UNIVERSITY OF CAMBRIDGE

# MATHEMATICAL TRIPOS Part IB 2026

## List of Courses

Analysis II
Complex Analysis
Complex Analysis or Complex Methods
Complex Methods
Electromagnetism
Fluid Dynamics
Groups, Rings and Modules
Linear Algebra
Markov Chains
Methods
Numerical Analysis
Optimisation
Quantum Mechanics
Statistics
Topological Spaces
Variational Principles

**Paper 1, Section I**
**10G Analysis II**

(a) Let $(f_n)$ be a sequence of real-valued functions defined on a set $E \subset \mathbb{R}$. State the definition of *uniform convergence* of $(f_n)$ to a function $f$ on $E$.

(b) State and prove the Weierstrass M-test for the uniform convergence of a series of functions on $E$.

(c) For $x > 1$, define

$$\zeta(x) = \sum_{n=1}^{\infty} \frac{1}{n^x}.$$

Show that the series defining $\zeta(x)$ converges for $x > 1$ and that $\zeta$ is continuous on $(1, \infty)$.

(d) Consider the series defined by

$$S(x) = \sum_{n=1}^{\infty} \frac{\cos(n^3 x)}{n^4}, \qquad T(x) = \sum_{n=1}^{\infty} \frac{\sin(n^3 x)}{n}$$

for $x \in \mathbb{R}$.

(i) Evaluate

$$\int_0^{\pi} S(x)\,\mathrm{d}x$$

by integrating the series $S(x)$ termwise. Justify why this operation is valid.

(ii) Does the series $T(x)$ converge uniformly on $(0, \pi)$? Justify your answer.

**Paper 2, Section I**
**2F Analysis II**

Consider a function $f\colon \mathbb{R}^2 \to \mathbb{R}$. What does it mean for $f$ to be *differentiable* at a point $p \in \mathbb{R}^2$, and what is its *derivative* at $p$? What are the *partial derivatives* of $f$ at $p$?

Show that, if the partial derivatives of $f$ exist in a ball about a point $p \in \mathbb{R}^2$ and are continuous at $p$, then $f$ is differentiable at $p$.

Suppose $f\colon \mathbb{R}^2 \to \mathbb{R}$ is given by

$$f(x,y) = \begin{cases} x^2 \sin(1/x) + y^2 \sin(1/y) & \text{if } xy \neq 0 \\ x^2 \sin(1/x) & \text{if } x \neq 0,\ y = 0 \\ y^2 \sin(1/y) & \text{if } y \neq 0,\ x = 0 \\ 0 & \text{if } x = y = 0 \end{cases}.$$

At which points $(x, y) \in \mathbb{R}^2$ is $f$ differentiable? At which points is it $C^1$? Justify your answers.

**Paper 2, Section II**

**10F Analysis II**

Carefully state the inverse function theorem.

Let $\Omega$ be a bounded open subset of $\mathbb{R}^n$; denote its closure in $\mathbb{R}^n$ by $\overline{\Omega}$ and its boundary in $\mathbb{R}^n$ by $\partial\Omega = \overline{\Omega} \setminus \Omega$. Let $f\colon \overline{\Omega} \to \mathbb{R}^n$ be continuous in $\overline{\Omega}$ and $C^1$ in $\Omega$. Suppose $0 \notin f(\partial\Omega)$ and that $\det Df(x) \neq 0$ for every $x \in f^{-1}(0)$.

(i) Show that the zeros of $f$ are isolated, i.e. that

$$\forall x \in f^{-1}(0) \;\; \exists r > 0 \;\; \text{such that} \;\; y \in B_r(x)\setminus\{x\} \implies f(y) \neq 0.$$

(ii) Recall that $\Omega$ is bounded and $0 \notin f(\partial\Omega)$. Show that the zero set of $f$ is finite and contained in $\Omega$, i.e. $f^{-1}(0) = \{x_1, \ldots, x_N\}$ for some $N < \infty$ and $x_1, \ldots, x_N \in \Omega$. [You may assume results from lectures if properly stated.]

Deduce that there exist $r_1, \ldots, r_N > 0$ such that

$$\sup_{x \in B_{r_k}(x_k)} \|[Df(x_k)]^{-1}[Df(x) - Df(x_k)]\|_{\mathrm{op}} \leqslant \frac{1}{4} \quad \forall k = 1, \ldots, N,$$

where $\|\cdot\|_{\mathrm{op}}$ is the operator norm on $\mathbb{R}^{n\times n}$ induced by a choice of norm $\|\cdot\|$ on $\mathbb{R}^n$.

(iii) Define $N(f) = \sum_{x \in f^{-1}(0)} \operatorname{sign}(\det Df(x))$. Here, as usual, $\operatorname{sign}\colon \mathbb{R} \setminus \{0\} \to \{-1, 1\}$ takes the value $+1$ for positive real numbers and $-1$ for negative real numbers.

Show that there exists $\delta > 0$ sufficiently small such that, for all $g\colon \overline{\Omega} \to \mathbb{R}^n$ continuous in $\overline{\Omega}$ and $C^1$ in $\Omega$,

$$\sup_{\Omega} \|g - f\| + \sup_{\Omega} \|Dg - Df\|_{\mathrm{op}} < \delta \implies N(g) = N(f).$$

[You may assume the following statement. Let $r > 0$, $x \in \mathbb{R}^n$ and $h\colon \overline{B_r(x)} \to \mathbb{R}^n$ be continuous in $\overline{B_r(x)}$ and $C^1$ in $B_r(x)$. If both $\sup_{B_r(x)} \|A^{-1}(Dh - A)\|_{\mathrm{op}} \leqslant \frac{1}{2}$ and $\sup_{B_r(x)} \|A^{-1}h\| \leqslant \frac{r}{2}$ for some invertible $A \in \mathbb{R}^{n\times n}$, then $h$ has a unique zero in $B_r(x)$ and $\operatorname{sign}(\det Dh(y)) = \operatorname{sign}(\det A)$ for all $y \in B_r(x)$.]

**Paper 3, Section II**

**11F Analysis II**

(i) Let $V$ be a vector space over the real numbers. What is a *norm* on $V$? What does it mean for two norms to be *equivalent*?

(ii) Consider the space $\mathcal{L}(\mathbb{R}^n, \mathbb{R}^n)$ of $n \times n$ real matrices. Given a norm $\|\cdot\|$ on $\mathbb{R}^n$, recall that the induced operator norm on $\mathcal{L}(\mathbb{R}^n, \mathbb{R}^n)$ is defined as follows: for $A \in \mathcal{L}(\mathbb{R}^n, \mathbb{R}^n)$,

$$\|A\|_{\mathrm{op}} = \sup_{v \neq 0} \frac{\|Av\|}{\|v\|}.$$

Define $\rho \colon \mathcal{L}(\mathbb{R}^n, \mathbb{R}^n) \to \mathbb{R}$ by

$$\rho(A) = \lim_{k \to \infty} \|A^k\|_{\mathrm{op}}^{1/k}.$$

Show that $\rho$ is independent of the choice of norm on $\mathbb{R}^n$. [You may assume the above limit exists. You may also assume results from lectures if properly stated.]

(iii) What is a *contraction* on a metric space? State the contraction mapping theorem.

Let $A \in \mathcal{L}(\mathbb{R}^n, \mathbb{R}^n)$ and suppose $A$ is a contraction with respect to some norm on $\mathbb{R}^n$. Show that $I - A$ is invertible. By considering Picard iterates for the problem $Ax + b = x$, where $x, b \in \mathbb{R}^n$, obtain a formula for $(I - A)^{-1}$.

(iv) Let $A \in \mathcal{L}(\mathbb{R}^n, \mathbb{R}^n)$ with $\rho(A) < 1$. Construct a norm $\|\cdot\|_\alpha$ on $\mathbb{R}^n$ that makes $A$ a contraction. [*Hint: Consider* $\|\cdot\|_\alpha := \sup_{m \in \mathbb{N} \cup \{0\}} \alpha^{-m} \|A^m \cdot\|$ *for suitably chosen* $\alpha > 0$ *and* $\|\cdot\|$ *an arbitrary norm on* $\mathbb{R}^n$.]

**Paper 4, Section II**

**2F Analysis II**

(a) Let $\Omega \subset \mathbb{R}^n$ and $x_0 \in \Omega$. Let $d\colon \Omega \times \Omega \to \mathbb{R}$ be defined by

$$d(x,y) = \begin{cases} 0\,, & \text{if } x = y\,, \\ \|x - x_0\| + \|y - x_0\|\,, & \text{if } x \neq y\,, \end{cases}$$

where $\|\cdot\|$ denotes the usual Euclidean norm on $\mathbb{R}^n$. Is $d$ a metric on $\Omega$? Justify your answer briefly.

(b) What does it mean to say $X \subset \mathbb{R}^n$ is *path-connected*?

We say $\Omega \subset \mathbb{R}^n$ is star-shaped with star-centre $x_0 \in \Omega$ if, for every $x \in \Omega$, the line segment from $x_0$ to $x$ lies in $\Omega$, i.e. $\{(1-t)x_0 + tx\colon t \in [0,1]\} \subset \Omega$. Show that star-shaped sets are path-connected.

(c) Let $f\colon \mathbb{R}^n \to \mathbb{R}^m$. Assume that $f$ is $C^1$, and that there exists a finite $M \geqslant 0$ such that $\sup_{z\in\mathbb{R}^n} \|Df(z)\| \leqslant M$. Show that, for $\Omega \subset \mathbb{R}^n$ star-shaped with star-centre $x_0$,

$$\|f(x) - f(y)\| \leqslant M d(x,y) \qquad \forall\, x, y \in \Omega,$$

for $d$ as defined in part (a). Deduce that if $Df$ vanishes on $\Omega$, then $f$ is constant in $\Omega$.

**Paper 4, Section II**

**10F Analysis II**

What does it mean for a metric space to be *bounded*? What does it mean for a metric space to be *sequentially compact*?

Let $(X, d)$ and $(Y, \rho)$ be metric spaces, with $Y$ having at least two elements. Let $f\colon X \to Y$ be a continuous function.

(i) Show that, if $X$ is sequentially compact, then $f$ is uniformly continuous.

(ii) Show that, if $\rho$ is the discrete metric, then $f$ must be locally constant, i.e. for each $x \in X$ there exists $\delta > 0$ such that

$$d(x, y) < \delta \implies f(x) = f(y).$$

(iii) Show that, if $X$ is sequentially compact *and* $\rho$ is the discrete metric, then $f$ must be uniformly locally constant, i.e. there exists $\delta > 0$ such that

$$d(x, y) < \delta \implies f(x) = f(y) \quad \forall x, y \in X.$$

Consider the Cantor set $C \subseteq [0, 1]$ which is constructed by iteratively deleting the open middle third from a set of line segments:

$$C = \bigcap_{n=0}^{\infty} C_n, \quad \text{where} \quad C_n = \begin{cases} [0,1], & n = 0, \\ \dfrac{C_{n-1}}{3} \cup \left(\dfrac{2}{3} + \dfrac{C_{n-1}}{3}\right), & n \geqslant 1. \end{cases}$$

Figure: $C_0$ through $C_5$.

Show that $C$ is sequentially compact with respect to the induced Euclidean metric. [You may assume results from lectures if carefully stated.]

Let $f\colon C \to \{0, 1\}$, where $\{0, 1\}$ is endowed with the discrete metric, be a continuous function. Show that there exists a finite $N$ such that, for each $x \in C$, $f(x)$ is entirely determined by the first $N$ digits of $x$ in base 2.

If we replace $C$ with $C \setminus \{0\}$, is the above conclusion still true? Justify your answer. [*Hint: Consider the function $g$ that, for each $x \in C \setminus \{0\}$, returns the binary digit immediately following the first 1 in the binary expansion of $x$, e.g. $g(0\ldots0100\ldots) = 0$ and $g(0\ldots0110\ldots) = 1$.*]

**Paper 3, Section II**

**13G Complex Analysis**

(a) Let $f$ be holomorphic in a domain $D$ of the complex plane. Let $f(x+iy) = u(x,y)+iv(x,y)$ for $x+iy \in D$, where $u(x,y)$ and $v(x,y)$ are real. Derive the Cauchy–Riemann equations for $u$ and $v$.

(b) Let $u(x,y) = e^x(x\cos y - y\sin y)$. Find a holomorphic function $f\colon \mathbb{C} \to \mathbb{C}$ such that $f(0) = 0$ and $\mathrm{Re}(f(x+iy)) = u(x,y)$ for all $x, y \in \mathbb{R}$. Prove that such a function $f$ is unique.

**Paper 4, Section I**

**3G Complex Analysis**

(i) State the argument principle on a domain bounded by a simple closed curve $\gamma$. Using the argument principle, or otherwise, state and prove Rouché's theorem.

(ii) Determine the number of zeros of the polynomial $p(z) = z^6 + 15z + 1$ in the annulus $\{z \in \mathbb{C}\colon 1 < |z| < 2\}$.

(iii) Let $U \subset \mathbb{C}$ be an open set. Let $f_n\colon U \to \mathbb{C}$ be injective holomorphic functions, and suppose that $f_n \to f$ uniformly as $n \to \infty$. Prove that the function $f\colon U \to \mathbb{C}$ is either constant or injective. [*Hint: Argue by contradiction and use Rouché's theorem.*]

**Paper 1, Section I**

**3 Complex Analysis OR Complex Methods**

*This is the joint question for Complex Analysis/Complex Methods. Attempt only ONE of the sub-questions. On your answer sheet, specify the question number as either "3.1G" or "3.2C".*

**(3.1G) Complex Analysis**

(a) Let $f$ be holomorphic on the punctured disc $\{z \in \mathbb{C} \colon 0 < |z - z_0| < r\}$ for some $z_0 \in \mathbb{C}$ and $r > 0$. Define the *residue* of $f$ at $z_0$. State and prove a formula for calculating the residue at a pole of order 2 as a derivative of an appropriate function.

(b) Consider the function

$$f(z) = \frac{1}{z^2(z+1)(z+2)}.$$

Find the Laurent series expansion of $f(z)$ about 0 in the annulus

$$A = \{z \in \mathbb{C} \colon 1 < |z| < 2\}.$$

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**(3.2C) Complex Methods**

For each of the following functions of a complex variable, state the nature of the point $z = 0$, classifying any singularities. Where it is meaningful, state the residue of the function at $z = 0$.

$$\sin(z)\,, \qquad \frac{1}{\sin(z)}\,, \qquad \frac{z}{\sin(z)}\,, \qquad \sin\left(\frac{1}{z}\right), \qquad z\sin\left(\frac{1}{z}\right), \qquad \frac{1}{\sin\left(\frac{1}{z}\right)}\,.$$

**Paper 1, Section II**

**11 Complex Analysis OR Complex Methods**

*This is the joint question for Complex Analysis/Complex Methods. Attempt only ONE of the sub-questions. On your answer sheet, specify the question number as either "11.1G" or "11.2C".*

**(11.1G) Complex Analysis**

(a) Let $D = \{z \in \mathbb{C} \colon |z| < 1\}$ denote the open unit disc of the complex plane. Let $f \colon D \to D$ be a holomorphic function such that $f(0) = 0$. Show that $|f(z)| \leqslant |z|$ for all $z \in D$. Deduce that if such a function $f$ also satisfies $|f(z_0)| = |z_0|$ for some $z_0 \in D \setminus \{0\}$, then there exists $a \in \mathbb{C}$ such that $f(z) = az$ for all $z \in D$.

(b) Let $\Omega$ be the following domain defined by the intersection of two discs:

$$\Omega = \{z \in \mathbb{C} : |z| < 1\} \cap \{z \in \mathbb{C} : |z - 1| < 1\}.$$

Find, with justification, a conformal bijection $f$ mapping $\Omega$ onto the upper half-plane $\mathbb{H} = \{w \in \mathbb{C} \colon \mathrm{Im}(w) > 0\}$.

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**(11.2C) Complex Methods**

Let $z = x + iy$ and $\zeta = \xi + i\eta$ be two complex variables. Consider the map $z \mapsto f(z)$ given by the function

$$f(z) = \zeta = \frac{z + \sqrt{z^2 - c^2}}{c},$$

where $c$ is a positive real constant and the branch cut is taken along the real axis from $-c$ to $+c$.

(i) Show that the map is conformal away from the branch cut.

(ii) Verify that

$$\frac{1}{\zeta} = \frac{z - \sqrt{z^2 - c^2}}{c}$$

and use this property to express $z$ in terms of $\zeta$.

(iii) For any $R > 1$, show that the circle $|\zeta| = R$ is the image under $f$ of an ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \qquad (*)$$

and express the semi-axes $a, b > 0$ of the ellipse in terms of $c$ and $R$. Show further that the exterior of the ellipse maps to the region $|\zeta| > R$.

**[QUESTION CONTINUES ON THE NEXT PAGE]**

**Paper 1, Section II**

(iv) Let $w = u + iv$ be a complex variable. Explain why the function

$$g(\zeta) = w = \frac{\zeta - R}{\zeta + R}$$

corresponds to a conformal map of the region $|\zeta| > R$ to the right half-plane $u > 0$. [Properties of Möbius maps may be quoted without proof.]

(v) Suppose we wish to find a solution of Laplace's equation in the exterior of the ellipse $(*)$ that equals $+1$ on the upper half of the ellipse $(y > 0)$, equals $-1$ on the lower half of the ellipse $(y < 0)$ and tends to zero as $\sqrt{x^2 + y^2} \to \infty$. By considering the function $\log w$ (with the standard branch cut) and quoting any relevant properties of analytic functions, express this solution in terms of a function of $z$.

**Paper 2, Section II**

**12 Complex Analysis OR Complex Methods**

*This is the joint question for Complex Analysis/Complex Methods. Attempt only ONE of the sub-questions. On your answer sheet, specify the question number as either "12.1G" or "12.2C".*

**(12.1G) Complex Analysis**

(a) Let $0 < r < R$ and $z_0 \in \mathbb{C}$. Let $f$ be a holomorphic function on the open disc $D(z_0, R)$. Let $\gamma$ be the circle of radius $r$ centred at $z_0$, oriented anticlockwise. State and prove Cauchy's integral formula expressing $f(w)$ for any $w \in D(z_0, r)$ in terms of an integral along $\gamma$. State, without proof, a similar formula for $f'(w)$.

[You may assume Cauchy's theorem in the following form. If $U$ is a star-shaped domain and $g\colon U \to \mathbb{C}$ is continuous everywhere and holomorphic at all but finitely many points, then for any closed piecewise $C^1$ path $\gamma$ in $U$ we have $\int_\gamma g(z)\,\mathrm{d}z = 0$.]

(b) Prove that if a holomorphic function $f\colon \mathbb{C} \to \mathbb{C}$ is bounded, then $f$ is constant.

(c) Evaluate the integral

$$\int_{-\infty}^{\infty} \frac{\cos^2 x}{x^2+1}\,\mathrm{d}x.$$

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**(12.2C) Complex Methods**

(i) Let $c > 1$ be a real number. Show that the function

$$f(z) = \frac{\sqrt{z^2-1}}{z+c}$$

of a complex variable $z = x + iy$ can be made single-valued by introducing a branch cut along the real axis from $-1$ to $1$, such that $f(z) \to 1$ as $z \to \infty$. Express the limiting values of $f(z)$ above and below the branch cut in terms of $x$.

(ii) Let

$$J = \int_C f(z)\,dz\,,$$

where $C$ is a closed curve that encloses the branch cut once in a positive sense but does not enclose the point $z = -c$. Using the substitution $z = 1/\zeta$, show that

$$J = \int_{C'} g(\zeta)\,d\zeta\,,$$

where

$$g(\zeta) = -\frac{1}{\zeta^2}\frac{\sqrt{1-\zeta^2}}{1+c\zeta}$$

and $C'$ is the image of $C$ under the map $z \mapsto \zeta$. Describe how the branch cut of $g(\zeta)$ relates to that of $f(z)$. Explain why $C'$ encloses the points $\zeta = 0$ and $\zeta = -1/c$ once in a negative sense.



**[QUESTION CONTINUES ON THE NEXT PAGE]**

**Paper 2, Section II**

(iii) Use the residue theorem to show that

$$J = -2\pi i \left(c - \sqrt{c^2 - 1}\right)$$

and deduce the value of the real integral

$$I = \int_{-1}^{1} \frac{\sqrt{1 - x^2}}{x + c}\, dx\,.$$

**Paper 3, Section I**

**3C Complex Methods**

Consider the real integral

$$I = \int_0^\infty \frac{dx}{1+x^p}\,,$$

where $p > 1$ is a real number. Use a wedge-shaped contour (including the lines $\arg z = 0$ and $\arg z = 2\pi/p$) and the residue theorem to show that

$$I = \frac{\pi}{p}\,\mathrm{cosec}\left(\frac{\pi}{p}\right)\,.$$

**Paper 4, Section II**

**12C Complex Methods**

(i) Define the Laplace transform $Y(s)$ of $y(t)$. Find the Laplace transforms of $\dot{y}(t)$ and $\ddot{y}(t)$, the first and second derivatives of $y(t)$, in terms of $Y(s)$. Find the Laplace transforms of $e^{\lambda t}$ (where $\lambda$ is a complex constant) and of the Dirac distribution $\delta(t)$. [In the latter case you may assume that the lower limit of the integral defining the Laplace transform tends to 0 from below.]

(ii) Consider the ordinary differential equation

$$\ddot{y}(t) + 2\gamma\dot{y}(t) + \omega^2 y(t) = f(t)$$

for the unknown function $y(t)$ in $t \geqslant 0$, with given initial values for $y(0)$ and $\dot{y}(0)$, where $\gamma$ and $\omega$ are positive constants and $f(t)$ is a given function with Laplace transform $F(s)$.

Show that the Laplace transform $Y(s)$ of $y(t)$ is given by

$$Y(s) = G(s)H(s)\,,$$

where

$$G(s) = \frac{1}{s^2 + 2\gamma s + \omega^2}$$

and $H(s)$ is a function that you should determine.

(iii) Use the convolution theorem to express $y(t)$ in terms of the functions $g(t)$ and $h(t)$, of which $G(s)$ and $H(s)$ are the Laplace transforms, respectively.

(iv) Determine the (manifestly real) forms of $g(t)$ for $t \geqslant 0$ in the three cases $\gamma > \omega$, $\gamma < \omega$ and $\gamma = \omega$. Also determine $h(t)$ in the case $y(0) = 0$ and deduce the contribution of the initial value $\dot{y}(0)$ to the solution $y(t)$.

**Paper 1, Section II**
**14D Electromagnetism**

(a) A steady current $I$ flows along a curve $\mathcal{C}$ formed by a perfectly conducting wire. Assume that the vector potential $\mathbf{A}$ at a point $\mathbf{x}$ away from the wire is given by

$$\mathbf{A}(\mathbf{x}) \;=\; \frac{\mu_0 I}{4\pi}\int_{\mathcal{C}} \frac{d\mathbf{x}'}{|\mathbf{x}-\mathbf{x}'|}\,.$$

Show that $\mathbf{A}(\mathbf{x})$ satisfies the Coulomb gauge condition $\nabla\cdot\mathbf{A} = 0$ with appropriate boundary conditions for $\mathcal{C}$. Deduce the Biot-Savart law for the magnetic field,

$$\mathbf{B}(\mathbf{x}) \;=\; \frac{\mu_0 I}{4\pi}\int_{\mathcal{C}} \frac{d\mathbf{x}'\times(\mathbf{x}-\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|^3}\,.$$

(b) An infinite wire carrying a current $I$ runs along the $z$-axis from $z=-\infty$ to the origin $(0,\,0,\,0)$, where it turns at right angles and continues along the negative $y$-axis to $y=-\infty$. Consider the semi-infinite segment of wire along the negative $z$-axis only and use the Biot-Savart law to show that its contribution to the magnetic field is given in cylindrical polar coordinates $(r,\phi,z)$ (with $r^2 \equiv x^2+y^2$) by

$$\mathbf{B}(\mathbf{x}) \;=\; \frac{\mu_0}{4\pi}\frac{I}{r}\left(1-\frac{z}{\sqrt{r^2+z^2}}\right)\hat{\boldsymbol{e}}_\phi\,.$$

By considering Cartesian coordinates, obtain the complete solution for $\mathbf{B}$ from the wire which also includes the semi-infinite segment along the negative $y$-axis. Briefly discuss the limiting values of the magnetic field along the line $x=0$, $y=1$ in the asymptotic limits $z\to\pm\infty$.

**Paper 2, Section I**
**4D Electromagnetism**

A point charge $q$ is placed at position $(0,0,d)$ above a conducting region $z\leqslant 0$ held at zero potential $\phi=0$. Use the method of images to find the electric field $\mathbf{E}$ for $z>0$ by placing another charge $-\,q$ at position $(0,0,-d)$. Show that the surface charge density on the conductor is given by

$$\sigma(r) = -\frac{q}{2\pi}\frac{d}{(r^2+d^2)^{3/2}}\,,$$

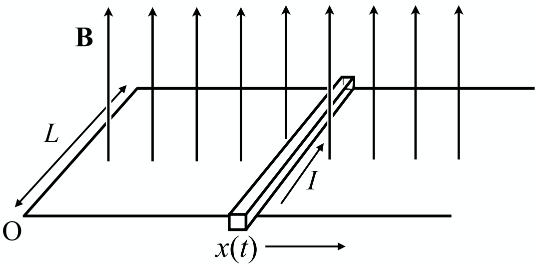
where $r^2 = x^2+y^2$. Integrate to find the total surface charge induced on the conductor.

**Paper 2, Section II**

**16D Electromagnetism**

(a) Consider a perfect conductor made up of a closed, but movable, circuit $\mathcal{C}$ in the presence of a magnetic field $\mathbf{B}$. State Faraday's Law of Induction, defining all quantities.

(b) Suppose that a rectangular circuit $\mathcal{C}$ lying in the $xy$-plane (see diagram) is made up of two parallel metal rails placed a distance $L$ apart and connected at one end by a straight wire, with all three elements perfectly conducting. Lying perpendicular to these rails is a movable conducting bar (mass $m$) with resistance $R$ that can travel without friction in the $x$-direction. A uniform time-varying magnetic field $\mathbf{B}$ points along the $z$-axis as shown. [*You may neglect any magnetic fields due to current flow.*]




(i) Find the current $I(t)$ that passes through the bar and hence find the Lorentz force acting on the bar. For a decaying magnetic field, falling as $B_z = B_0/(1+t)$, $(t \geqslant 0)$, show that the acceleration of the bar at $x(t)$ satisfies the differential equation:

$$\ddot{x} + \frac{\alpha}{(1+t)^2}\,\dot{x} - \frac{\alpha}{(1+t)^3}\,x = 0\,, \qquad (*)$$

where you should specify $\alpha$ in terms of $B_0$, $L$, $R$ and $m$.

(ii) For simplicity, set $\alpha = 1$ in $(*)$. Solve for the motion of the bar, assuming it starts initially at position $x(0) = x_0$ and at rest $\dot{x}(0) = 0$.

[*Hint: Set $u = 1 + t$. Note that $x_1(u) = u$ is a solution, so seek a second in the form $x_2(u) = u\,w(u)$.*]

(iii) Describe the motion of the bar and calculate its asymptotic velocity for $t \gg 1$. What is the total mechanical work done on the bar? Given the orientation of the magnetic field $\mathbf{B}$, specify the direction of the induced current $I$ around the circuit. Briefly explain why this and the subsequent motion are consistent with Lenz's Law.

**Paper 3, Section II**

**15D Electromagnetism**

(a) A particle of mass $m$ and charge $q$ moves relativistically in the electromagnetic fields $\mathbf{E}$ and $\mathbf{B}$. The covariant Lorentz force law is given by $\frac{dP^\mu}{d\tau} = qF^{\mu\nu}U_\nu$. Define all the quantities in this expression and show that it implies both

$$\frac{d\mathbf{p}}{dt} = q(\mathbf{E} + \mathbf{u} \times \mathbf{B}), \quad \text{and} \quad \frac{d}{dt}\left(\gamma(u)mc^2\right) = q\,\mathbf{E}\cdot\mathbf{u}\,,$$

with $u = |\mathbf{u}|$. State the two Lorentz invariants that can be obtained from $\mathbf{E}$ and $\mathbf{B}$.

(b) Suppose that the electric and magnetic fields are perpendicular in the laboratory frame and given by $\mathbf{E} = E\,\hat{\mathbf{e}}_x$, $\mathbf{B} = B\,\hat{\mathbf{e}}_z$, with $E < cB$. Assume a Lorentz boost with velocity $\mathbf{v}$ (with $v = |\mathbf{v}|$ and $\hat{\mathbf{v}} = \mathbf{v}/v$) of uniform electric $\mathbf{E}$ and magnetic $\mathbf{B}$ fields is given by

$$\begin{aligned} \mathbf{E}' &= \gamma(v)(\mathbf{E} + \mathbf{v} \times \mathbf{B}) - (\gamma(v) - 1)\,(\mathbf{E}\cdot\hat{\mathbf{v}})\,\hat{\mathbf{v}}\,, \\ \mathbf{B}' &= \gamma(v)\left(\mathbf{B} - \frac{1}{c^2}\mathbf{v} \times \mathbf{E}\right) - (\gamma(v) - 1)\,(\mathbf{B}\cdot\hat{\mathbf{v}})\,\hat{\mathbf{v}}\,. \end{aligned}$$

(i) Show that there exists an inertial frame in which the electric field vanishes $\mathbf{E}' = 0$ and find the velocity $\mathbf{v}$ of that frame relative to the laboratory frame. Determine the transformed magnetic field $\mathbf{B}'$.

(ii) In the frame where $\mathbf{E}' = 0$, assume that the particle velocity is perpendicular to the uniform magnetic field $\mathbf{B}'$. Show that this results in a circular trajectory and find the angular frequency and radius.

(iii) Transform this circular trajectory back to the laboratory frame and show that the particle undergoes a superposition of circular motion and a uniform drift with velocity approximately (for small $|\mathbf{v}| \ll c$) given by

$$\mathbf{u}_d \approx \frac{\mathbf{E} \times \mathbf{B}}{B^2}\,.$$

Briefly explain why this drift is independent of the particle's mass and charge. How is this solution distinct from the helical trajectory possible in the $\mathbf{E}' = 0$ and non-zero $\mathbf{B}'$ frame?

**Paper 4, Section I**

**5D Electromagnetism**

A linearly polarized electromagnetic plane wave, described by the real part of the fields

$$\mathbf{E}_{\text{inc}}(\mathbf{x},t) = E_0\,\hat{\mathbf{e}}_x\,e^{i(kz+\omega t)}, \qquad \mathbf{B}_{\text{inc}}(\mathbf{x},t) = -\frac{E_0}{c}\,\hat{\mathbf{e}}_y\,e^{i(kz+\omega t)}, \quad (\omega, k > 0)\,,$$

propagates through vacuum for $z > 0$ and is incident normal to a perfect conductor occupying the region $z \leqslant 0$.

What is the dispersion relation required for this wave to satisfy Maxwell's equations?

Write down the boundary conditions that the electric and magnetic fields must satisfy at the surface of the conductor.

Assuming that a reflected plane wave is produced in the region $z > 0$, use the boundary conditions to determine the reflected electric $\mathbf{E}_{\text{ref}}$ and magnetic $\mathbf{B}_{\text{ref}}$ fields.

What is the resulting surface current induced by the wave?

**Paper 1, Section II**

**15A Fluid Dynamics**

An inviscid, incompressible fluid flow is governed by the Euler equations

$$\rho\left[\frac{\partial \boldsymbol{u}}{\partial t} + (\boldsymbol{u}\cdot\nabla)\boldsymbol{u}\right] = -\nabla p\,, \qquad \nabla\cdot\boldsymbol{u} = 0\,,$$

where the density $\rho$ is a constant.

(a) Derive the evolution equation satisfied by the vorticity $\boldsymbol{\omega} = \nabla\times\boldsymbol{u}$ and express it in a form in which the incompressibility of the flow is explicit.

(b) Consider the velocity field $\boldsymbol{u}$ whose Cartesian components are

$$\boldsymbol{u} = (-\frac{1}{2}\alpha(t)x, -\frac{1}{2}\alpha(t)y, \alpha(t)z),$$

where $\alpha(t)$ is a given function of time.

(i) Show that the flow is incompressible.

(ii) Suppose that a spatially uniform but temporally varying rotation is added to the flow so that

$$\boldsymbol{\omega}(t) = (0, 0, \omega(t)).$$

Using the vorticity equation derived in part (a), obtain an equation for the rate of change of $\omega(t)$.

(iii) Solve the obtained equation for $\omega(t)$ in terms of $\alpha(t)$ and the initial value $\omega(0) = \omega_0$. Hence describe how the vorticity magnitude changes when $\alpha(t) > 0$ and explain your result in terms of vortex stretching.

**Paper 2, Section I**

**5A Fluid Dynamics**

Consider the 2D flow with velocity field

$$\boldsymbol{u}(x, y, t) = \big(\alpha y + yt,\ 1\big),$$

where $\alpha \geqslant 0$ is a parameter, $x$ and $y$ are spatial coordinates, and $t \geqslant 0$ is time.

(i) Compute the streamfunction $\psi(x, y, t)$ for this flow.

(ii) Sketch the approximate streamlines at $t = 0$ and $t = 1$.

(iii) Explain why the flow is approximately steady when $0 \leqslant t \ll \alpha$ and determine the streamlines in this limit.

**Paper 3, Section I**

**7A Fluid Dynamics**

A spherical bubble of radius $a$ translates with time-dependent velocity $\boldsymbol{V}(t)$ in an inviscid fluid of constant density $\rho$ in the absence of gravity. The resulting irrotational flow has the velocity potential

$$\phi(\boldsymbol{x},t) = \frac{a^3}{2r^3}\boldsymbol{V}(t)\cdot\boldsymbol{x},$$

where $\boldsymbol{x}$ is the position vector and $r = |\boldsymbol{x}|$ is its modulus.

(i) Obtain the pressure acting on the bubble surface.

(ii) Hence obtain the total force $\boldsymbol{F}$ acting on the bubble.

(iii) When an arbitrary time-dependent external force $\boldsymbol{G}(t)$ acts on a bubble of mass $m$, show that the motion of the bubble satisfies $m_*\boldsymbol{a} = \boldsymbol{G}(t)$, where $\boldsymbol{a}$ is the acceleration of the bubble and $m_*$ is a constant. Obtain an expression for $m_*$ and interpret the result.

**Paper 3, Section II**

**16A Fluid Dynamics**

Consider the incompressible, 2D parallel viscous flow of a Newtonian fluid of constant density $\rho$ and dynamic viscosity $\mu > 0$ in a long, straight, horizontal channel of height $H > 0$. Use Cartesian coordinates $(x, y, z)$ with channel walls located at $y = 0$ and $y = H$. The flow is driven by a pressure gradient in the $x$-direction and has Cartesian components

$$\mathbf{u} = (u(y,t), 0, 0).$$

(i) By considering the balance of momentum in an elementary volume, derive the governing equation

$$\rho \frac{\partial u}{\partial t} = -\frac{\partial p}{\partial x} + \mu \frac{\partial^2 u}{\partial y^2},$$

for the rate of change of momentum in the $x$-direction.

(ii) Consider steady flow. Show that $\partial p/\partial x$ must be independent of $x$, and hence can be expressed as $\partial p/\partial x = -G$, for a constant $G$ which may be assumed to be positive. Solve for the velocity profile $u(y)$ subject to the no-slip boundary conditions at $y = 0$ and $y = H$.

(iii) Determine the volumetric flow rate per unit width

$$Q = \int_0^H u(y)\, dy$$

in terms of $G$, $H$, and $\mu$.

(iv) Now suppose that at time $t = 0$ the pressure gradient is suddenly removed, so that $\partial p/\partial x = 0$ for $t > 0$. Solve for the time-dependent velocity $u(y, t)$, assuming that the initial condition is the steady solution found in part (ii). You may assume a solution in the form of a Fourier sine series.

**Paper 4, Section II**

**16A Fluid Dynamics**

A layer of inviscid fluid of constant density $\rho$ and equilibrium depth $H$ lies on a horizontal surface in a frame rotating with angular velocity $\Omega\hat{\mathbf{z}}$. Gravity acts vertically downward with acceleration $g$. Assume the motion is hydrostatic and independent of the vertical coordinate.

(a) Starting from the Euler equations in a rotating frame,

$$\rho\left(\frac{D\mathbf{u}}{Dt} + 2\boldsymbol{\Omega}\times\mathbf{u}\right) = -\nabla p + \rho\mathbf{g},$$

derive the shallow-water equations in the form

$$\frac{\partial\mathbf{u}}{\partial t} + \mathbf{u}\cdot\nabla\mathbf{u} + f\hat{\mathbf{z}}\times\mathbf{u} = -g\nabla h, \qquad \frac{\partial h}{\partial t} + \nabla\cdot(h\mathbf{u}) = 0,$$

where $h(x,y,t)$ is the free-surface height and $f = 2\Omega$.

(b) Define the potential vorticity

$$q = \frac{\zeta + f}{h}, \qquad \zeta = \hat{\mathbf{z}}\cdot(\nabla\times\mathbf{u}).$$

Show that $q$ satisfies

$$\frac{Dq}{Dt} = 0.$$

(c) The shallow-water equations are inviscid and unforced.

(i) Show that the quantity

$$E = \frac{1}{2}\rho h|\mathbf{u}|^2 + \frac{1}{2}\rho g h^2$$

satisfies a local conservation law of the form

$$\frac{\partial E}{\partial t} + \nabla\cdot\mathbf{J} = 0,$$

for some flux $\mathbf{J}$ to be determined. Interpret the quantity $E$.

(ii) Hence show that for a domain $A$, on whose boundary $\mathbf{u}\cdot\mathbf{n} = 0$, the total amount of $E$,

$$\int_A \left(\frac{1}{2}\rho h|\mathbf{u}|^2 + \frac{1}{2}\rho g h^2\right)\, dA,$$

is conserved.

(iii) Consider an initial state consisting of a localized height perturbation in an otherwise motionless fluid. Explain why the system cannot evolve to a state of rest with uniform depth if the total amount of $E$ is conserved.

**Paper 1, Section II**

**9E Groups, Rings and Modules**

Let $R$ be a non-zero commutative ring with a multiplicative identity.

(i) Recall that an element $r \in R$ is nilpotent if $r^n = 0$ for some $n > 0$. Let $N$ be the set of nilpotent elements of $R$.

Show that $N$ is an ideal of $R$. What are the nilpotent elements in $R/N$?

(ii) Show that a polynomial $f(X) = a_0 + a_1 X + \cdots + a_n X^n \in R[X]$ is nilpotent if and only if $a_0, a_1, \ldots, a_n \in N$.

(iii) Show that if $u \in R$ is a unit and $b \in N$ then their sum $u + b$ is a unit. Deduce that if a polynomial $f(X) = a_0 + a_1 X + \cdots + a_n X^n \in R[X]$ is such that $a_0$ is a unit in $R$ and $a_1, \ldots, a_n \in N$, then $f(X)$ is a unit.

(iv) Show that if $f(X) \in R[X]$ is a zero divisor then there is some non-zero $r \in R$ such that $rf(X) = 0$. [*Hint: Suppose not, and choose a non-zero $g(X) \in R[X]$ of minimal degree such that $f(X)g(X) = 0$.*]

Consider a polynomial $f(X) = a_0 + a_1 X + \cdots + a_n X^n \in \mathbb{Z}[X]$ and let $m \geqslant 2$. Write $f(X) \bmod m$ when we consider its coefficients modulo $m$. Deduce that if $f(X) \bmod m$ is a zero divisor in $(\mathbb{Z}/m\mathbb{Z})[X]$ then $\gcd(a_0, \ldots, a_n, m) > 1$.

**Paper 2, Section I**

**1E Groups, Rings and Modules**

(a) Let $n \geqslant 3$. Let $R = \mathbb{Z}[\sqrt{-n}]$. By factorising $n$ or $n+1$, as appropriate, show that $R$ is not a UFD.

(b) Can the ideal $I = (4, 2X, X^2)$ of $\mathbb{Z}[X]$ be generated by two elements? Justify your answer.

For each $n \geqslant 3$, write down (without proof) an ideal of $\mathbb{Z}[X]$ that is generated by $n$ elements but not by $n-1$ elements.

**Paper 2, Section II**

**9E Groups, Rings and Modules**

In this question, $p$ is a prime number.

(a) State Sylow's first theorem. State and prove Sylow's second theorem.

(b) Let $G$ be a finite group.

(i) Suppose that $p^d$ divides $|G|$ for some $d > 0$. Show that each subgroup of $G$ with order $p^{d-1}$ has index $p$ in a subgroup of $G$. [*Hint: Cauchy's theorem may be assumed. If $d > 1$, let $H$ be a subgroup of $G$ of order $p^{d-1}$ and consider the left multiplication action of $H$ on $G/H$.*]

(ii) Deduce that if $p^d$ divides $|G|$ then there is a subgroup of $G$ of order $p^d$.

(iii) Deduce further that if $H$ is a $p$-subgroup of $G$ with normaliser $N(H)$, then the indices satisfy

$$|G : H| \equiv |N(H) : H| \pmod{p}.$$

(c) Let $G$ be a finite group and let $p$ be a divisor of $|G|$. Is it true that $p$-subgroups of a common non-maximal order are all conjugate? Briefly justify your answer.

**Paper 3, Section I**

**1E Groups, Rings and Modules**

Let $A$ be the abelian group generated by two elements $x$, $y$, subject to the relation $6x + 9y = 0$.

(i) Define $A$ as an appropriate quotient of a free abelian group of rank 2.

(ii) Prove that $A$ itself is not a free abelian group.

(iii) Determine, with justification, the exact structure of $A$ as a direct sum of cyclic groups.

**Paper 3, Section II**

**10E Groups, Rings and Modules**

In this question, $R$ is a non-zero commutative ring with a multiplicative identity.

(a) Prove that an $R$-module is finitely-generated if and only if it is a quotient of a free module $R^k$, for some $k > 0$.

(b) Let $M$ be a finitely-generated $R$-module, let $I$ be an ideal of $R$ and let $\varphi$ be an $R$-module homomorphism from $M$ to itself such that

$$\varphi(M) \subseteq I \cdot M = \{m \in M : m = rm' \text{ with } r \in I,\ m' \in M\}.$$

(i) Prove that $\varphi$ satisfies an equation of the form

$$\varphi^n + a_{n-1}\varphi^{n-1} + \cdots + a_1\varphi + a_0 = 0,$$

where each $a_i \in I$ and for some $n$. [You may assume that if $A$ is a matrix over $R$, then the adjugate matrix satisfies $\operatorname{adj}(A)A = (\det A)E$, with $E$ the identity matrix.]

(ii) Deduce that if $M$ satisfies $I \cdot M = M$, then there exists some $x \in R$ such that $x - 1 \in I$ and $xM = 0$.

Give an example of a finitely-generated $\mathbb{Z}$-module $M$ and a proper ideal $I$ of $\mathbb{Z}$ satisfying the hypothesis $I \cdot M = M$, and for your example, give an explicit such element $x$.

(iii) Define $J$ as the intersection of all the maximal ideals of $R$. Show that $r \in J$ if and only if $1 - rs$ is a unit in $R$ for all $s \in R$. [You may assume that every non-unit of $R$ is contained in a maximal ideal.]

Deduce from part (ii) that if $M$ is a finitely-generated $R$-module and $I$ is an ideal of $R$ contained in $J$, then $I \cdot M = M$ implies $M = 0$.

(iv) Suppose that $M$ is a finitely-generated $R$-module. Let $f : M \to M$ be a surjective $R$-homomorphism. Show that $f$ is injective. [*Hint: You may wish to view $M$ as a $R[X]$-module via the rule $p(X).m = p(f)(m)$ for $p(X) \in R[X]$ and $m \in M$ (you are not asked to verify this).*]

**Paper 4, Section II**

**9E Groups, Rings and Modules**

In this question rings are commutative with a multiplicative identity.

(a) What is a *Noetherian* ring? You should state your definition using two equivalent statements about ideals.

(i) Show that the image of a Noetherian ring under a ring homomorphism is always Noetherian. Use this to give an example of a Noetherian integral domain that is not a UFD. Is every UFD Noetherian? In both cases give brief justification of your answer.

(ii) Let $R$ be a Noetherian ring. Show that each surjective ring homomorphism $R \to R$ is an injection (and thus is an isomorphism). Give an example of a non-Noetherian ring $S$ and a surjective homomorphism $S \to S$ which is not injective. Is it always true that every injective ring homomorphism from a Noetherian ring to itself is an isomorphism? In both cases give brief justification of your answer.

(b) Let $\mathrm{Int}(\mathbb{Z}) = \{f \in \mathbb{Q}[X] : f(\mathbb{Z}) \subseteq \mathbb{Z}\}$ be the subring of $\mathbb{Q}[X]$ consisting of integer-valued polynomials. You may assume that $\mathrm{Int}(\mathbb{Z})$ is also an (additive) abelian group. Let $f_0(X) = 1$. For any $n \in \mathbb{N}$ define the polynomial

$$f_n(X) = \frac{X(X-1)\cdots(X-n+1)}{n!}.$$

(i) Verify that $f_n(X) \in \mathrm{Int}(\mathbb{Z})$.

(ii) Show that $\{f_n(X) : n \geqslant 0\}$ is a $\mathbb{Z}$-basis of $\mathrm{Int}(\mathbb{Z})$, and hence that $\mathrm{Int}(\mathbb{Z})$ is a free $\mathbb{Z}$-module.

(iii) Is $\mathrm{Int}(\mathbb{Z})$ Noetherian? Justify your answer. [*Hint: You may wish to consider polynomials of the form $f_p(X)$ where $p$ is any prime number.*]

**Paper 1, Section I**

**1G Linear Algebra**

Let $V = P_2(\mathbb{R})$ be the vector space of polynomials with real coefficients of degree at most 2.

(a) Show that

$$\langle p, q\rangle = \int_0^1 p(t)q(t)\,\mathrm{d}t$$

defines an inner product on $V$.

(b) Let $W$ be the subspace of $V$ spanned by the polynomials $\{1, t\}$. Find an orthonormal basis for $W$.

**Paper 1, Section II**

**8E Linear Algebra**

(a) Describe (without proof) what it means to put an $n \times n$ complex matrix into *Jordan normal form*. Explain (without proof) the sense in which the Jordan normal form is unique.

(b) Let $V_2$ be the set of complex polynomials in $z, w$ of degree at most 2 in each variable. You may assume this is a 9-dimensional vector space over $\mathbb{C}$, with basis $\{z^i w^j : 0 \leqslant i, j \leqslant 2\}$.

(i) For $f \in V_2$, consider the map $D : V_2 \to V_2$ given by

$$D(f)(z, w) = \frac{\partial f}{\partial z} + \frac{\partial f}{\partial w}.$$

Explain briefly why 0 is the only eigenvalue of $D$. Find the corresponding eigenspace.

(ii) Determine the Jordan normal form of $D$.

(c) Let $A$ be an $n \times n$ complex matrix and let $g(X)$ be a polynomial with complex coefficients.

(i) By considering the Jordan normal form of $A$, or otherwise, show that if the eigenvalues of $A$ are $\lambda_1, \ldots, \lambda_n$ then the eigenvalues of $g(A)$ are $g(\lambda_1), \ldots, g(\lambda_n)$.

(ii) Let $B = \begin{pmatrix} a & d & c & b \\ b & a & d & c \\ c & b & a & d \\ d & c & b & a \end{pmatrix}$ be a complex matrix. Express $B$ in the form $g(A)$ for a certain polynomial $g$ that you should determine, and with $A = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$. Find the eigenvalues of $B$.

[*Hint: Compute powers of $A$.*]

**Paper 2, Section II**

**8E Linear Algebra**

In this question $V$ is a finite-dimensional vector space over $\mathbb{C}$.

(a) Suppose $\alpha : V \to V$ is an endomorphism and let $W$ be a subspace of $V$. We say that $W$ is $\alpha$-stable if $\alpha(W) \leqslant W$. Assume that $W$ is $\alpha$-stable.

(i) Let $m_\alpha(X)$ be the minimal polynomial of $\alpha$. Show that the induced linear maps $\alpha_W : W \to W$ and $\alpha_{V/W} : V/W \to V/W$ have minimal polynomials such that their least common multiple is a factor of $m_\alpha(X)$ and their product is divisible by $m_\alpha(X)$.

(ii) Write down an example to show that $m_\alpha(X)$ is not always equal to the least common multiple. Write down an example to show that $m_\alpha(X)$ is not always equal to the product.

(iii) Deduce that if $\alpha$ is diagonalisable, then $\alpha_W$ and $\alpha_{V/W}$ are diagonalisable. Is the converse of this statement true? Briefly justify your answer.

(b) Let $r \geqslant 2$. Suppose that $\alpha_1, \ldots, \alpha_r$ are (pairwise) commuting endomorphisms on $V$ and that each $\alpha_i$ is diagonalisable. Using induction on $r$, or otherwise, show that they are simultaneously diagonalisable, namely, there is a basis of $V$ consisting of simultaneous eigenvectors for all the $\alpha_i$. [*Hint: You may wish to begin by showing that certain eigenspaces are $\alpha_i$-stable for all $i$.*]

(c) Let $\mathcal{A} = \{\alpha_i\}_{i \in I}$, where $I$ is an index set that is not necessarily finite, denote a family of pairwise commuting endomorphisms of $V$. Suppose that each $\alpha_i$ is diagonalisable on $V$. Are the members of $\mathcal{A}$ simultaneously diagonalisable? Justify your answer.

**Paper 3, Section II**
**9G Linear Algebra**

(a) Let $V$ be a finite-dimensional real vector space with inner product $\langle \cdot, \cdot \rangle$.

(i) Define the *dual space* $V^*$ and the *dual basis* associated with a basis $\{v_1, \ldots, v_n\}$ of $V$. Prove that the map $\phi \colon V \to V^*$ given by $\phi(u)(v) = \langle u, v \rangle$ is an isomorphism between $V$ and $V^*$.

(ii) Prove that the map $\psi \colon V \to V^{**}$ defined by $\psi(v)(\phi) = \phi(v)$ is an isomorphism between $V$ and $V^{**}$.

(iii) Define the *adjoint* $T^*$ of a linear operator $T \colon V \to V$. Prove that, for every linear operator $T$, the adjoint $T^*$ exists and is unique.

(b) Let $V$ be a finite-dimensional real vector space, and let $T \colon V \to V$ be a linear operator.

(i) Prove that if $T$ is diagonalisable, then there exists an inner product on $V$ with respect to which $T$ is self-adjoint.

(ii) We say that a set $S \subset V$ is $T$-invariant if $T(S) \subset S$. Prove that if $T \colon V \to V$ is self-adjoint with respect to some inner product, then every $T$-invariant subspace has a $T$-invariant orthogonal complement.

**Paper 4, Section I**
**1E Linear Algebra**

Let $\mathbb{F}_p$ be the field of integers modulo $p$, where $p$ is a prime number. Let $V$ be a vector space of dimension $n$ over $\mathbb{F}_p$.

How many vectors are there in $V$? How many (ordered) bases does $V$ have? How many bijective endomorphisms of $V$ are there? How many $k$-dimensional subspaces are there in $V$? Justify your answers.

**Paper 4, Section II**

**8G Linear Algebra**

(a) Let $A$ be a complex Hermitian matrix. Prove that all the eigenvalues of $A$ are real.

(b) Let $C$ be a positive definite Hermitian matrix. Prove that there exists a positive definite Hermitian matrix $B$ such that $B^2 = C$.

(c) Let

$$H = \begin{pmatrix} 2 & i \\ -i & 2 \end{pmatrix}.$$

Find a unitary matrix $U$ such that $U^{-1}HU$ is diagonal.

(d) Let $T$ be a self-adjoint operator on a finite-dimensional inner product space $(V, \langle \cdot\,, \cdot \rangle)$. Let $\lambda_{\max}$ be the largest eigenvalue of $T$. Prove that

$$\lambda_{\max} = \sup_{v \in V \setminus \{0\}} \frac{\langle Tv, v \rangle}{\langle v, v \rangle}.$$

**Paper 1, Section II**

**18H Markov Chains**

(a) Let $(X_n)_{n\geqslant 0}$ be a Markov chain with transition matrix $P$. What is a *stopping time* of $(X_n)_{n\geqslant 0}$? What is the *strong Markov property*?

(b) The chatbot for dogs, ChatGPET 1.0, produces sequences consisting of three sounds: a bark, a growl and a whine. Given a sound $X_0$ of this form initiated by a dog, at time $t$ for $t = 1, 2, \ldots$, the chatbot produces a sound $X_t$ such that $X_0, X_1, \ldots$ is a Markov chain with transition matrix

$$P = \begin{pmatrix} 0 & 1/2 & 1/2 \\ 2/3 & 0 & 1/3 \\ 2/3 & 1/3 & 0 \end{pmatrix}.$$

Here rows 1, 2 and 3 correspond to bark, growl and whine, respectively. Suppose $X_0 = \text{bark}$. Determine the expected time for ChatGPET 1.0 to produce:

(i) a growl;

(ii) a growl immediately followed by a whine.

(c) It was found that dogs find too many barks in the conversation over-stimulating, but this can be mitigated by a number of whines. The newest version, ChatGPET 2.0, therefore includes the following additional safety feature: writing $N_b$ for the number of barks produced by the chatbot, and $N_w$ for the number of whines, when $N_b - N_w \geqslant m$ for some $m \in \mathbb{N}$, the conversation is terminated. By appropriately augmenting the state space of the Markov chain in part (b), find the expected length of the sequence produced by ChatGPET 2.0.

[*Hint: It may help to set $\beta, \gamma, \omega$ to be the expected times for $N_b - N_w$ to increase by 1 when starting with a bark, growl and whine respectively.*]

**Paper 2, Section II**

**18H Markov Chains**

Let $P$ be a transition matrix on state space $I$. What does it mean for a distribution $\pi$ to be an *invariant distribution*? Consider an irreducible Markov chain on a finite state space with invariant distribution $\pi$. State the relationships between $\pi$ and

(i) the expected return time to a state $i$, and

(ii) the expected time spent in a state $i$ between visits to a state $j$.

A burglar is trying to guess a PIN given by a pair of digits in $\{0, \ldots, 7\}^2$. Rather than trying each possibility systematically, the burglar, in a panic, instead tries a random sequence of PINs. At each step he chooses one of the 2 components with probability $1/2$, and uniformly at random changes the digit to one of the 7 other possibilities. Suppose the burglar initially tries $(6, 0)$.

(a) What is the expected number of steps before the burglar tries $(6, 0)$ again?

(b) What is the expected number of times that both digits will be odd before it is $(6, 0)$ again?

(c) What is the expected number of steps that the burglar would take to reach $(6, 7)$?

(d) After the first burglar has left, a second burglar arrives who modifies the first burglar's strategy as follows: after selecting a component, rather than choosing from the remaining digits uniformly at random, he picks each of 6 and 7 (if it is not the current digit) twice as often as the remaining options. Thus, for instance, the transition from $(6, 0)$ to $(7, 0)$ has probability $\frac{1}{2} \times \frac{2}{8}$, and that from $(0, 0)$ to $(7, 0)$ has probability $\frac{1}{2} \times \frac{2}{9}$. The second burglar initially tries $(0, 0)$. What is the expected number of times he tries $(6, 7)$ before he tries $(0, 0)$ again?

**Paper 3, Section I**

**8H Markov Chains**

Suppose $(X_n)_{n\geqslant 0}$ is a Markov chain with state space $I$ and transition matrix $P$. What does it mean for $C \subseteq I$ to be a *communicating class*? What does it mean for $C$ to be *closed*?

Consider the Markov chain with state space $\{1,\ldots,6\}$ and transition matrix

$$P = \begin{pmatrix} 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{6} & 0 & \frac{1}{6} \\ \frac{1}{3} & \frac{2}{3} & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{3} & \frac{1}{3} & 0 & \frac{1}{3} & 0 \\ 0 & 0 & 0 & \frac{1}{3} & \frac{2}{3} & 0 \\ 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 & 0 \end{pmatrix}.$$

Determine the communicating classes. Which of these are closed?

What is the probability that the chain ever reaches 2, given that it starts in 1?

**Paper 4, Section I**

**7H Markov Chains**

After each minute, a viewer watching television either continues to watch the same channel with probability $0 < 1-\alpha < 1$, or picks randomly from any of the other $m-1$ channels available. You observe the viewer at a time when they are watching the BBC1 channel. Determine the probability that they are watching the BBC2 channel (one of the other channels available) in an hour's time.

**Paper 1, Section II**

**12D Methods**

(i) Define the two functions $f(x)$ and $F(x)$ on the domain $x \in [0,1)$ as:

$$f(x) \;=\; 1-2x\,, \qquad F(x) \;=\; x(1-x)\,.$$

Now consider the odd extension of these functions to the domain $[-1,1)$, labelled respectively as $\bar{f}(x)$, $\bar{F}(x)$. Determine the coefficients for the Fourier series for both of these odd functions on $[-1,1)$. Compare the rates of convergence of the series and offer a brief explanation.

(ii) Consider the eigenfunction problem

$$\mathcal{L}\,y \;\equiv\; -\,y'' - 2y' \;=\; \lambda y\,,$$

with boundary conditions $y(0)=0$ and $y(1)=0$. Demonstrate that the eigenvalues are given by $\lambda = n^2\pi^2+1$ for $n=1,2,3,\ldots$ and determine the associated eigenfunctions $y_n(x)$. Put $\mathcal{L}\,y = \lambda y$ in Sturm-Liouville form and obtain the orthogonality relation of these eigenfunctions.

Now consider the inhomogeneous problem,

$$\mathcal{L}\,y \;\equiv\; -\,y'' - 2y' \;=\; e^{-x}f(x)\;, \qquad y(0)=y(1)=0\,,$$

where the function $f(x) = 1-2x$ is the same as in part (i). Assuming the set of eigenfunctions $\{y_n(x)\}$ is complete, seek a series solution of the form $y(x) = \sum_n a_n\, y_n(x)$, using an appropriate expansion for the source term. Explicitly evaluate the series coefficients $a_n$ for this solution.

**Paper 2, Section I**

**3D Methods**

The Fourier series for the real-valued function $f(x)$ on $(-1,1)$ is given by

$$f(x) \;=\; \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos n\pi x + b_n \sin n\pi x\right).$$

(i) Derive Parseval's identity, assuming the standard orthogonality relations

$$\int_{-1}^{1} \cos n\pi x \, \cos m\pi x \, dx \;=\; \delta_{mn}\,, \qquad \int_{-1}^{1} \sin n\pi x \, \sin m\pi x \, dx \;=\; \delta_{mn}\,,$$

$$\int_{-1}^{1} \sin n\pi x \, \cos m\pi x \, dx \;=\; 0\,,$$

where $n, m$ are non-zero integers.

(ii) The Fourier series for $f(x) = x^2$ is given by

$$x^2 = \frac{1}{3} + \frac{4}{\pi^2} \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \cos n\pi x, \qquad -1 < x < 1.$$

Use Parseval's identity in this case to evaluate the series,

$$\sum_{n=1}^{\infty} \frac{1}{n^4}\,.$$

**Paper 2, Section II**
**14D Methods**

The wave equation in polar coordinates for a stretched membrane $u(r,\theta,t)$ on the unit disc is given by

$$\frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2},$$

with Dirichlet boundary conditions $u(1,\theta,t)=0$ and initial conditions given by $u(r,\theta,0)=\phi(r,\theta)$, and $\frac{\partial u}{\partial t}(r,\theta,0)=\psi(r,\theta)$.

(i) By using the method of separation of variables $u(r,\theta,t)=R(r)\Theta(\theta)T(t)$, show that the radial equation (Bessel's equation) can be written in Sturm-Liouville form as

$$-\frac{d}{dr}\left[r\frac{dR}{dr}\right]+\frac{m^2}{r}R=\lambda r R\,, \qquad (*)$$

where $m$ (an integer) and $\lambda$ are separation constants.

(ii) After making the substitution $z=\sqrt{\lambda}r$ in $(*)$ to eliminate $\lambda$, use the Frobenius method $R=z^p\sum_{k=0}^{\infty}a_k\,z^k$ to find a recurrence relation between the $a_k$ and the $a_{k-2}$ coefficients of the power series solution which is non-singular at the origin ($r=0$).

(iii) You may assume the following expression for the resulting power series expansion of the Bessel function $J_m(z)$ [*you are not required to derive this*]:

$$J_m(z)=\left(\frac{z}{2}\right)^m\sum_{n=0}^{\infty}\frac{(-1)^n}{n!\,(n+m)!}\left(\frac{z}{2}\right)^{2n}\,, \qquad (m\geqslant 0).$$

Deduce the following recurrence relations

$$\frac{d}{dz}(z^m J_m(z)) = z^m J_{m-1}(z) \qquad \text{and} \qquad \frac{d}{dz}(z^{-m}J_m(z)) = -z^{-m}J_{m+1}(z)\,.$$

(iv) The $J_m(z)$ are oscillatory functions with zeros at positions $z=j_{m1}, j_{m2}, j_{m3}, \ldots$ with $0<j_{m1}<j_{m2}<\ldots$. Assume that $J_m(j_{mn}r)$ are the radial eigenfunctions for the wave equation with Dirichlet boundary conditions on the unit disc, with normalisation given by $\mathcal{N}_{mn}=\int_0^1[J_m(j_{mn}r)]^2\,rdr=\frac{1}{2}[J'_m(j_{mn})]^2$.

Write down a general expansion for a purely radial solution for the membrane wave equation specified above on the unit disc. Give expressions for the specific solution satisfying the initial conditions $\phi(r,\theta)=1-r$ and $\psi(r,\theta)=0$ and integrate to find the eigenfunction coefficients explicitly.

**Paper 3, Section I**

**5A Methods**

The Fourier transform of a function $f(x)$ on the real line is defined by

$$\tilde{f}(k) = \int_{-\infty}^{\infty} f(x) \exp(-ikx)\, dx.$$

(i) Compute the Fourier transform of the function

$$g_\sigma(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{x^2}{2\sigma^2}\right)$$

by obtaining a differential equation for $\tilde{g}_\sigma(k)$ and solving it.

(ii) State without proof the convolution theorem for two functions $f(x)$ and $g(x)$. Deduce that the convolution $g_{\sigma_1}(x)$ and $g_{\sigma_2}(x)$ is given by

$$\int_{-\infty}^{\infty} g_{\sigma_1}(y) g_{\sigma_2}(x-y)\, dy = g_{\sigma_3}(x),$$

where $\sigma_3^2 = \sigma_1^2 + \sigma_2^2$.

**Paper 3, Section II**

**14A Methods**

(a) Verify that

$$\nabla^2 \hat{G}(\mathbf{x}, \mathbf{y}) = \delta(\mathbf{x} - \mathbf{y}) \,,$$

where

$$\hat{G}(\mathbf{x}, \mathbf{y}) = -\frac{1}{4\pi|\mathbf{x} - \mathbf{y}|}, \qquad \mathbf{x}, \mathbf{y} \in \mathbb{R}^3.$$

(b) Let $\Omega = \{z > 0\}$ and suppose $\varphi$ satisfies

$$\nabla^2 \varphi = 0 \text{ in } \Omega, \qquad \varphi(x_1, x_2, 0) = h(x_1, x_2), \qquad \varphi(\mathbf{x}) \to 0 \text{ as } |\mathbf{x}| \to \infty.$$

(i) State Green's second identity.

(ii) Let $\mathcal{G}(\mathbf{x}, \mathbf{y})$ satisfy

$$\nabla^2 \mathcal{G} = \delta(\mathbf{x} - \mathbf{y}) \text{ in } \Omega, \qquad \mathcal{G} = 0 \text{ on } z = 0.$$

Using Green's second identity, show that

$$\varphi(\mathbf{x}) = \int_{\mathbb{R}^2} h(y_1, y_2) \frac{\partial \mathcal{G}}{\partial n}(\mathbf{x}, \mathbf{y}) \, dy_1 dy_2.$$

(iii) Show that $\mathcal{G}$ may be constructed from $\hat{G}$ using the method of images and write $\mathcal{G}$ explicitly.

(c) Suppose

$$h(y_1, y_2) = \frac{1}{2\pi\sigma^2} \exp\left( -\frac{y_1^2 + y_2^2}{2\sigma^2} \right), \qquad \sigma > 0 \,.$$

Show that on the axis $x_1 = x_2 = 0$,

$$\varphi(0, 0, z) = \frac{z}{2\pi\sigma^2} \int_0^\infty \frac{e^{-u/(2\sigma^2)}}{(u + z^2)^{3/2}} \, du \,.$$

**Paper 4, Section II**

**14A Methods**

Consider the first-order quasilinear PDE

$$a(x, y)u_x + b(x, y)u_y = c(x, y, u).$$

(a) Briefly describe the method of characteristics for such an equation. Write down the associated characteristic system and state what initial data must be prescribed to determine a local solution.

(b) Suppose that

$$\frac{\partial a}{\partial x} + \frac{\partial b}{\partial y} = 0\,.$$

so that, locally, there exists a function $H(x, y)$ such that

$$(a, b) = (H_y, -H_x)\,.$$

(i) Show that the characteristic curves satisfy

$$\dot{x} = H_y\,, \qquad \dot{y} = -H_x\,.$$

(ii) Prove that $H(x, y)$ is constant along characteristics.

(c) Consider the PDE

$$(2xy)\,u_x + (x^2 - y^2)\,u_y = 0\,.$$

(i) Verify that the vector field determining the characteristic has zero divergence.

(ii) Find a function $H(x, y)$ such that

$$(2xy,\ x^2 - y^2) = (H_y, -H_x)\,.$$

(iii) Determine and sketch the characteristic curves.

(iv) Hence obtain the general solution of the PDE.

**Paper 1, Section I**

**5C Numerical Analysis**

A $\nu$-stage explicit Runge–Kutta method for solving the ordinary differential equation (ODE) $d\boldsymbol{y}/dt = \boldsymbol{f}(t, \boldsymbol{y})$ has the form

$$
\begin{aligned}
\boldsymbol{k}_1 &= f(t_n, \boldsymbol{y}_n)\,, \\
\boldsymbol{k}_2 &= f(t_n + c_2 h, \boldsymbol{y}_n + h c_2 \boldsymbol{k}_1)\,, \\
\boldsymbol{k}_3 &= f(t_n + c_3 h, \boldsymbol{y}_n + h(a_{3,1}\boldsymbol{k}_1 + a_{3,2}\boldsymbol{k}_2))\,, \qquad a_{3,1} + a_{3,2} = c_3\,, \\
&\;\vdots \\
\boldsymbol{k}_\nu &= f\left(t_n + c_\nu h, \boldsymbol{y}_n + h\sum_{j=1}^{\nu-1} a_{\nu,j}\boldsymbol{k}_j\right), \qquad \sum_{j=1}^{\nu-1} a_{\nu,j} = c_\nu\,, \\
\boldsymbol{y}_{n+1} &= \boldsymbol{y}_n + h\sum_{l=1}^{\nu} b_l \boldsymbol{k}_l\,,
\end{aligned}
$$

where $h > 0$ is the step size.

(i) Apply the method to the ODE $dy/dt = \lambda y$, where $\lambda$ is a complex constant, to show that

$$y_{n+1} = r(z) y_n\,,$$

where $z = h\lambda$ and $r(z)$ is a polynomial of degree $\nu$ (which you do not need to determine).

(ii) Show that, for the method to be of order $\nu$, $r(z)$ must have the form of the truncated exponential series

$$r(z) = \sum_{l=0}^{\nu} \frac{z^l}{l!}\,.$$

(iii) For the methods considered in part (ii) in the cases $\nu = 2$ and $\nu = 3$, determine which part (if any) of the imaginary axis belongs to the linear stability domain.

**Paper 1, Section II**

**16C Numerical Analysis**

(i) Let $p \in \mathbb{P}_n$ interpolate the function $f \in C^{n+1}[-1,1]$ at the $n+1$ distinct points $x_0, \cdots, x_n \in [-1,1]$. Prove that the interpolation error for any $x \in [-1,1]$ can be expressed as

$$f(x) - p(x) = \frac{f^{(n+1)}(\xi)}{(n+1)!}\,\Pi(x)\,,$$

where

$$\Pi(x) = \prod_{i=0}^{n}(x - x_i)$$

and $\xi \in [-1,1]$ is some number that depends on $x$. [You may assume Rolle's theorem.]

(ii) Let $\theta = \arccos x$ and let $T_{n+1}(x) = \cos((n+1)\theta)$ be the Chebyshev polynomial of degree $n+1$. In the case of the Chebyshev points,

$$x_i = \cos\theta_i\,, \qquad \theta_i = \left(i + \frac{1}{2}\right)\frac{\pi}{n+1}\,, \qquad 0 \leqslant i \leqslant n\,,$$

which are the zeros of $T_{n+1}(x)$, show that

$$|\Pi(x)| \leqslant \left(\frac{1}{2}\right)^n$$

for all $x \in [-1,1]$.

[*Hint: The identity* $\cos((n+1)\theta) + \cos((n-1)\theta) = 2\cos\theta\cos(n\theta)$ *can be rewritten as a recurrence relation for the (non-monic) polynomial* $T_n(x)$*. Hence, or otherwise, deduce the coefficient of* $x^n$ *in* $T_{n+1}(x)$*.*]

(iii) For the function

$$f(x) = \cos(kx)\,,$$

where $k$ is a positive constant, show that $f(x) - p(x) \to 0$ as $n \to \infty$ for all $x \in [-1,1]$ in the case of the Chebyshev points.

(iv) In the case of equally spaced points,

$$x_i = -1 + \left(i + \frac{1}{2}\right)\frac{2}{n+1}\,, \qquad 0 \leqslant i \leqslant n\,,$$

show that

$$\Pi(1) > \left(\frac{2}{e}\right)^n.$$

[*Hint: Use a graphical or similar argument to show that*

$$\log\left(\frac{3}{2}\right) + \log\left(\frac{5}{2}\right) + \cdots + \log\left(\frac{2n+1}{2}\right) > \int_1^{n+1} \log x\, dx\,.]$$

**Paper 2, Section II**

**17C Numerical Analysis**

Let $f \in C^5[-1,1]$. The two-point Gauss–Legendre quadrature rule is

$$\int_{-1}^{1} f(x)\,dx \approx f(-s) + f(s)\,, \qquad \text{where} \quad s = \frac{1}{\sqrt{3}}\,.$$

The linear functional

$$L(f) = \int_{-1}^{1} f(x)\,dx - f(-s) - f(s)$$

quantifies the error in the quadrature rule.

(i) Work out $L(p)$ explicitly when $p \in \mathbb{P}_5$, i.e. $p(x) = \sum_{j=0}^{5} c_j x^j$. What is the highest degree $n$ of polynomial for which the quadrature rule is exact?

(ii) State the Peano kernel theorem. Evaluate the Peano kernel for the error in the two-point Gauss–Legendre quadrature rule. Assuming that the Peano kernel is non-negative, deduce that

$$|L(f)| \leqslant \frac{||f^{(n+1)}||_\infty}{N}\,,$$

where $||f||_\infty = \max_{x\in[-1,1]} |f(x)|$ and $N$ is an integer which you should determine.

(iii) Show that the upper bound in part (ii) is sharp in the case of a quartic polynomial.

**Paper 3, Section II**

**17C Numerical Analysis**

Consider the $s$-step, $s$-order backward differentiation formula (BDF)

$$\sum_{l=0}^{s} \rho_l \boldsymbol{y}_{n+l} = h\sigma_s \boldsymbol{f}(t_{n+s}, \boldsymbol{y}_{n+s}) , \qquad \rho_s = 1$$

as a method for solving the ordinary differential equation $d\boldsymbol{y}/dt = f(t, \boldsymbol{y})$, where $h$ is the step size such that $t_n = nh$. Let

$$\rho(w) = \sum_{l=0}^{s} \rho_l w^l , \qquad \sigma(w) = \sigma_s w^s .$$

(i) Show that to satisfy the order condition

$$\rho(e^z) - z\sigma(e^z) = \mathcal{O}(z^{s+1}) \qquad \text{as} \quad z \to 0 ,$$

we require

$$\rho(w) = \sigma_s \sum_{l=1}^{s} \frac{1}{l} w^{s-l}(w-1)^l , \qquad \sigma_s = \left( \sum_{l=1}^{s} \frac{1}{l} \right)^{-1} .$$

(ii) Determine $\rho(w)$ and $\sigma_s$ explicitly for the BDF methods with $s = 2$ and $s = 3$. Use the Dahlquist equivalence theorem to show that these two methods are convergent.

(iii) Show that the BDF method with $s = 2$ is A-stable. [*Hint: Find the boundary of the linear stability domain.*]

**Paper 4, Section I**

**6C Numerical Analysis**

(i) Write down the form of a Givens rotation matrix $\Omega^{[p,q]}$.

(ii) Let $A$ be an $n \times n$ matrix. For any $i$ and $j$ such that $1 \leqslant j < i \leqslant n$, explain how to choose $p$, $q$ and the angle of the Givens rotation such that $\left(\Omega^{[p,q]} A\right)_{i,j} = 0$. Describe the effect of the Givens rotation on the rows of $A$.

(iii) Outline the Givens algorithm to obtain a $QR$ factorization of a $3 \times 3$ matrix $A$, showing symbolically the structure of the matrix at each stage of the algorithm.

**Paper 1, Section I**

**7H Optimisation**

Solve the following optimisation problem using the simplex algorithm:

$$\begin{array}{ll} \text{maximise} & x_1 + x_2 + x_3 \\ \text{subject to} & (x_1 - x_2 + x_3)^2 \leqslant 4, \\ & 2x_1 + 3x_2 + x_3 \leqslant 4, \qquad x_1, x_2, x_3 \geqslant 0. \end{array}$$

Suppose the constraint $2x_1 + 3x_2 + x_3 \leqslant 4$ is replaced by $2x_1 + 3x_2 + x_3 \leqslant 4 + \epsilon$. Give an expression for the optimal solution that is valid for all sufficiently small non-zero $\epsilon$.

**Paper 2, Section I**

**7H Optimisation**

What is the *maximum flow problem* in a network?

Describe the Ford–Fulkerson algorithm.

Suppose all edge capacities are rational numbers. Show that the algorithm is guaranteed to terminate in a finite number of steps.

**Paper 3, Section II**

**19H Optimisation**

Let $f : \mathbb{R}^d \to \mathbb{R}$ be a twice differentiable convex function.

(i) Describe the procedure of *gradient descent* for minimising $f$ with step-size $\eta > 0$.

(ii) Suppose in what follows that $\nabla^2 f(x) \preceq \beta I$ for $\beta > 0$, where $\nabla^2 f(x)$ denotes the Hessian of $f$ at $x$, and for symmetric matrices $A, B \in \mathbb{R}^{d\times d}$, $A \preceq B$ means that $B - A$ is positive semi-definite. Show that

$$f(z) - f(y) \leqslant \nabla f(y)^\top (z-y) + \frac{\beta}{2}\|z-y\|^2.$$

(iii) Show further that

$$f(x) - f(y) \leqslant \nabla f(x)^\top (x-y) - \frac{1}{2\beta}\|\nabla f(x) - \nabla f(y)\|^2.$$

[*Hint: Consider* $z = y - \frac{1}{\beta}(\nabla f(y) - \nabla f(x))$ *and recall that* $f(x) - f(z) \leqslant \nabla f(x)^\top (x-z)$.]

Conclude that

$$0 \leqslant (\nabla f(x) - \nabla f(y))^\top (x-y) - \frac{1}{\beta}\|\nabla f(x) - \nabla f(y)\|^2.$$

(iv) Suppose in what follows that additionally $\nabla^2 f(x) \succeq \alpha I$ for $\beta > \alpha > 0$. Writing $\varphi(x) = f(x) - \frac{\alpha}{2}\|x\|^2$, explain briefly why $\varphi$ is convex and why $\nabla^2 \varphi(x) \preceq (\beta - \alpha)I$. Using part (iii), show that

$$(\nabla f(x) - \nabla f(y))^\top (x-y) \geqslant \frac{\alpha\beta}{\beta+\alpha}\|x-y\|^2 + \frac{1}{\beta+\alpha}\|\nabla f(x) - \nabla f(y)\|^2.$$

(v) Let $x^*$ be the minimiser of $f$ and let $x_t$ be the $t$ th iterate of the gradient descent procedure from part (i). Show that when $\eta = \frac{2}{\alpha+\beta}$, we have

$$\|x_{t+1} - x^*\|^2 \leqslant \left(\frac{\beta-\alpha}{\beta+\alpha}\right)^{2t} \|x_1 - x^*\|^2.$$

[*Hint: Use part (iv) with* $y = x^*$.]

**Paper 4, Section II**

**18H Optimisation**

(i) What does it mean for a point $x$ to be an *extreme point* of a convex set $C \subset \mathbb{R}^d$?

(ii) Consider the optimisation problem

$$\min_{x \in \mathbb{R}^n} \sum_{i=1}^{n} |x_i| \qquad \text{subject to } Ax = b, \tag{$*$}$$

where $A \in \mathbb{R}^{m \times n}$ and $b \in \mathbb{R}^m$. Show how this may be converted to a linear program in standard form: that is, find a linear program of the form

$$\min_{y \geqslant 0} \; c^\top y \qquad \text{subject to } My = a, \tag{$**$}$$

where $c, M$ and $a$ are related to $(A, b)$ in ways that you should specify, and where an optimal $y^*$ for $(**)$ may be converted to an optimal $x^*$ for $(*)$. [*Hint: Consider decomposing* $x = x^+ - x^-$ *where* $x^+, x^- \geqslant 0$.]

(iii) Suppose there exists an optimal solution to $(*)$. Show that there must exist an optimal solution with no more than $m$ non-zero components.

(iv) For $\gamma > 0$ and some $f : \mathbb{R}^m \to \mathbb{R}$, consider the optimisation problem

$$\min_{x \in \mathbb{R}^n} \left\{ f(Ax) + \gamma \sum_{i=1}^{n} |x_i| \right\}.$$

Assuming an optimal solution exists, show that there must also exist an optimal solution with no more than $m$ non-zero components.

[*You may assume that if a linear program of the form $(**)$ has an optimal solution, then there must also exist an optimal solution that is an extreme point of the feasible set.*]

**Paper 1, Section II**

**13B Quantum Mechanics**

(i) Why does the Schrödinger equation not apply to massless particles?

(ii) A one-dimensional infinite potential well has a vanishing potential for $-a < x < a$ and an infinite potential elsewhere. Show that the stationary states $\psi_n$ of a quantum mechanical particle of mass $m > 0$ can be written

$$\psi_n(x) = A \sin\left(\omega(x+a)\right),$$

where you should determine each of $\omega$, $A$ and the energies $E_n$ of the stationary states explicitly in terms of $m$, $a$, $\hbar$ and $n = 1, 2, \ldots$ Show explicitly that the $\psi_n$ are orthonormal. You may assume that they are also complete.

(iii) Consider the preparation of a particular state $\psi$ in this potential well:

$$\psi(x) = \begin{cases} \sqrt{\frac{2}{a}} \sin\left(\frac{2\pi x}{a}\right), & 0 < x < a, \\ 0, & -a < x \leqslant 0. \end{cases}$$

Many such states are prepared, each in a similar infinite one-dimensional potential well, in different experiments. Calculate the average $E_{av}$ of the energies measured immediately after their preparation.

(iv) By calculating $E_{av}$ in terms of the energies of the Hamiltonian eigenstates, show that

$$\pi^2 = B \sum_{n=0}^{\infty} \frac{(2n+1)^2}{[(2n+1)^2 - 16]^2}$$

for some constant $B$, which you should find.

[*Hint: You may find the following identity helpful,*

$$\sin\theta_1 \sin\theta_2 = \frac{\cos(\theta_1 - \theta_2) - \cos(\theta_1 + \theta_2)}{2}. \qquad ]$$

**Paper 2, Section II**

**15B Quantum Mechanics**

Muonic hydrogen is like ordinary hydrogen, where the electron of mass $m_e$ has been replaced by a muon, of identical charge and of mass $m_\mu > m_e$ but $m_\mu \ll m_p$, where $m_p$ is the mass of a proton.

(i) Write down the time-independent Schrödinger equation for the wavefunction of muonic hydrogen.

(ii) Explain the origin of each term in this equation, briefly.

(iii) What assumptions must you make for the equation that you have written in part (i) to hold?

(iv) The first excited state of muonic hydrogen where the muon has zero orbital angular momentum can be written

$$\psi(r) = A\left(1 - \frac{r\alpha}{2}\right)\exp(-\alpha r/2),$$

where $A$ is a normalisation constant. Assuming that this form of $\psi$ satisfies your answer to part (i), find $E$ in terms of $\alpha$ and then determine $\alpha$ in terms of the other constants.

(v) Verify that $\psi$ satisfies the equation you have given in answer to part (i).

(vi) Sketch $\psi(r)/A$ as a function of $r$. On the same graph, sketch and label the radial wavefunction divided by its normalising factor for the first excited state of ordinary hydrogen where the electron has zero orbital angular momentum.

[*Hint: The time-independent Schrödinger equation for the wavefunction of the hydrogen atom is*

$$-\frac{\hbar^2}{2m_e r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\psi}{\partial r}\right) + \frac{1}{2m_e r^2}L^2\psi - \frac{e^2}{4\pi\epsilon_0 r}\psi = E\psi\,. \qquad ]$$

**Paper 3, Section I**

**6B Quantum Mechanics**

For three-dimensional space with indices $a, b, c, d \in \{1, 2, 3\}$, write the angular momentum operator $L_a$ in terms of the position operator $x_b$ and the momentum operator $p_c$.

State the commutation relations between $x_a$ and $p_b$, between $x_a$ and $x_b$, and between $p_a$ and $p_b$.

Thus calculate the commutator between $L_a$ and $L_b$, writing the right-hand side in terms of a linear function of the operators introduced above.

Find the commutator between $L_a$ and $p_d$, writing the right-hand side as a linear function of the operators introduced above. Are the operators $L_1$ and $p_2$ simultaneously diagonalisable?

**Paper 4, Section I**

**4B Quantum Mechanics**

Consider the Gaussian wavepacket wavefunction

$$\psi(x,t)=\frac{A}{\sqrt{1+i\hbar t/m}}\exp\left(-\frac{x^2}{2(1+i\hbar t/m)}\right),$$

where $A$ is a normalising constant which you may take to be real. Show that $\psi(x,t)$ satisfies the time-dependent Schrödinger equation. What is the form of the potential energy?

Find $A$ if $\psi(x,t)$ is normalised.

Calculate the uncertainty in the measurement of the particle position $\Delta x=\sqrt{\langle x^2\rangle-\langle x\rangle^2}$ as a function of $t$. What does this imply about whether the state is stationary or not?

[*Hint: You may assume that, for real* $\alpha>0$, $\int_{-\infty}^{\infty}dx\,e^{-\alpha x^2}=\sqrt{\pi/\alpha}$.]

**Paper 4, Section II**

**15B Quantum Mechanics**

Consider a wavefunction $\psi(x,t)$ describing a particle of mass $m$ moving in one dimension.

(i) Write down expressions for the probability density $\rho$ and current density $j$.

(ii) Starting from the time-dependent Schrödinger equation, show that $\rho$ and $j$ satisfy

$$\frac{\partial \rho}{\partial t} + \frac{\partial j}{\partial x} = 0.$$

For

$$\psi(x,t) = \Psi(x) e^{-iEt/\hbar},$$

show that the current is a constant in $x$.

(iii) A particle of mass $m$ moving in one dimension is incident from the left with energy $E = 3U_0/2$ on a potential well $U(x)$ given by

$$U(x) = \begin{cases} 0\,, & 0 \leqslant x \leqslant a, \\ U_0\,, & x < 0,\ x > a\,, \end{cases}$$

where $U_0$ is a positive constant. Write down the conditions that should be imposed on the wavefunction at $x = 0$ and $x = a$. Use these conditions to determine that the probability of transmission to the right is

$$\frac{1}{1 + F \sin^2(G\sqrt{U_0})},$$

where $F$ and $G$ are constants which you should determine.

(iv) Keeping $E = 3U_0/2$, for what values of $U_0$ is the probability of transmission minimised?

**Paper 1, Section I**

**6H Statistics**

Let $X_1, \ldots, X_n \overset{\text{i.i.d.}}{\sim} U[\theta, 1]$ where $\theta < 1$.

(a) What does it mean for a statistic $T$ to be *sufficient* for $\theta$? Find such a sufficient statistic $T$.

(b) State the Rao–Blackwell theorem.

(c) Find, with justification, an unbiased estimator $\hat{\theta}$ of $\theta$ with mean squared error

$$\mathbb{E}(\hat{\theta} - \theta)^2 = \frac{(1-\theta)^2}{n(n+2)}.$$

[*Hint: First consider an unbiased estimator of $\theta$ based on just a single observation. The variance of the minimum of $n$ independent $U[0,1]$ random variables is*

$$\frac{n}{(n+1)^2(n+2)}\,. \qquad ]$$

**Paper 1, Section II**

**17H Statistics**

Consider the normal linear model $Y = X\beta + \varepsilon$ where $X$ is a known matrix of predictors, $\beta \in \mathbb{R}^p$ is an unknown vector of parameters, and $\varepsilon \sim N_n(0, \sigma^2 I)$ is a vector of normal errors with $\sigma > 0$. Let $X = (X_0, X_1)$, where $X_0 \in \mathbb{R}^{n\times p_0}$ and $X_1 \in \mathbb{R}^{n\times p_1}$ with $p_0 + p_1 = p$, and correspondingly write $\beta^\top = (\beta_0^\top, \beta_1^\top)$ where $\beta_0 \in \mathbb{R}^{p_0}$ and $\beta_1 \in \mathbb{R}^{p_1}$. We wish to test the null hypothesis $H_0$ that $\beta_1 = 0$ against the alternative that $\beta_1 \neq 0$. We assume throughout that $X_0$ has full column rank.

(i) Write down the maximum likelihood estimator $(\hat\beta_0, \hat\sigma^2)$ of $(\beta_0, \sigma^2)$ in the model given by the null hypothesis $H_0$.

(ii) Assuming that $X$ has full column rank, write down the $F$-test statistic for testing $H_0$.

(iii) Let $R = Y - X_0\hat\beta_0$ be the residuals from fitting the model given by $H_0$. Show that under the null, the distribution of $R/\sigma$ is the same as that of $(I - P_0)Z$ where $Z \sim N_n(0, I)$ and $P_0 \in \mathbb{R}^{n\times n}$ is a matrix you should specify.

(iv) Let $X_{1j} \in \mathbb{R}^n$ be the $j$th column of the matrix $X_1$ and suppose $\|X_{1j}\| = \sqrt{n}$ for all $j$. It is known that if $\zeta \sim N(0,1)$, then $\mathbb{P}(|\zeta| > t) \leqslant e^{-t^2/2}$ for $t \geqslant 0$. Show that for each $j$,

$$\mathbb{P}\left(\frac{1}{\sqrt{n}}\left|X_{1j}^\top (I - P_0)Z\right| > t\right) \leqslant e^{-t^2/2}.$$

(v) Let $X_{1j} \in \mathbb{R}^n$ be the $j$th column of the matrix $X_1$. We do not assume that $X$ has full column rank, but suppose that $\sigma$ is known. Consider the test statistic

$$T = \frac{1}{\sigma\sqrt{n}} \max_{j=1,\ldots,p_1} |X_{1j}^\top R|.$$

Let $\alpha \in (0,1)$. Show that the test that rejects $H_0$ when $T \geqslant \sqrt{2\log(p_1/\alpha)}$ has size at most $\alpha$.

[*Hint: Given events $A_1, \ldots, A_{p_1}$, a union bound yields* $\mathbb{P}(\cup_{j=1}^{p_1} A_j) \leqslant \sum_{j=1}^{p_1} \mathbb{P}(A_j)$. ]

(vi) Now consider the case where $\sigma > 0$ is unknown (and $X$ does not necessarily have full column rank). Find, with justification, a test statistic for the null $H_0$ against the alternative $\beta_1 \neq 0$ that under the null has the same distribution as

$$\max_{j=1,\ldots,p_1} \frac{|X_{1j}^\top (I - P_0)Z|}{\|(I - P_0)Z\|}.$$

## Paper 2, Section I

### 6H Statistics

Suppose $X_1, \ldots, X_n \overset{\text{i.i.d.}}{\sim} N(\mu, 1)$. Find the maximum likelihood estimator $\hat{\mu}$ of $\mu$, and write down its mean squared error.

Now adopt a Bayesian perspective and take a $N(0, \tau^{-2})$ prior for $\mu$, with $\tau > 0$. Find the posterior distribution. What is the posterior mean $\tilde{\mu}$?

Show that the mean squared error of $\tilde{\mu}$ is strictly smaller than that of $\hat{\mu}$ if and only if

$$\mu^2 \; < \; 2\tau^{-2} + \frac{1}{n}.$$

**Paper 3, Section II**
**18H Statistics**

(a) (i) Suppose a random variable $X$ has unknown density $p$. State the Neyman–Pearson lemma for testing the null $H_0 : p = f$ against the alternative $H_1 : p = g$, where $f$ and $g$ are continuous densities with the same support.

(ii) Suppose $X_1, \ldots, X_n \overset{\text{i.i.d.}}{\sim} N(\mu, 1)$. Find, with justification, a uniformly most powerful test for testing

$$H_0 : \mu = 0, \quad \text{against} \quad H_1 : \mu > 0,$$

that has size $\alpha \in (0,1)$. Express the critical region in terms of the cumulative distribution function $\Phi$ of the $N(0,1)$ distribution. Argue moreover that your test is also a uniformly most powerful size $\alpha$ test for testing

$$H_0 : \mu \leqslant 0, \quad \text{against} \quad H_1 : \mu > 0.$$

(b) (i) Consider the setting of part (a) (i) but where we modify the null hypothesis $H_0$ to be that $p \in \{f_1, f_2\}$, with the alternative unchanged (again $f_1, f_2$ are continuous with the same support as $g$). Given $\pi \in [0,1]$, let $\phi$ be the size $\alpha$ likelihood ratio test for the null $H_{0,\pi}$ that $p = \pi f_1 + (1-\pi) f_2$, against $H_1$. Suppose that in fact $\phi$ has size $\alpha$ for the null hypothesis $H_0$. Explain why $\phi$ must be uniformly most powerful for $H_0$ against $H_1$.

(ii) A certain drug requires the logarithm $\mu$ of the relative concentrations of two chemicals to be within a range $[-\mu_0, \mu_0]$, for $\mu_0 > 0$. A new cheaper method for manufacturing the drug has been proposed for which it is claimed that $\mu = 0$, but there is concern that $\mu \notin [-\mu_0, \mu_0]$. The value of $\mu$ cannot be measured directly, but a single measurement $X \sim N(\mu, 1)$ is available. Consider therefore testing

$$H_0 : \mu \notin [-\mu_0, \mu_0], \quad \text{against} \quad H_1 : \mu = 0\,,$$

based on $X$. Given $\alpha \in (0,1)$, let $z > 0$ be such that

$$\Phi(z - \mu_0) - \Phi(-z - \mu_0) = \alpha\,.$$

Show that the test which rejects $H_0$ when $X \in [-z, z]$ is uniformly most powerful.

**Paper 4, Section II**

**17H Statistics**

(a) Suppose $X_1, \ldots, X_n \overset{\text{i.i.d.}}{\sim} \mathrm{Exp}(\theta)$ where the rate parameter $\theta > 0$ is unknown. Find the maximum likelihood estimator $\hat{\theta}$ of $\theta$.

(b) Show that $\sum_{i=1}^{n} X_i \sim \mathrm{Gamma}(n, \theta)$. If $Y \sim \mathrm{Gamma}(\beta, \lambda)$, where $\beta, \lambda > 0$, what is the distribution of $\gamma Y$ for $\gamma > 0$?

(c) Let $\alpha \in (0,1)$. Find, with justification, a $(1-\alpha)$-level confidence interval for $\theta$ of the form $[l\hat{\theta}, u\hat{\theta}]$, where $l$ and $u$ are deterministic quantities depending only on $n$ and $\alpha$.

(d) Now suppose a Bayesian approach is taken, and we use a $\mathrm{Gamma}(\beta, \lambda)$ prior for $\theta$. Find the posterior distribution of $\theta$ and the posterior mean $\widetilde{\theta}$.

(e) Find deterministic quantities $l'$ and $u'$ depending only on $\beta, n$ and $\alpha$ such that the posterior probability that $\theta$ lies in $[\,l'\,\widetilde{\theta},\, u'\,\widetilde{\theta}\,]$ is $1-\alpha$.

[*Hint: A* $\mathrm{Gamma}(\beta, \lambda)$ *distribution has density* $f(y) = \lambda^{\beta} y^{\beta-1} e^{-\lambda y} / \Gamma(\beta)$ *and mean* $\beta/\lambda$. *Moreover if* $Y \sim \mathrm{Gamma}(\beta, \lambda)$ *then* $\mathbb{E}e^{tY} = \{\lambda/(\lambda - t)\}^{\beta}$ *if* $t < \lambda$ *and* $\infty$ *otherwise.*]

**Paper 1, Section I**

**2F Topological Spaces**

Let $(X, \mathcal{T})$ be a topological space. What is a *neighbourhood* of a point $x \in X$? What does it mean for a sequence in $X$ to *converge* to some $x \in X$?

Recall that $(X, \mathcal{T})$ is metrisable if $\mathcal{T}$ is induced by a metric $d$ on $X$. Given metrics $d$ and $d'$ on $X$, show that the following are equivalent:

(1) $d$ and $d'$ induce the same topology on $X$;

(2) the convergent sequences with respect to $d$ and $d'$ are the same, i.e. for every sequence $(x_n)$ in $X$, $\exists x \in X$ such that $x_n \to x$ with respect to $d$ $\iff$ $\exists y \in X$ such that $x_n \to y$ with respect to $d'$.

It follows that the convergent sequences uniquely determine the topology of a metrisable space. Does the same hold for a general topological space? Briefly justify your answer.

**Paper 2, Section I**

**11F Topological Spaces**

(a) What does it mean for a topological space to be *connected*?

Consider $\mathbb{R}$ with the usual Euclidean topology. Let $f\colon \mathbb{R} \to \mathbb{R}$ be a bijection with the property that, for every connected $X \subset \mathbb{R}$, $f(X)$ is connected. Show that $f$ must be monotone. [You may assume the following fact: $g\colon \mathbb{R} \to \mathbb{R}$ is monotone $\iff$ for every triple $(a, b, c)$ with $a < b < c$, $g(b)$ lies between $g(a)$ and $g(c)$.]

Show that $f$ must be continuous.

(b) What does it mean for a topological space to be *path-connected*? Show that a path-connected space is connected.

Consider the unit square $Q = [0,1] \times [0,1]$ with the usual Euclidean topology. The boundary of $Q$ is $\partial Q = ([0,1] \times [0,1]) \setminus ((0,1) \times (0,1))$. Let $\gamma$ and $\delta$ be two continuous paths on $\partial Q$ starting at $(0,0)$ and ending at $(1,1)$ that only intersect at these two points.

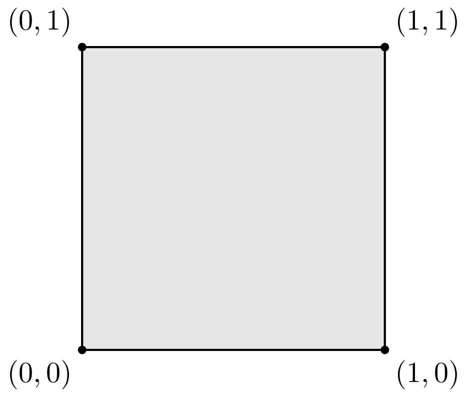



Figure: the unit square $Q$ with its four corners.

(i) Give an explicit example of one possible such pair $(\gamma, \delta)$.

(ii) Let $\alpha, \beta\colon [0,1] \to Q$ be continuous paths on $Q$ satisfying

$$\alpha(0) = (0,0), \quad \alpha(1) = (1,1),$$
$$\beta(0) = (0,1), \quad \beta(1) = (1,0).$$

Show that the images of $\alpha$ and $\beta$ must intersect in $Q$. [You may assume the following fact: there is no continuous $F : Q \to S^1$ such that the images of $F \circ \gamma$ and of $F \circ \delta$ are opposite hemispheres of the unit circle $S^1$.]

(iii) Suppose that $A \subset Q$ is path-connected, contains $(0,0)$ and $(1,1)$ and does not contain the other two corners of $Q$. Deduce that $(0,1)$ and $(1,0)$ must lie in different path components of $Q \backslash A$.

[*You may assume results from the lectures about intervals in* $\mathbb{R}$ *if stated clearly.*]

**Paper 3, Section II**

**2F Topological Spaces**

Let $(X, \mathcal{T})$ be a topological space, and $A \subset X$. What is the *subspace topology* on $A$? If $Y = X/\sim$, for $\sim$ an equivalence relation on $X$, what is the *quotient topology* on $Y$? If $(Y, \mathcal{T}')$ is another topological space, what is a *homeomorphism* between $X$ and $Y$?

Consider $\mathbb{R}^2$ with the Euclidean topology, and let $S = [-1,1]^2 \subset \mathbb{R}^2$. Let $\sim$ be the equivalence relation on $S$ generated by $(x,y) \sim (-x,y)$ for all $(x,y) \in S$, and let $Y = S/\sim$. Determine the quotient map $q\colon S \to Y$.

By first showing that $Y$ is homeomorphic to $S^+ = [0,1] \times [-1,1]$, or otherwise, show that $Y$ is homeomorphic to $S$. [You may assume results from lectures if carefully stated.]

**Paper 3, Section II**

**12F Topological Spaces**

What is a *base* for a topology on a topological space?

Let $X$ be a topological space whose topology is generated by a metric $\rho$. Fix $k \in \mathbb{N}$, $k \geqslant 2$ and let $Y = \{1, \ldots, k\}$. Consider the product space $X^k$, whose elements are of the form $(f(1), \ldots, f(k))$ for some $f\colon Y \to X$. Determine the product topology on $X^k$.

Explain briefly why

$$d(f,g) = \sum_{n=1}^{k} \frac{1}{2^n} \min\{1, \rho(f(n), g(n))\},$$

where $f, g\colon Y \to X$, defines a metric on the space of functions from $Y$ to $X$. Show that the product topology on $X^k$ is the same as that induced by $d$. What are the convergent sequences with respect to this topology?

Let $X^{\mathbb{N}} = X \times X \times \ldots$ with a topology that has as base the family of finite intersections of sets of the form

$$\{\pi_n^{-1}(U_n)\colon n \in \mathbb{N} \text{ and } U_n \subset X \text{ open in } X\},$$

where $\pi_n$ is the projection map from $X^{\mathbb{N}}$ to the $n$th factor. Is this topology induced by a metric? Justify your answer. [*Hint: It may help to rewrite the base for the topology in terms of functions from* $\mathbb{N}$ *to* $X$ *as in the previous part.*]

**Paper 4, Section II**

**11F Topological Spaces**

For $c = (c_1, c_2) \in \mathbb{R}^2$ and $r > 0$, let $S_r(c) = (c_1 - \frac{r}{2}, c_1 + \frac{r}{2}) \times (c_2 - \frac{r}{2}, c_2 + \frac{r}{2})$ be the open square of centre $c$ and side length $r$. Consider the space

$$P = S_2((0,0)) \setminus \left( \overline{S_{\frac{1}{4}}((-\tfrac{1}{2}, 0))} \cup \overline{S_{\frac{1}{4}}((\tfrac{1}{2}, 0))} \right) \subset \mathbb{R}^2$$

endowed with the subspace topology.

(i) Define the *interior* and *closure* of a subset of a topological space. Identify the topological boundary of $P$ inside $\mathbb{R}^2$, which is defined as $\partial P = \overline{P} \setminus \mathring{P}$.

(ii) Construct a polyhedron homeomorphic to $\overline{P}$. How many vertices, edges and triangles does it have?

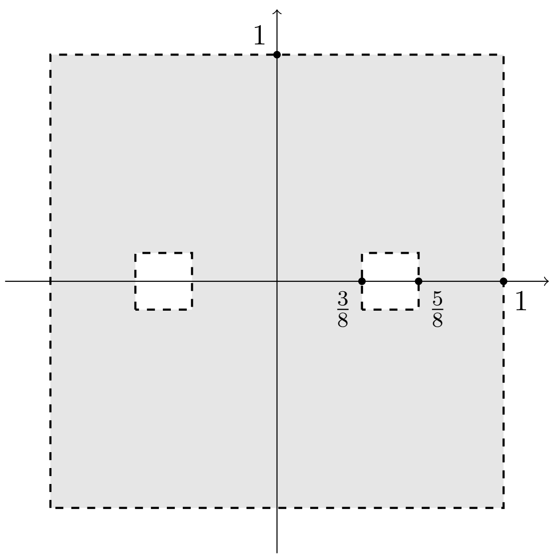



Figure: the set $P$.

Define $\sim$ to be the smallest equivalence relation on $(0 \times \overline{P}) \sqcup (1 \times \overline{P})$ such that

$$(0, x) \sim (1, x) \quad \text{for all} \quad x \in \partial P.$$

Consider the space $D = ((0 \times \overline{P}) \sqcup (1 \times \overline{P}))/\sim$ endowed with the quotient topology.

(iii) Show carefully that $D$ is compact and connected. [You may assume results from lectures if clearly stated.]

(iv) What does it mean for a topological space to be *locally Euclidean*? What is a *topological manifold*? Explain briefly why $D$ is locally Euclidean. [You do not need to construct explicit homeomorphisms.]

(v) Define the *Euler characteristic* of a topological surface. What is the image of the triangulation you found in part (ii) under the quotient map? Is it a triangulation of $D$? Justify your answer.

(vi) State the classification theorem for topological surfaces. Assume that $D$ is homeomorphic to a connected sum of $k$ tori. Write down, without proof, the value of $k$.

**Paper 1, Section I**
**4B Variational Principles**

State Fermat's principle.

Consider a light ray travelling between two points in the horizontal-vertical $x - y$ plane in a medium such that the speed of light is $c(x)$. Starting from Fermat's principle, derive Snell's law, namely that

$$\frac{\sin\theta}{c(x)}$$

is a constant, where $\theta$ is the angle the ray makes with the horizontal at any point.

[You may state then use a suitable Euler-Lagrange equation without proof.]

If $c(x) = 1/\sqrt{3-x}$ and a light ray travels through the origin at an angle $\pi/4$ to the horizontal, what is its angle to the horizontal at $x = 1$?

**Paper 2, Section II**
**13B Variational Principles**

Use Lagrange multipliers to do the following:

(a) A solid cylinder of radius $a$ and length $l$ has volume $V$ and total surface area $A$.

(i) Show that, for fixed volume $V$, the extremal surface area is

$$A = B(2\pi)^{1/3}V^{2/3},$$

determining the integer $B$ in the process.

(ii) The cylinder is inscribed within a sphere of fixed radius. The ratio $l/a$ is chosen to maximise the surface area of the cylinder. Show that

$$A = 2\pi\sqrt{C}a^2,$$

where $C$ is an integer which you should determine.

(b) Find all stationary points of

$$f(x,y,z) = -2x^3 + y^2 + 3z^2$$

on the plane $x = -y - z$.

[You need *not* identify the nature of any extrema.]

**Paper 3, Section I**

**4B Variational Principles**

Let $C$ be the unit circle containing the unit disk $D$ centred on the origin. Derive the Euler-Lagrange equation for the function $u(x, y)$ which is fixed to be equal to 1 on $C$ and gives a stationary value of

$$I[u] = \int_D L\left(x, y, u, \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}\right) \, dx \, dy.$$

Find the Euler-Lagrange equation satisfied by the function $u(x, y)$ which satisfies the different boundary condition $u(x, y) = x^y$ on $C$ and gives a stationary value of

$$\int_D \left[u^2 + \frac{\partial u}{\partial x}\frac{\partial u}{\partial y}\right] \, dx \, dy.$$

**Paper 4, Section II**

**13B Variational Principles**

Consider the integral

$$I = \int_{t_0}^{t_1} dt \; g(x, \dot{x}, \ddot{x}, t)$$

where $x(t_0)$, $x(t_1)$, $\dot{x}(t_0)$, $\dot{x}(t_1)$ are all fixed. Derive the Euler-Lagrange equation which extremises $I$.

Find $x(t)$ for which $x(t) \to 0$ and $\dot{x}(t) \to 0$ when $t \to \infty$, $x(0) = \dot{x}(0) = 1$ and the integral

$$\int_0^\infty dt \; \left(16x^2 + 8(\dot{x})^2 + (\ddot{x})^2 + \pi^e \dot{x}\ddot{x}\right)$$

is stationary.

## END OF PAPER