangle$ indicates a sum over nearest neighbour edges and $\langle ijk \rangle$ represents a sum over elemental triangles (as depicted in the figure), and $B, J, K \in \mathbb{R}$ are parameters.



(a) Assuming the validity of mean field theory, calculate the canonical partition function $Z(N, \beta)$, written in the form of a sum over possible values of the total magnetisation $M = \sum_i s_i$.

(b) Using Stirling's formula: $\ln N! \approx N \ln N - N$, show that for large N the partition function can be approximated as an integral

$$Z_N(\beta, h) \approx \int_{-1}^{1} \exp[-N f(m)] \; dm,$$

where the average magnetisation is $m = M/N$, and

$$f(m) = P(m) + \frac{1+m}{2} \ln \frac{1+m}{2} + \frac{1-m}{2} \ln \frac{1-m}{2},$$

where $P(m)$ is a polynomial in m that you should determine.

(c) Show that this integral is dominated at a value of m satisfying the equation

$$m = \tanh[Q(m)]\,,$$

where $Q(m)$ is a polynomial in m that you should determine.

[*Hint: You may wish to use the identity* $\dfrac{1+\tanh(x)}{1-\tanh(x)} = e^{2x}$.]

(d) Suppose now that $B = J = 0$ and $K > 0$ is fixed. Prove that, if the system is gradually cooled off, there must be a phase transition from $m = 0$ to $m \neq 0$ at some critical value β_*. [You do not need to calculate the value of β_*.]

Paper 4, Section II

35E Statistical Physics

(a) For any gas with internal energy E, temperature T, volume V and pressure p, use the Maxwell relation derived from varying the free energy $F(V,T)$ to show that

$$\left(\frac{\partial E}{\partial V}\right)_T = T^2 \left[\frac{\partial}{\partial T}\left(\frac{p}{T}\right)\right]_V .$$

For the remaining parts of this problem, consider a classical but non-ideal monatomic gas, obeying the van der Waals equation of state

$$p = \frac{NkT}{V - bN} - a\frac{N^2}{V^2} ,$$

where a and b are constants.

(b) Calculate the internal energy E of the non-ideal gas. Assume that the heat capacity of the gas is $C_V = \frac{3}{2}kN$ in the dilute gas limit, $V/N \to \infty$ (with N constant), and show that this formula for C_V continues to be valid even at finite V/N.

(c) Consider a Carnot engine, whose cylinder is filled with the same non-ideal gas as in the previous part.

(i) Under isothermal expansion of the volume from V_1 to V_2 at some temperature T, how much heat ΔQ is obtained from the thermal bath?

(ii) Under adiabatic expansion (or contraction) of the volume V, how does the temperature T vary with the volume V?

(iii) Suppose a complete, reversible Carnot cycle is performed, involving a hot bath with temperature T_H and a cold bath with temperature T_C. Give an expression for the efficiency η of the engine, and discuss whether it can depend on the parameters a and b. Justify your answer with reference to physical principles.

Paper 1, Section II

30L Stochastic Financial Models

Consider a one-period market with d risky assets, with time-t price vector S_t for $t \in \{0,1\}$ and interest rate r. An investor has utility function U, assumed concave, differentiable and strictly increasing, and initial wealth x. Let

$$\mathcal{X} = \{(1+r)x + \theta^\top (S_1 - (1+r)S_0) : \theta \in \mathbb{R}^d\}$$

be the set of possible time-1 wealths attainable by the investor. For a random variable Z, let

$$u(Z) = \sup_{X \in \mathcal{X}} \mathbb{E}[U(X+Z)]$$

be the indirect utility of the payout Z, where $U(X+Z)$ is assumed integrable for all choices of X and Z. [You may assume that this supremum is attained at some $X_Z \in \mathcal{X}$, and in particular that $u(0) = \mathbb{E}[U(X_0)]$ for some $X_0 \in \mathcal{X}$.]

(a) Show that

(i) if $Z_0 \leqslant Z_1$ almost surely and $\mathbb{P}(Z_0 < Z_1) > 0$, then $u(Z_0) < u(Z_1)$,

(ii) $u(pZ_1 + (1-p)Z_0) \geqslant pu(Z_1) + (1-p)u(Z_0)$ for all $p \in (0,1)$.

(b) In this set-up, how is an *indifference price* $\pi(Y)$ of a contingent claim with payout Y defined? Show that the indifference price exists and is unique. [You may use standard properties of concave functions without proof.]

(c) Show that

$$\mathbb{E}[U'(X_0)S_1] = (1+r)\mathbb{E}[U'(X_0)]S_0.$$

[You may interchange differentiation and expectation without justification.]

(d) How is the *marginal utility price* $\pi_0(Y)$ of a contingent claim with payout Y defined? Show that

$$\pi_0(Y) \geqslant \pi(Y).$$

[You may use standard properties of concave functions.]

(e) Suppose that S_1 and Y are jointly Gaussian, and let $\mu = \mathbb{E}(S_1)$ and $V = \operatorname{cov}(S_1)$. Assume that V is positive definite. Show that

$$\pi_0(Y) = \frac{1}{1+r}\left(\mathbb{E}(Y) - (\mu - (1+r)S_0)^\top V^{-1} \operatorname{cov}(S_1, Y)\right).$$

[You may use standard properties of Gaussian random vectors.]

Paper 2, Section II

30L Stochastic Financial Models

(a) What does it mean to say that an adapted integrable process $X = (X_n)_{n \geqslant 0}$ is a *supermartingale*?

(b) Let $R_0 = 0$ and $R_n = \xi_1 + \cdots + \xi_n$ for $n \geqslant 1$, where $(\xi_n)_{n \geqslant 1}$ are independent and identically distributed with $\mathbb{P}(\xi_1 = 1) = p$ and $\mathbb{P}(\xi_1 = -1) = 1 - p$ for a constant $p \in (1/2, 1)$. Determine for which values of $c \in (0, \infty)$ the process defined by $X_n = c^{R_n}$ is a supermartingale in the filtration generated by $(R_n)_{n \geqslant 0}$.

(c) Suppose that X is a supermartingale and U is an increasing concave function. Set $Y_n = U(X_n)$. Assuming that Y is integrable, show that Y is a supermartingale.

Let $H = (H_n)_{n \geqslant 1}$ be a bounded previsible process and let $X = (X_n)_{n \geqslant 0}$ be an integrable adapted process. The adapted process $H \bullet X$ is defined by $(H \bullet X)_0 = 0$ and, for $n \geqslant 1$,

$$(H \bullet X)_n = \sum_{k=1}^{n} H_k (X_k - X_{k-1}).$$

(d) In the case where H is non-negative and X is a supermartingale, show that $\mathbb{E}[(H \bullet X)_n] \leqslant 0$ for all $n \geqslant 0$.

(e) Conversely, in the case where $\mathbb{E}[(H \bullet X)_n] \leqslant 0$ for all $n \geqslant 0$ whenever H is non-negative, show that X is a supermartingale.

Paper 3, Section II

29L Stochastic Financial Models

Consider a market with one risky asset with time-n price S_n and zero interest rate $r = 0$. Assume that the price evolves as

$$S_n = S_{n-1}(1 + R_n)$$

where the random variables $(R_n)_{n\geqslant 1}$ are independent and identically distributed. Let X_n denote an investor's wealth at time n, and suppose it evolves as

$$X_n = X_{n-1} - C_n + \eta_n R_n$$

where C_n denotes the investor's consumption in the time interval $(n-1, n]$, and η_n denotes the total amount invested in the market in the same time interval. Assume that the process $(C_n, \eta_n)_{n\geqslant 1}$, taking values in $\mathbb{R}^2$, is previsible with respect to the filtration generated by $(R_n)_{n\geqslant 1}$.

Given initial wealth X_0 and a non-random time horizon $N \geqslant 1$, the investor aims to maximise

$$\mathbb{E}\left(\sum_{n=1}^{N} U(C_n) + U(X_N)\right)$$

for a given utility function U, assumed to be bounded from above.

(a) Formulate the investor's problem as a stochastic optimal control problem: define its *value function* V and write down its Bellman equation.

From now on assume that $U(x) = -e^{-x}$ for all x. Assume also that R_1 takes both positive and negative values with positive probability.

(b) Rewrite the Bellman equation in terms of $G = -U$ and $F = -V$.

(c) Show that there is a solution of the form $F(n, x) = B_n e^{-A_n x}$ for some constants $A_n, B_n > 0$.

(d) Determine A_n and show that, for $\gamma = \inf_h \mathbb{E}(e^{-hR_1})$,

$$B_n = (N - n + 1)\gamma^{(N-n)/2}.$$

(e) Find the optimal consumption C_n^* over $(n-1, n]$ in terms of the optimized time-$(n-1)$ wealth X_{n-1}^* and γ.

Paper 4, Section II

29L Stochastic Financial Models

Let $W = (W_t)_{t \geqslant 0}$ be a Brownian motion and let $X_t = W_t + ct$ for some constant c.

(a) Use the reflection principle and the Cameron–Martin theorem to show, for all $x \geqslant 0$ and $T \geqslant 0$,

$$\mathbb{P}\left(\max_{0 \leqslant t \leqslant T} X_t \geqslant x\right) = \mathbb{P}(X_T \geqslant x) + e^{2cx}\mathbb{P}(-X_T \geqslant x).$$

Consider a market with a stock with time-t price $S_t = S_0 e^{\mu t + \sigma W_t}$ and interest rate r, where S_0, σ, r are positive constants and μ a real constant. Let T be a fixed maturity date, and let $\pi(Y)$ denote the time-0 Black–Scholes price of a European contingent claim with time-T payout Y.

(b) Compute $\pi(\log S_T)$.

(c) A trader aims to replicate (i.e. to delta-hedge) the payout $\log S_T$. At time 0, how many shares of the stock should the trader hold?

(d) Show that

$$\pi\left(\max_{0 \leqslant t \leqslant T} S_t + \min_{0 \leqslant t \leqslant T} S_t\right) = \pi\left(S_0 + \frac{\sigma^2}{2}\int_0^T S_t dt + S_T\right).$$

[*Hint: You may find the following identities useful. For $Y \geqslant 0$ and $\theta \geqslant 0$,*

$$\mathbb{E}(e^{\theta Y}) = 1 + \int_0^\infty \theta e^{\theta x}\mathbb{P}(Y \geqslant x)dx, \quad \mathbb{E}(e^{-\theta Y}) = \int_0^\infty \theta e^{-\theta x}\mathbb{P}(Y \leqslant x)dx. \quad]$$

Paper 1, Section I

2I Topics in Analysis

State Brouwer's fixed point theorem in the plane and also the equivalent theorem concerning continuous retractions. Explain (without detailed calculations) why the two statements are equivalent.

Let $S^2 = \{(x,y,z) \in \mathbb{R}^3 : x^2+y^2+z^2 = 1\}$. Show that if $f : S^2 \to S^2$ is a continuous map such that $f(p) \neq p$ for each $p \in S^2$, then f is surjective. [You may assume without proof that for each $p \in S^2$ and $0 < \varepsilon < 2$ the set $S^2 \setminus B_\epsilon(p)$ is homeomorphic to a closed disc in $\mathbb{R}^2$, where $B_\epsilon(p) = \{q \in S^2 : d(p,q) < \varepsilon\}$ and d is the Euclidean distance in $\mathbb{R}^3$.]

Paper 2, Section I

2I Topics in Analysis

State Runge's theorem on uniform approximation of holomorphic functions by polynomials.

Let $\Delta = \{z \in \mathbb{C} : |z| < 1\}$ and let $L^+ = \{it : t \in \mathbb{R}, t \geqslant 0\} \subset \mathbb{C}$. Prove that there is a sequence of complex polynomials which converges to $1/z$ uniformly on each compact subset of $\Delta \setminus L^+$.

Paper 2, Section II

11I Topics in Analysis

(a) Let $\gamma : [0,1] \to \mathbb{C} \setminus \{0\}$ be a continuous map with $\gamma(0) = \gamma(1)$. Define the *winding number* $w(\gamma;0)$ about the origin.

Show that the complex conjugate path $\overline{\gamma}$ satisfies $w(\overline{\gamma};0) = w(\gamma^-;0)$, where $\gamma^-(t) = \gamma(1-t)$.

(b) State precisely and prove a theorem about homotopy invariance of the winding number. [You may assume that if $\sigma : [0,1] \to \mathbb{C}$ is continuous, $\sigma(0) = \sigma(1)$ and $|\sigma(t)| < |\gamma(t)|$ for each $0 \leqslant t \leqslant 1$, then $w(\gamma+\sigma;0) = w(\gamma;0)$.]

(c) Let $D = \{z \in \mathbb{C} : |z| \leqslant 1\}$ and $C = \{z \in \mathbb{C} : |z| = 1\}$. If $g : D \to C$ is continuous, prove that for each non-zero integer n, there exists a point $z \in C$ such that $z^n = -g(z)$.

Paper 3, Section I

2I Topics in Analysis

Define a *dense subset* of a metric space.

State a version of the Baire category theorem.

Let A_j be a sequence of subsets of $[0,1]$ such that for each $N \geqslant 1$, $\cup_{j=N}^{\infty} A_j$ is open and dense in $[0,1]$. Prove that the set S of points $x \in [0,1]$ such that $x \in A_j$ for infinitely many j is dense. Must S be open? Must it be true that $\cap_{j=1}^{\infty} A_j \neq \emptyset$? Justify your answers.

Paper 4, Section I

2I Topics in Analysis

(i) Let $t_1, t_2, \ldots, t_n \in [-1,1]$ be any set of n distinct numbers. Show that there exist unique numbers $A_1, A_2, \ldots, A_n$ such that

$$\int_{-1}^{1} p(t)dt = \sum_{j=1}^{n} A_j p(t_j)$$

holds for every polynomial p of degree $n-1$ or less.

(ii) For $n = 1, 2, \ldots$, let p_n be the Legendre polynomial of degree n. Suppose that $t_1, t_2, \ldots, t_n \in [-1,1]$ are the roots of p_n, and $A_1, A_2, \ldots, A_n$ are the numbers corresponding to p_n as in part (i). Prove that the integration formula in part (i) is now valid for any polynomial of degree $2n-1$ or less.

[You may assume without proof that p_n has n distinct roots in $[-1,1]$.]

(iii) Let $t_1, t_2, \ldots, t_n$ and $A_1, A_2, \ldots, A_n$ be as in part (ii). Show that $A_j > 0$ for each $j = 1, 2, \ldots, n$ and that $\sum_{j=1}^{n} A_j = 2$. Show further that if p is any polynomial of degree $2n-1$ or less, then

$$\left| \sum_{j=1}^{n} A_j p(t_j) - \int_{-1}^{1} f(t)dt \right| \leqslant 4 \sup_{t \in [-1,1]} |f(t) - p(t)|.$$

holds for all continuous functions f.

Paper 4, Section II

12I Topics in Analysis

(a) State Liouville's theorem on approximation of algebraic numbers by rationals.

(b) Consider the continued fraction associated with a positive irrational number α,

$$\alpha = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \dots}}}$$

where a_n, $n = 1, 2, \ldots$, are positive integers and a_0 is a non-negative integer. Assuming that the nth convergent p_n/q_n is determined from

$$\begin{pmatrix} p_n & q_n \end{pmatrix} = \begin{pmatrix} a_n & 1 \end{pmatrix} \begin{pmatrix} a_{n-1} & 1 \\ 1 & 0 \end{pmatrix} \cdots \begin{pmatrix} a_0 & 1 \\ 1 & 0 \end{pmatrix},$$

show that $q_n p_{n-1} - q_{n-1} p_n = (-1)^n$ for all $n \in \{1, 2, 3, \ldots\}$. Show further that

$$\left| \alpha - \frac{p_n}{q_n} \right| \leqslant \frac{1}{q_n q_{n+1}}.$$

Deduce that for each irrational α there are infinitely many fractions k/m (with k, m co-prime integers) such that

$$\text{(i)} \quad \left| \alpha - \frac{k}{m} \right| < \frac{1}{m^2};$$

$$\text{(ii)} \quad \left| \alpha - \frac{k}{m} \right| < \frac{1}{2m^2}.$$

(c) Suppose that an infinite continued fraction is constructed according to the following rule: if p_n/q_n is the n-th convergent, then $a_{n+1} = q_n^n$. Let α be the real number associated with this continued fraction. Prove that α is transcendental.

Paper 1, Section II

40C Waves

An infinite rigid wavy wall moves with constant speed V in the positive x direction, such that the equation of the wall is given by

$$y = y_{\mathrm{wall}}(x,t) \equiv \mathrm{Re}[\epsilon \exp(ik(x - Vt))] \ ,$$

where $0 < \epsilon \ll 1$ and k is a constant.

(a) The region $y > y_{\mathrm{wall}}$ is filled with a uniform acoustic medium in which the steady sound speed and mean density are c_1 and ρ_1 respectively. You may assume that the wall imposes a boundary condition on the medium at $y = 0$.

For the case $V > c_1$, show that the y component of the time-averaged acoustic energy flux is

$$\frac{\rho_1 k^2 V^3 \epsilon^2}{2\beta_1} \ ,$$

where $\beta_1 = \sqrt{(V/c_1)^2 - 1}$. Explain briefly, without further detailed calculation, how your answer would change if $V < c_1$.

[You may use without proof the fact that the acoustic energy flux vector, $\mathbf{I}$, is given by $\mathbf{I} = p'\mathbf{u}'$, where p' is the acoustic pressure and $\mathbf{u}'$ is the acoustic velocity.]

(b) Now suppose that the uniform medium is replaced by a two-layer medium, in which the steady sound speed and density in $h > y > y_{\mathrm{wall}}$ are now c_0 and ρ_0 respectively, while in $y \geqslant h$ the steady sound speed and density still take the values c_1 and ρ_1 respectively.

For the case $c_1 < V < c_0$, calculate the y component of the time-averaged acoustic energy flux in $y \geqslant h$. Simplify your answer in the limiting cases (i) $k\beta_0 h \gg 1$ and (ii) $k\beta_0 h \ll 1$, where $\beta_0 = \sqrt{1 - (V^2/c_0^2)}$. Comment briefly on your answers in these two limits.

Paper 2, Section II

40C Waves

Small-amplitude disturbances in a homogeneous elastic solid, with density ρ and Lamé moduli λ and μ, are governed by the equation

$$\rho\frac{\partial^2\mathbf{u}}{\partial t^2} = (\lambda+2\mu)\nabla(\nabla\cdot\mathbf{u}) - \mu\nabla\times(\nabla\times\mathbf{u})\ ,$$

where $\mathbf{u}(\mathbf{x},t)$ is the displacement.

(a) Consider harmonic plane-wave solutions of the form

$$\mathbf{u}(\mathbf{x},t) = \mathrm{Re}[\mathbf{A}\exp(i\mathbf{k}\cdot\mathbf{x} - i\omega t)]\,,$$

with constant amplitude vector $\mathbf{A}$. Show that

$$\omega^2\mathbf{A} = c_P^2\,\mathbf{k}(\mathbf{k}\cdot\mathbf{A}) - c_S^2\,\mathbf{k}\times(\mathbf{k}\times\mathbf{A})\ ,$$

giving expressions for the P-wave and S-wave speeds c_P and c_S respectively. Explain mathematically how such waves can be classified into longitudinal P-waves and transverse SV- and SH-waves.

(b) The half-space $y<0$ is filled with the elastic solid described above, while the slab $0<y<h$ is filled with a homogeneous elastic solid with Lamé moduli λ' and μ'. The boundary at $y=h$ is rigid. A harmonic plane SH-wave propagates from $y<0$ towards the interface $y=0$, with displacement

$$(0,0,1)\mathrm{Re}[\exp(ilx+imy-i\omega t)]\ .$$

Write down the relationship between l, m and ω, and determine the complex amplitude of the SH-wave reflected back into $y<0$. Show that this complex amplitude has unit modulus, and comment on this result physically.

Paper 3, Section II

39C Waves

The response of a linear wave system to a perturbation at time $t = 0$ can be expressed in the form

$$\eta(x,t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(k) \exp(ikx - i\omega(k)t) \, \mathrm{d}k \, ,$$

where $\omega(k)$ and $F(k)$ are known functions of k, and $\omega(k)$ is real for real k.

(a) By using the method of stationary phase, calculate the leading-order behaviour of $\eta(x,t)$ as $t \to \infty$ with x/t fixed.

[*You may need the result,* $\int_{-\infty}^{\infty} e^{iu^2} \, \mathrm{d}u = \sqrt{\pi} e^{i\pi/4}$.]

(b) The surface of deep water is given by $y = \eta(x,t)$ and extends over $-\infty < x < \infty$. At $t = 0$, the surface is disturbed with initial conditions

$$\eta(x,0) = x \exp(-x^2) \, , \qquad \frac{\partial \eta}{\partial t}(x,0) = 0 \, .$$

Given that the motion of the water is incompressible, with velocity potential $\phi(x,y,t)$ satisfying

$$\nabla^2 \phi = 0 \, ,$$

and with boundary conditions

$$\frac{\partial \phi}{\partial y} = \frac{\partial \eta}{\partial t} \, , \qquad \frac{\partial \phi}{\partial t} = \frac{\partial^2 \eta}{\partial x^2},$$

on $y = 0$, in suitable units, show that the dispersion relation is

$$\omega^2 = |k|^3 \, .$$

Determine the behaviour of $\eta(Vt,t)$ as $t \to \infty$ for a fixed value of $V > 0$.

[*You may need the result,* $\int_{-\infty}^{\infty} e^{-u^2} \, \mathrm{d}u = \sqrt{\pi}$.]

Describe briefly the differences between what would be observed on the water surface by observers travelling at different speeds.

Paper 4, Section II

39C Waves

The Rankine-Hugoniot relations describing the jump conditions across a steady shock in a perfect fluid with constant specific heats are

$$[\rho \mathbf{u} \cdot \mathbf{n}] = 0 \,,$$
$$[\rho \mathbf{u}(\mathbf{u} \cdot \mathbf{n}) + p\mathbf{n}] = 0 \,,$$
$$[\rho(\mathbf{u} \cdot \mathbf{n})(h + \frac{1}{2}\mathbf{u}^2)] = 0 \,,$$

where ρ, p and $\mathbf{u}$ are the fluid density, pressure and velocity, $\mathbf{n}$ is the direction normal to the shock, $h = c^2/(\gamma - 1)$ is the enthalpy, γ is the adiabatic index, c is the local sound speed, and $[X]$ represents the jump in the quantity X across the shock.

(a) Briefly explain the physical meaning of each of these jump conditions.

(b) In the case of a normal shock, show that

$$M_2^2 = \frac{2 + (\gamma - 1)M_1^2}{2\gamma M_1^2 - (\gamma - 1)} \,,$$

where M_1 is the Mach number upstream of the shock, and M_2 is the Mach number downstream.

(c) Find the equation relating the temperatures upstream and downstream of a normal shock, given that

$$\frac{\rho_2}{\rho_1} = \frac{M_1^2(\gamma + 1)}{2 + (\gamma - 1)M_1^2} \qquad (\dagger)$$

in such a shock, where ρ_1 and ρ_2 are upstream and downstream densities, respectively.

(d) Now suppose that the shock is oblique, inclined at angles θ_1 and θ_2 to the upstream and downstream flows respectively (so that in the case of a normal shock $\theta_1 = \theta_2 = \pi/2$). Show how equation (†) is modified in an oblique shock, and derive a single equation for $\tan\theta_2$ in terms of θ_1 and M_1. Determine the maximum possible flow deflection when $M_1 \to \infty$.

$\Big[$*Recall that:* $\tan(A - B) = \dfrac{\tan A - \tan B}{1 + \tan A \tan B}$. $\Big]$

END OF PAPER