UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part IA 2026

List of Courses

Paper 1, Section I
3E Analysis I

(a) Define the *radius of convergence* R for both real and complex power series.

(b) Consider the complex power series $\sum_{n=0}^{\infty} a_n z^n$ defined by coefficients a_n given by $a_{2n} = (-1)^n, \ a_{2n+1} = 0$. Show that

(i) $\frac{1}{(z-i)(z+i)} = \sum_{n=0}^{\infty} a_n z^n$ for $|z| < 1$,

(ii) $\sup_{|z|<1} |\sum_{n=0}^{\infty} a_n z^n| = \infty$.

What is the radius of convergence R of this series?

(c) Suppose $\sum_{n=0}^{\infty} b_n x^n$ is a *real* power series with radius of convergence $R = 1$. Must $\sup_{x \in (-1,1)} |\sum_{n=0}^{\infty} b_n x^n| = \infty$?

Paper 1, Section I
4E Analysis I

(a) State the Bolzano–Weierstrass theorem for sequences in $\mathbb{R}$.

(b) Let $a_n \in \mathbb{C}$ be a sequence. State what it means for the sequence to *converge* to some $a \in \mathbb{C}$.

(c) Let $a_n \in \mathbb{C}$ be a sequence. Show that either there is a subsequence a_{n_k} which converges to some $a \in \mathbb{C}$, or else a subsequence such that $|a_{n_k}| \geqslant k^2$. [You may use the Bolzano–Weierstrass theorem for sequences in $\mathbb{R}$ from (a) without proof.] Are these two alternatives exclusive?

Paper 1, Section II
9E Analysis I

(a) We say that a function $f : \mathbb{R} \to \mathbb{R}$ is *real analytic* if for all $x_0 \in \mathbb{R}$, there exists an $\epsilon > 0$, and a coefficient sequence $a_n \in \mathbb{R}$, $n = 0, 1, \ldots$ such that for all $x \in (-\epsilon, \epsilon)$, the function $f(x_0 + x)$ may be represented as the convergent series

$$f(x_0 + x) = \sum_{n=0}^{\infty} a_n x^n. \tag{$*$}$$

Show that if $f(x) = P(x) : \mathbb{R} \to \mathbb{R}$ is a polynomial, then $f(x)$ is real analytic.

(b) State Taylor's theorem. [You may use any of the forms for the remainder term from the course.]

(c) Show that if $f : \mathbb{R} \to \mathbb{R}$ is infinitely differentiable and if for every $-\infty < a < b < \infty$, there exists a $C(a, b) > 0$ such that $|f^{(n)}(x)| \leqslant C^{n+1} n!$ for all $n \geqslant 0$ and all $x \in [a, b]$, then it follows that f is real analytic. [Hint: *Apply Taylor's theorem in sufficiently small intervals* $(x_0 - \epsilon, x_0 + \epsilon)$ *to show that* $f(x_0 + x)$ *may be written as* $(*)$ *with* $a_n = \frac{1}{n!} f^{(n)}(x_0)$.]

(d) Give an example of a real analytic function $f : \mathbb{R} \to \mathbb{R}$ which is not a polynomial. [Justify your answer.]

Paper 1, Section II
10E Analysis I

(a) State and prove Rolle's theorem for real valued functions $f : [a, b] \to \mathbb{R}$. [Be sure to state carefully the precise differentiability assumptions necessary.]

(b) Infer from the above that if $f : [a, b] \to \mathbb{R}$ is continuous and $f'(x)$ is defined and strictly positive for all $x \in (a, b)$, then $f(b) > f(a)$.

(c) Let $f : \mathbb{R} \to \mathbb{R}$ be a twice continuously differentiable function such that $f(0) = 0$, $f'(0) = 1$, $f''(0) = 0$, and such that $f'''(x)$ exists for all $x > 0$ and is strictly negative. Show that there must exist a unique $x_0 > 0$ such that $f(x_0) = 0$ and $f(x) > 0$ for all $x \in (0, x_0)$.

(d) Give the standard definitions of $\sin x$ and $\cos x$ in terms of power series. [In what follows, you may assume without proof that these series converge for all $x \in \mathbb{R}$ and are differentiable, that $\sin' x = \cos x$, $\cos' x = -\sin x$ and that the trigonometric identities $\cos^2 x + \sin^2 x = 1$, $\sin(x + y) = \sin x \cos y + \sin y \cos x$ hold.] Show that there must exist an $x_0 \in \mathbb{R}$, $x_0 > 0$ such that $\sin(x_0) = 0$, $\sin(x) > 0$ for all $0 < x < x_0$.

[Note that the number x_0 is usually denoted as π.]

(e) Show that $\sin(x + 2x_1) = \sin x$ for all $x \in \mathbb{R}$ if and only if $x_1 = n x_0$ for some $n \in \mathbb{Z}$, with x_0 $(= \pi)$ the unique number whose existence was shown in (d).

Paper 1, Section II

11E Analysis I

(a) State the Fundamental Theorem of Calculus (both parts, concerning derivatives of integrals and integrals of derivatives).

(b) Let $f : [0, a) \to \mathbb{R}$ be continuous and satisfy

$$|f(x)| \leqslant |f(0)| + \int_0^x g(\tilde{x})|f(\tilde{x})|d\tilde{x}$$

for some nonnegative continuous function $g : [0, a) \to \mathbb{R}$. Show that

$$|f(x)| \leqslant |f(0)| \exp \int_0^x g(\tilde{x})d\tilde{x}.$$

If $f(0) = 0$, deduce that $f(x) = 0$ for all $0 \leqslant x < a$. [You may use standard properties of the exponential function $\exp : \mathbb{R} \to \mathbb{R}_+$ proven in the course, and the fact that it admits a continuously differentiable inverse, denoted $\log : \mathbb{R}_+ \to \mathbb{R}$, with $(\log x)' = 1/x$.]

(c) Let f_1, f_2 be two continuously differentiable solutions to the equation

$$f'(x) = (1 + \sqrt{x})f^2(x)$$

on some interval $[0, a)$, with $f_1(0) = f_2(0)$. Deduce that $f_1 = f_2$ identically on $[0, a)$. [Hint: *Consider the differential equation satisfied by the difference.*]

Paper 1, Section II

12E Analysis I

For each of the statements below, determine whether it is true or false, justifying appropriately your answer with a proof or counterexample. You may refer to standard results in the course.

(a) The function $f:[0,1]\to\mathbb{R}$ defined by $f(x)=0$ if $x\in\mathbb{Q}\cap[0,1]$, $f(x)=1$ if $x\notin\mathbb{Q}\cap[0,1]$ is Riemann integrable.

(b) Suppose $f:[a,b]\to\mathbb{R}$ is Riemann integrable. Then f is continuous at all but finitely many points of $[a,b]$.

(c) Suppose $f_i:[a,b]\to\mathbb{R}$, $i=1,\ldots$, is a sequence of Riemann integrable functions such that $\lim_{i\to\infty}f_i(x)$ exists for all $x\in[a,b]$. Define $f(x)=\lim_{i\to\infty}f_i(x)$. Then $f(x)$ is Riemann integrable.

(d) If $f:[a,b]\to\mathbb{R}$ is Riemann integrable, $f\geqslant 0$ and $\int_a^b f(x)dx=0$, then $f=0$ identically.

(e) Let $F:[a,b]\to\mathbb{R}$ be differentiable at all but countably many points and suppose that F' extends to a Riemann integrable function $f:[a,b]\to\mathbb{R}$, i.e. there exists a Riemann integrable $f:[a,b]\to\mathbb{R}$ such that, *for all x such that $F'(x)$ is defined*, we have $f(x)=F'(x)$. Then

$$\int_a^b f(x)dx = F(b)-F(a).$$

(f) If $f:(a,b)\to\mathbb{R}$ is continuous and there exists a constant $0<C<\infty$ such that $\int_{\tilde a}^{\tilde b}|f|^p dx\leqslant C^p$ for all $a<\tilde a<\tilde b<b$ and all integers $p\geqslant 1$, then $\sup_{(a,b)}|f(x)|\leqslant C$.

Paper 2, Section I

1C Differential Equations

Consider the integral

$$I(x) = \int_0^1 y^2 \, e^{xy} \, dy \qquad \text{for } x \in \mathbb{R}.$$

By differentiating under the integral sign show that

$$x^2 \frac{d^2 I(x)}{dx^2} - 12 I(x) = e^x (x-4).$$

Paper 2, Section I

2C Differential Equations

(a) Find two solutions of

$$x^2 y'' + 4xy' + (2 + x^2) y = 0, \qquad \text{for } x > 0,$$

and prove explicitly by computing their *Wronskian* $W(x)$ that they are independent. [*Hint: You may find the substitution* $y(x) = z(x)/x^2$ *helpful.*]

(b) Find a particular integral for

$$x^2 y'' + 4xy' + (2 + x^2) y = x^2, \qquad \text{for } x > 0.$$

Paper 2, Section II

5C Differential Equations

Consider the second order differential equation

$$y'' + (\alpha - x^2)y = 0,$$

where α is a real positive constant parameter.

(i) Find the value of the parameter α such that

$$y_0(x) = e^{-x^2/2}$$

is a solution of the equation.

(ii) Substitute

$$y(x) = e^{-x^2/2} f(x),$$

and find a second order differential equation for f.

(iii) Find a recurrence relation between the coefficients of the power series solutions for $f(x)$

$$f(x) = \sum_{n=0}^{\infty} a_n x^n.$$

(iv) Impose the initial condition $y'(0) = 0$. For what values of α is there a solution with $f(x)$ a polynomial of degree $2N$?

(v) Hence, or otherwise, find the solution of

$$y'' + (9 - x^2)y = 2e^{-x^2/2},$$

satisfying $y'(0) = 0$ and $y(0) = 1$.

Paper 2, Section II

6C Differential Equations

Consider the function

$$U(x,y) = x^3 - 3x + xy^2 + y^2.$$

(i) Find the stationary points of $U(x,y)$ and identify the local minimum. Sketch the contours $U(x,y) = \text{constant}$.

(ii) A particle with position $(x(t), y(t))$ at time t moves with velocity

$$(\dot{x}(t), \dot{y}(t)) = -\nabla U(x,y).$$

Show that $U(x(t), y(t))$ is non-increasing function of t along any solution trajectory. If $x = 2$ and $y = 1$ at $t = 0$, what happens to the solution as $t \to \infty$? [Justify your answer with reference to your sketched contours.]

(iii) Find the particle's trajectory $y(x)$ in the vicinity of the local minimum found in part (i).

(iv) Consider the modified system where a periodic external force is applied along the y direction

$$\begin{aligned} \frac{dx}{dt} &= -\frac{\partial U(x,y)}{\partial x} \\ \frac{dy}{dt} &= -\frac{\partial U(x,y)}{\partial y} + \cos(\omega t), \end{aligned}$$

with ω a real constant. Show that $U(x(t), y(t))$ is no longer necessarily non-increasing. Find how the solutions $(x(t), y(t))$ change in the vicinity of the minimum due to the periodic forcing term.

Paper 2, Section II

7C Differential Equations

Let $u(x,t)$ be a real-valued function satisfying the following equation for x and $t \in \mathbb{R}$:

$$\frac{\partial^2 u}{\partial t^2} + \gamma \frac{\partial u}{\partial t} - c^2 \frac{\partial^2 u}{\partial x^2} + \omega^2 u = f(x,t), \qquad (*)$$

where γ, ω and c are real positive parameters and $f(x,t)$ is a real function.

(a) For $f(x,t) = f(t)$ consider spatially uniform solutions of $(*)$ of the form $u(x,t) = y(t)$.

(i) Show that y satisfies

$$\ddot{y} + \gamma \dot{y} + \omega^2 y = f(t).$$

(ii) Assume $f(t) = 0$. Write down the spatially uniform solutions $y(t)$ and identify the underdamped, overdamped and critically damped solutions according to the values of γ and ω. State the long-time behaviour of the solutions $y(t)$.

(iii) Consider the case of light damping in which $\gamma = \sqrt{3}\omega$ and let

$$f(t) = I\delta\left(t - \frac{\pi}{2\omega}\right), \qquad I > 0,$$

where δ is the Dirac δ-function. Find a particular solution of the equation for $t \in [0, \pi/\omega]$ such that $y = 0$ at $t = 0$ and $t = \pi/\omega$.

(b) For $f(x,t) = 0$ consider solutions of $(*)$ of the form $u(x,t) = \phi(\xi)$, where $\xi = x - X(t)$ with $X(t)$ a smooth real function of t, such that $\phi(\xi) \to 0$ and $\phi'(\xi) \to 0$ as $|\xi| \to \infty$.

Derive the equation that such solutions satisfy. Deduce that either $\phi \equiv 0$ (trivial solution), or $\ddot{X} + \gamma \dot{X} = 0$ for all t. In the second case, solve for $X(t)$ and describe the long-time behaviour of the solution as $t \to \infty$.

Paper 2, Section II

8C Differential Equations

(i) A bacterial culture is inside a test tube at room temperature θ_0. The tube is placed in a hot incubator and kept there for a time T, then removed and placed on a bench at room temperature for the same time T.

The culture temperature $\theta(t)$ changes at a rate proportional to the temperature difference between the culture and its surroundings, with proportionality constant $\alpha > 0$. The surroundings are at temperature $\theta_1 > \theta_0$ (inside the incubator) during warming and θ_0 (room temperature) during cooling.

Write down the differential equations for $\theta(t)$ during the warming and cooling phases, and find the unique solutions using the initial conditions.

(ii) The bacteria have an enzyme and the concentration of the enzyme $N(t)$ decreases during the warming and cooling process at a rate proportional to its current concentration and a temperature-dependent denaturation rate

$$\beta(\theta(t)) = \beta_{\max} \frac{\theta(t) - \theta_0}{\theta_1 - \theta_0},$$

with $\beta_{\max}$ a positive constant.

Find an implicit equation for the time T required so that the concentration of enzyme is reduced by a factor of 100 by the end of the entire warming and cooling cycle.

(iii) Show that for small values of α such that $\alpha T \ll 1$, the required time is proportional to $1/\sqrt{\alpha}$, and find the constant of proportionality in terms of $\beta_{\max}$.

Paper 4, Section I
3B Dynamics and Relativity

(i) Determine the mass, time, and length dimension of Newton's constant G.

(ii) Use dimensional analysis to derive Kepler's third law.

(iii) General relativity is a theory of gravity that involves two fundamental constants: G and the speed of light c. Use dimensional analysis to relate the radius R of a black hole to its mass M.

(iv) The theory of elementary particles involves Planck's constant $\hbar$ and the speed of light c. Use dimensional analysis to relate the linear size (i.e. length) λ of a particle to its mass m.

(v) Combining these results, what is the threshold mass above which an elementary particle become a black hole?

[Throughout this question, you may ignore overall numerical constants.]

Paper 4, Section I
4B Dynamics and Relativity

Define the *proper time*. Explain why it is appropriate to define the 4-velocity in terms of proper time.

In an inertial frame, Bob traces out a path in Minkowski space given by

$$X^\mu = (ct, R\sin(\omega t), R\cos(\omega t), 0) \ .$$

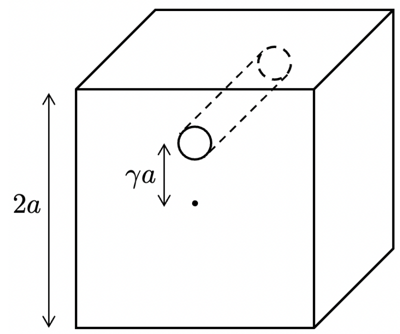
What requirements on ω and R ensure this is a time-like trajectory?

Bob starts his journey at $t = 0$. How old is he, in terms of the parameters R, ω, and c, when he next returns to his starting point in this frame?

Compute the 4-velocity U^μ. What is the Lorentz invariant $U \cdot U$?

Paper 4, Section II
9B Dynamics and Relativity

A cube of uniform density has sides of length $2a$. A cylindrical hole of radius r is drilled through the cube, parallel to one edge, equidistant from two other edges, and a distance γa from the centre, as shown in the figure. Here $\gamma < 1$ and $r < (1-\gamma)a$.



The cube oscillates about a frictionless, cylindrical horizontal rod of radius r that passes through the hole. What is the period of small oscillations?

[You may quote the parallel axis theorem without proof.]

Paper 4, Section II

10B Dynamics and Relativity

A particle of mass m moves in three dimensions in a potential $V(\mathbf{x})$.

(a) Write down the equation of motion of the particle. Show that there is a conserved energy E.

(b) What is meant by a *central potential*? Show that the angular momentum $\mathbf{L} = m\mathbf{x} \times \dot{\mathbf{x}}$ is conserved when the potential is central. Explain why conservation of angular momentum implies that the motion is restricted to a plane.

(c) Suppose now that the particle has electric charge q and, in addition to the central potential, moves in a magnetic field $\mathbf{B}(\mathbf{x})$. Write down the equation of motion of the particle and show that the energy E is still conserved.

The magnetic field is given by

$$\mathbf{B}(\mathbf{x}) = \frac{\beta}{r^2}\hat{\mathbf{r}}$$

with β constant, $r = |\mathbf{x}|$ and $\hat{\mathbf{r}} = \mathbf{x}/r$. Show that

$$\mathbf{J} = \mathbf{L} - \kappa\hat{\mathbf{r}}$$

is conserved for some constant κ that you should determine. Hence show that the motion of the particle is restricted to lie on a cone. Find an expression for the opening angle α of the cone.

(d) Show that the problem considered in part (c) reduces to motion in the radial direction along the cone with the effective potential given by

$$V_{\text{eff}}(r) = \frac{J^2 \sin^2\alpha}{2mr^2} + V(r)\ .$$

Paper 4, Section II
11B Dynamics and Relativity

(a) A non-inertial frame S' rotates with angular velocity $\boldsymbol{\omega}$ with respect to an inertial frame S. Show that, from the perspective of S', the equation of motion for a particle of mass m, subjected to a force $\mathbf{F}$, takes the form

$$m\ddot{\mathbf{x}} = \mathbf{F} - 2m\boldsymbol{\omega} \times \dot{\mathbf{x}} - m\dot{\boldsymbol{\omega}} \times \mathbf{x} - m\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{x}) \ .$$

(b) A horizontal square hoop ABCD is made of fine smooth wire and has side length $2a$. When stationary, the coordinates of the corners are $A = (0,0,0)$, $B = (2a,0,0)$, $C = (2a,2a,0)$ and $D = (0,2a,0)$. The hoop rotates about a vertical axis through A with time-dependent angular velocity

$$\boldsymbol{\omega}(t) = -\frac{\alpha}{t}\hat{\mathbf{z}}$$

where $\alpha > 0$. A small bead can slide along the side BC. Let s be the distance of the bead from the vertex B. At time $t = t_0 > 0$, the bead is at rest at the point $s = s_0$.

For what values of α is there a choice of s_0 such that the bead remains stationary on the wire? The bead is now nudged gently so it moves away from s_0. What is the most general form of the solution $s(t)$? Where does the bead end up?

Paper 4, Section II
12B Dynamics and Relativity

(a) A pion with rest mass m_π decays to an electron of rest mass m_e and a neutrino which we approximate as massless. Find an expression for the speed of the electron in the pion rest frame.

(b) A positron and electron have the same rest mass m_e. A moving positron with energy E annihilates with a stationary electron, producing two photons. What is the minimum E for which the two photons may move at right angles to each other.

(c) A Higgs boson, with rest mass m, moves in the x-direction. It decays into two photons with energies E_1 and E_2. Show that the angle θ between the photons is given by

$$\sin\frac{\theta}{2} = \frac{mc^2}{2\sqrt{E_1 E_2}} \ .$$

Find an expression for the speed of the Higgs boson in terms of θ if, in the rest frame of the Higgs boson, the photons are emitted in opposite directions along the y-axis.

Paper 3, Section I
1D Groups

In the following groups determine the number of elements of order 4. Justify your answers.

(a) $C_2 \times C_4$ (where C_n denotes the cyclic group of order n).

(b) S_4 (where S_n denotes the symmetric group of degree n).

(c) S_6.

(d) A_6 (where A_n denotes the alternating group of degree n).

Paper 3, Section I
2D Groups

Define the centre, $Z(G)$, of a group G and show that it is an abelian normal subgroup of G.

Are the following true or false? Justify your answers.

(a) If K is a normal subgroup of order 2, then $K \leqslant Z(G)$.

(b) If $G/Z(G)$ is cyclic, then G is abelian.

(c) There exists a nontrivial group with trivial centre.

Paper 3, Section II
5D Groups

Let G be a finite group and $g \in G$. Define the *order* of g. Show that if $g^n = e$, then the order of g divides n.

State and prove Lagrange's theorem.

Deduce that the order of g divides the order of G.

Let N be a normal subgroup of G. Explain how G/N inherits the structure of a group.

Suppose N has index m in G and the order of N is n. Further, suppose m and n are coprime. Show the following:

(a) $N = \{a \in G : a^n = e\}$.

(b) $N = \{b^m : b \in G\}$.

Paper 3, Section II

6D Groups

(i) Define $\mathrm{GL}_2(\mathbb{C})$ and $\mathrm{SL}_2(\mathbb{C})$. Show that $\mathrm{SL}_2(\mathbb{C})$ is a normal subgroup of $\mathrm{GL}_2(\mathbb{C})$ and identify the quotient group $\mathrm{GL}_2(\mathbb{C})/\mathrm{SL}_2(\mathbb{C})$.

(ii) Define the Möbius group $\mathcal{M}$. Explain how $\mathcal{M}$ can be identified as a quotient group of $\mathrm{GL}_2(\mathbb{C})$.

(iii) Show that $\mathcal{M}$ acts transitively on $\mathbb{C} \cup \{\infty\}$. Show that any non-identity element of $\mathcal{M}$ has one or two fixed points. Identify the stabiliser of ∞ and denote it by S.

(iv) Show that every element of $\mathcal{M}$ is conjugate to an element of S.

Paper 3, Section II

7D Groups

Let G be a finite group. Define what it means to say G acts on a set X. Explain how, if $|X| = n$, this yields a homomorphism from G to S_n, where S_n denotes the symmetric group of degree n.

Let H be a subgroup of G of index n. Suppose $|G|$ does not divide $n!$. Show that H contains a nontrivial normal subgroup of G.

Let L be a subgroup of G of index p, where p is the smallest prime divisor of $|G|$. Show that L is a normal subgroup of G.

If M is a subgroup of G of index p, where p is prime, does it follow that M is normal?

Paper 3, Section II

8D Groups

Let S_n denote the symmetric group of degree $n \geqslant 2$. Explain how a permutation $\sigma \in S_n$ can be written as a product of disjoint cycles.

Show that S_n can be generated by transpositions.

Define A_n, the alternating group of degree n.

For any even m with $2 \leqslant m \leqslant n$, show that A_n can be generated by m-cycles. [*Hint: Write $(a,b)(a,c)$ as a product of two m-cycles.*]

For any even m with $2 \leqslant m \leqslant n$, show that S_n can be generated by m-cycles.

Can S_n be generated by m-cycles if m is odd? Justify your answer.

Paper 4, Section I

1F Numbers and Sets

Let $u_n = (3+\sqrt{5})^n + (3-\sqrt{5})^n, \quad$ for $n = 0, 1, 2, \ldots$.

(i) Show that u_n is an integer for all $n = 0, 1, 2, \ldots$.

(ii) Show that u_n is divisible by 2^n for all $n = 0, 1, 2, \ldots$.

[*Hint: Find integers $a, b \in \mathbb{Z}$ such that $u_n = au_{n-1} + bu_{n-2}$ for $n \geqslant 2$.*]

Paper 4, Section I

2D Numbers and Sets

(a) Find all solutions of

$$5z \equiv 2 \bmod 17 \ \text{ and } \ 3z \equiv 5 \bmod 19.$$

(b) Suppose Alice sends Bob a message encrypted using RSA with $(N, e) = (33, 7)$. How should Bob decode the received message $29 \bmod 33$? Explain your answer and explain why RSA is believed to be secure.

Paper 4, Section II

5F Numbers and Sets

(i) Find which of the following numbers is irrational with proper justification.

(a) $\log 2$ (here log is logarithm to the base e)

(b) $\log_2 3$

(c) $ae + be^{-1}$ for a, b non-zero integers

(d) the real root of the equation $x^3 + 4x - 3$

(ii) Let $n \geqslant 2$ be an integer. Let

$$H_n := 1 + \frac{1}{2} + \cdots + \frac{1}{n}\,.$$

Show that H_n cannot be an integer.

[You may use the fact that e is transcendental.]

Paper 4, Section II

6D Numbers and Sets

Let $\mathbb{Z}_n = \{0, 1, \ldots, n-1\}$ with arithmetic modulo n.

(i) Suppose $n = p$ is a prime. Show that all non-zero elements of $\mathbb{Z}_p$ are units.

(ii) Show that if $n = p$ is an odd prime and $a \neq 0$ is a square modulo n, then a has exactly 2 square roots. Does this hold if n is composite?

(iii) Prove that

$$(n-1)! \equiv -1 \bmod n$$

if and only if n is a prime.

(iv) Evaluate $16! \bmod 19$.

(v) Suppose $n = p$ is a prime. Show that if $a \neq 0$ is a square modulo p, then $a^{\frac{p-1}{2}} \equiv 1 \bmod p$ and if a is not a square, then $a^{\frac{p-1}{2}} \equiv -1 \bmod p$.

[*Fermat's Little Theorem can be used as long as it is clearly stated.*]

Paper 4, Section II

7E Numbers and Sets

For each of the statements (a)–(e) below, determine whether it is true or false, justifying appropriately your answer with a proof or counterexample. You may refer without proof to standard results in the course.

(a) Let $f : A \to B$ be injective. There exists a $C \subset B$ and an injective $g : C \to A$ such that $g \circ f = \mathrm{id}$.

(b) Suppose $f : A \to B$ is surjective and $g : B \to C$ is injective. Then $g \circ f$ is injective.

(c) Suppose $f : A \to B$ is injective and $g : B \to C$ is surjective. Then $g \circ f$ is surjective.

(d) Suppose $f : A \to B$ is not injective and $g : B \to C$ is a map. Then $g \circ f$ is not injective.

To introduce the final statement (e) to be given below, let A and B be non-empty finite sets with $|A| = |B| + 1 \geqslant 4$, and define $F_{A,B} = \{f : A \to B : f \text{ surjective}\}$ and $G_{B,A} = \{g : B \to A : g \text{ injective}\}$. [Note that these sets are non-empty.] Define a map $\Psi : F_{A,B} \to G_{B,A}$ taking $f \mapsto g$ where g is defined as follows: For each $b \in B$, pick $a \in A$ such that $f(a) = b$ and set $g(b) = a$. [Note that Ψ indeed defines a map from the sets indicated.]

(e) The map Ψ is injective.

Does the validity of (e) change if we assume $|A| = |B| + 1 = 2$? What can you say in the case $|A| = |B| + 1 = 3$?

Paper 4, Section II

8E Numbers and Sets

(a) For given $n \geqslant 1$, let $\mathcal{P}_n$ denote the set of all polynomials in n variables $x_1, \ldots, x_n$ with rational coefficients, not all vanishing. (For instance, an example of a member of the set $\mathcal{P}_2$ is the polynomial $p(x_1, x_2) = x_1^3 x_2 + \frac{1}{2} x_2^2$, while the 0 polynomial is excluded from the set.) Show that the set $\mathcal{P}_n$ is countable for all $n \geqslant 1$.

(b) Now define the set

$$X_n := \{(x_1, x_2, \ldots, x_n) \in \mathbb{R}^n : \exists p \in \mathcal{P}_n : p(x_1, x_2, \ldots, x_n) = 0\}\,.$$

Is X_n countable? Does your answer depend on n?

(c) Let S be a set with the following properties:

(i) $\mathbb{R} \supset S \supset \mathbb{Q}$

(ii) If $x \in S$, then $\sqrt{|x|} \in S$

(iii) If $x \in S$ and $0 \neq y \in S$, then $xy \in S$, $x + y \in S$, $-y \in S$ and $y^{-1} \in S$

(iv) If $\tilde{S} \subset \mathbb{R}$ satisfies (i), (ii) and (iii), then $\tilde{S} \supset S$

Show that the set S is uniquely defined and is countable.

[Hint: *Exhibit* S *as a union* $S = \cup_{n \geqslant 0} S_n$ *of countable sets* S_n *satisfying (iii) and the property that if* $x \in S_n$ *then* $\sqrt{|x|} \in S_{n+1}$. *To construct* S_n, *again try and exhibit these as countable unions* $S_n = \cup_{m \geqslant 0} S_{n,m}$.]

Paper 2, Section I

3F Probability

Let X and Y have the joint density function f given by

$$f(x,y) = \frac{\sqrt{3}}{4\pi} e^{-(x^2-xy+y^2)/2}, \quad x,y \in \mathbb{R}.$$

(i) Find the marginal density of X. Find the marginal density of Y. Are X and Y independent?

(ii) Find the conditional density of Y given X. What is the expectation of Y given X?

Paper 2, Section I

4F Probability

Suppose we have two decks of n cards, numbered $1, 2, \ldots, n$. The two decks are shuffled so that the order of cards in each deck is uniformly distributed and independent of the order in the other deck. We say a match occurs at position i if the ith cards from each deck have the same number. Let S_n denote the total number of matches.

(i) Find $\mathbb{E}S_n$. Find $\mathrm{var}(S_n)$.

(ii) Show that for any non-negative random variable X with finite variance,

$$\mathbb{P}(X > 0) \geqslant \frac{(\mathbb{E}X)^2}{\mathbb{E}(X^2)}.$$

(iii) Show that

$$\mathbb{P}(S_n > 0) \geqslant \frac{1}{2}.$$

[You may use standard inequalities from lectures if you state them clearly.]

Paper 2, Section II
9F Probability

A *graph* on a set V is a set of some unordered pairs of (distinct) elements of V: we call these the *edges* of the graph and the elements of V are called the *vertices*. A *random graph* with n vertices $v_1, v_2, \ldots, v_n$ is drawn by adding an edge with probability p between v_i and v_j for all $i \neq j$ independently.

A triangle is a set of three distinct vertices v_i, v_j, v_k that are all connected to each other. Let T be the number of triangles in this random graph.

(i) Find the expectation $\mathbb{E}(T)$.

(ii) Let $p = 1/n^{\alpha}$. Show that if $\alpha > 1$, then $\mathbb{P}(T = 0) \to 1$ as $n \to \infty$.

(iii) Show that if $p \in (0,1)$ is such that $\operatorname{var}(T)/\mathbb{E}(T)^2 \to 0$ as $n \to \infty$, then $\mathbb{P}(T = 0) \to 0$ as $n \to \infty$.

(iv) Now let $p = 1/n^{\alpha}$ with $0 < \alpha < 1$. Show that $\mathbb{P}(T = 0) \to 0$ as $n \to \infty$.

[You may use standard inequalities from lectures if you state them clearly.]

Paper 2, Section II
10F Probability

Consider a biased coin whose probability of head H is p and tail T is $1-p$ for some $p \in (0,1)$. Successive tosses of the coin are independent.

(i) What is the expected number of tosses to get a head H?

(ii) What is the expected number of tosses to get two consecutive heads HH?

(iii) What is the expected number of tosses to get the sequence HT?

(iv) What is the expected number of tosses to get n consecutive heads?

(v) Suppose the coin is tossed repeatedly until the total number of heads is n. Let $p_k(n)$ denote the probability that k tosses are required. Find $p_k(n)$.

For any real numbers $-\infty < a < b < \infty$, find integers $k_a(n), k_b(n)$ such that

$$\sum_{k=k_a(n)}^{k_b(n)} p_k(n) \to \frac{1}{\sqrt{2\pi}} \int_a^b e^{-x^2/2} dx\,, \quad \text{as } n \to \infty.$$

[You may quote standard results from lectures. You may use that the variance of a Geometric(p) random variable is $(1-p)/p^2$.]

Paper 2, Section II
11F Probability

(a) Let $(S_n : n \geqslant 0)$ be a simple symmetric random walk on $\mathbb{Z}$ with $S_0 = 0$, that is defined as

$$\mathbb{P}(S_n - S_{n-1} = 1) = \mathbb{P}(S_n - S_{n-1} = -1) = \frac{1}{2} \text{ for all } n \geqslant 1\,.$$

For any $a \in \mathbb{Z}, a \geqslant 1$, find the probability that (S_n) hits a before hitting -1. Let T denote the time at which (S_n) first hits -1 or a, i.e.,

$$T = \min\{n \geqslant 0 : S_n \in \{-1, a\}\}\,.$$

Find $\mathbb{E}(T)$.

(b) Now, for $n \geqslant 2$, let $\mathbb{Z}_n$ be the circle graph consisting of the vertices $0, 1, \ldots, n-1$ and edges between vertices $k, k+1$ for $k = 0, 1, \ldots, n-2$ and an edge between $n-1$ and 0. Let $(Y_n : n \geqslant 0)$ be a simple symmetric random walk on $\mathbb{Z}_n$ with $Y_0 = 0$.

(i) Let T_k denote the first time the random walk (Y_n) has hit k distinct vertices, for $k = 1, 2, \ldots, n$. Clearly $T_1 = 0$. Find $\mathbb{E}(T_{k+1} - T_k)$ for $k = 1, 2, \ldots, n-1$. Find $\mathbb{E}(T_n)$, where T_n is the first time the random walk has hit all the vertices.

(ii) Let Z denote the last new vertex to be hit by the random walk (Y_n). Find the distribution of Z.

[You may use the following facts: Conditional on $Y_{T_k} = i$, $(Y_{T_k+n} : n \geqslant 0)$ is again distributed as a simple symmetric random walk starting at i. Similarly, if τ_i is the first time (Y_n) hits i, $(Y_{\tau_i+n} : n \geqslant 0)$ is distributed as a simple symmetric random walk starting at i.]

Paper 2, Section II

12F Probability

In a branching process, the probability that an individual has exactly k offspring is given by p_k, for $k = 0, 1, 2, \ldots$, and all individuals reproduce independently. Let X_n denote the size of the nth generation. Assume that $X_0 = 1$ and $p_0 > 0, p_2 > 0$ and let $\mu := \mathbb{E}X_1$. Let $F_n(s)$ be the generating function of X_n. Thus

$$F_1(s) = \mathbb{E}(s^{X_1}) = \sum_{k=0}^{\infty} p_k s^k, \quad |s| \leqslant 1 .$$

(i) Show that $F_{n+1}(s) = F_n(F_1(s))$.

(ii) Compute $\mathbb{E}X_n$.

(iii) Show that if $\mu \leqslant 1$, the branching process eventually becomes extinct with probability 1.

(iv) Now let $p_k = 2^{-k-1}$ for $k = 0, 1, 2 \ldots$. Show that the branching process becomes extinct with probability 1.

Find $F_n(s)$. Compute the probability that the nth generation is empty.

Paper 3, Section I
3A Vector Calculus

Consider a curve $\mathbf{x}(t)$ in $\mathbb{R}^3$. Define the *arc length* s, *tangent vector* $\mathbf{t}$, *principal normal vector* $\mathbf{n}$ and *curvature* κ.

Show that $\mathbf{t}$ is perpendicular to $\mathbf{n}$.

Compute $\mathbf{t}$, $\mathbf{n}$ and κ for the curve

$$\mathbf{x}(t) = \left(e^t \cos t,\ e^t \sin t,\ \sqrt{2} e^t \right) .$$

Paper 3, Section I
4A Vector Calculus

State the *divergence theorem* for a smooth vector field $\mathbf{F}(\mathbf{x})$ in $\mathbb{R}^3$ and a volume V enclosed by a piecewise smooth closed surface $S = \partial V$.

By explicitly calculating the volume and surface integrals, verify the divergence theorem for the case where

$$\mathbf{F}(\mathbf{x}) = \mathbf{x}\, e^{-\sqrt{x^2+y^2+z^2}} ,$$

and V is a sphere of radius a centred on the origin.

Paper 3, Section II

9A Vector Calculus

(a) Using suffix notation, prove the vector identity

$$\boldsymbol{\nabla} \times (\mathbf{F} \times \mathbf{G}) = (\boldsymbol{\nabla} \cdot \mathbf{G})\mathbf{F} - (\boldsymbol{\nabla} \cdot \mathbf{F})\mathbf{G} + (\mathbf{G} \cdot \boldsymbol{\nabla})\mathbf{F} - (\mathbf{F} \cdot \boldsymbol{\nabla})\mathbf{G}\,.$$

[You may use without proof any standard results involving the antisymmetric tensor ϵ_{ijk}.]

(b) Define what is meant by an *irrotational* vector field. Prove that if $\mathbf{E} = \boldsymbol{\nabla} f$, where f is a smooth scalar field, then $\mathbf{E}$ is irrotational.

(c) Consider cylindrical polar coordinates (ρ, ϕ, z). For each of the following vector fields $\mathbf{E}$, explicitly compute

$$I = \oint_C \mathbf{E} \cdot d\mathbf{x}\,,$$

where C is a circle of radius a about the z-axis at $z = 0$ traversed in the direction of increasing ϕ. For each case, find a scalar field f such that $\mathbf{E} = \boldsymbol{\nabla} f$.

(i) $\mathbf{E} = \frac{1}{\rho}\,\mathbf{e}_\phi$ for $\rho \neq 0$.

(ii) $\mathbf{E} = \left[(-2\rho^2 + 1)\sin\phi\,\mathbf{e}_\rho + \cos\phi\,\mathbf{e}_\phi - 2z\rho\sin\phi\,\mathbf{e}_z\right] e^{-\rho^2 - z^2}$.

[*Hint: In cylindrical polar coordinates* $\boldsymbol{\nabla} f = \frac{\partial f}{\partial \rho}\mathbf{e}_\rho + \frac{1}{\rho}\frac{\partial f}{\partial \phi}\mathbf{e}_\phi + \frac{\partial f}{\partial z}\mathbf{e}_z$.]

In the case of part (ii), how could we have inferred the value of I without explicit computation? What about the value of I in part (i)?

Paper 3, Section II

10A Vector Calculus

(a) State *Stokes' theorem* for a smooth vector field $\mathbf{F}$ in $\mathbb{R}^3$ and a piecewise smooth open surface S with a piecewise smooth boundary curve $C = \partial S$. [Carefully specify any directions/orientations.]

(b) Consider $\mathbf{F} = (-y^3, x^3, -x^3 + y^3)$. Calculate $\boldsymbol{\nabla} \times \mathbf{F}$ and explicitly verify that $\boldsymbol{\nabla} \cdot (\boldsymbol{\nabla} \times \mathbf{F}) = 0$.

(c) Consider an open surface S defined in two pieces: $x^2+y^2 = a^2$ for $0 \leqslant z \leqslant h$ and the disc $x^2 + y^2 \leqslant a^2$ for $z = h$, where a and h are positive constants. Sketch S and illustrate the direction of $d\mathbf{S}$ for each piece of the surface.

(d) Verify Stokes' theorem for $\mathbf{F}$ in part (b) with S in part (c) by explicitly calculating the surface and line integrals.

(e) Now consider an open surface S' defined by $x^2 + y^2 + 3z^2 + 5z^4 = a^2$ for $z \geqslant 0$, where a is a positive constant. Calculate

$$\int_{S'} (\boldsymbol{\nabla} \times \mathbf{F}) \cdot d\mathbf{S}\,,$$

for $\mathbf{F}$ given in part (b).

Paper 3, Section II

11A Vector Calculus

(a) Consider a scalar field $\psi(\mathbf{x})$ that satisfies

$$\nabla^2\psi - m^2\psi = \rho \text{ in } V, \quad \partial\psi/\partial\mathbf{n} = f \text{ on } S,$$

where V is a region of $\mathbb{R}^3$ with boundary S a closed surface, ρ and f are given functions on V and S respectively, and m is a real constant. Show that ψ is unique for $m \neq 0$. Does the conclusion change if $m = 0$?

(b) The electric field $\mathbf{E}(\mathbf{x})$ satisfies $\boldsymbol{\nabla}\cdot\mathbf{E} = \rho/\epsilon_0$ and $\boldsymbol{\nabla}\times\mathbf{E} = \mathbf{0}$, where $\rho(\mathbf{x})$ is the charge density and ϵ_0 is a constant (the permittivity of free space).

(i) Briefly explain how an electric potential $\phi(\mathbf{x})$ can be defined and show this satisfies Poisson's equation.

(ii) Consider a sphere of radius b with charge density

$$\rho(\mathbf{x}) = \begin{cases} 0 & \text{for } r < a \\ \rho_0/r & \text{for } a \leqslant r \leqslant b\,, \end{cases}$$

where $b > a > 0$ and ρ_0 are constants, and $r = |\mathbf{x}|$. Find the electric potential ϕ in the three regions $r < a$, $a \leqslant r \leqslant b$ and $r > b$, assuming that $\phi = \phi(r)$ is spherically symmetric, ϕ and ϕ' are continuous everywhere, and $\phi \to 0$ as $r \to \infty$.

(iii) Compute the electric field outside a sphere of radius $c < b$ with uniform charge density (i.e. ρ constant) and total charge Q. For $r > b$ show that this electric field is equivalent to the electric field outside the sphere in part (ii) if that sphere also has total charge Q.

[*Hint:* $\nabla^2\phi(r) = \frac{1}{r}\frac{d^2}{dr^2}(r\phi)$.]

Paper 3, Section II

12A Vector Calculus

In this question consider real Cartesian tensors in $\mathbb{R}^3$.

(a) Define a *tensor* T of rank k, explaining carefully the meaning of the notation in your definition.

(b) If A is a tensor of rank p and B is a tensor of rank q, define what is meant by the contraction of one index of A with one index of B. What is the rank of the resulting tensor? For the particular case $p = 3$ and $q = 3$, show that this contraction of A and B does result in a tensor.

(c) Define an *isotropic* tensor and write down the most general isotropic tensors of rank 0, 1, 2 and 3.

(d) For each of the following integrals, briefly indicate why the integral is an isotropic tensor, state the rank of the tensor and compute the integral,

(i) $\int_S x_i e^{-r^2} dS$,

(ii) $\int_S x_i x_j e^{-r^2} dS$,

(iii) $\int_S x_i x_j x_k e^{-r^2} dS$,

where S is the surface of a sphere of radius a centred on the origin.

(e) Consider the tensor

$$T_{ij}(\mathbf{u}, \mathbf{v}) = \int_S (x_i - u_i)(x_j - v_j) e^{-r^2} dS \,,$$

where $\mathbf{u}$ and $\mathbf{v}$ are fixed vectors. Show that it has the form $T_{ij} = \alpha \delta_{ij} + \beta u_i v_j$, where α and β are scalars. Determine α and β.

Paper 1, Section I
1A Vectors and Matrices

(a) For $z \in \mathbb{C}$ define the principal value of $\log z$.

(b) Find the principal value of $\log(1+i)$ and all other possible values.

(c) Find all solutions $z \in \mathbb{C}$ of $(z-i)^5 = 2$ and sketch them on an Argand diagram.

Paper 1, Section I
2C Vectors and Matrices

The matrix

$$A = \begin{pmatrix} 1 & -1 & 0 \\ 2 & 2 & 1 \end{pmatrix}$$

represents a linear map $\Phi : \mathbb{R}^3 \to \mathbb{R}^2$ with respect to the bases

$$B = \left\{ \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \right\} \qquad C = \left\{ \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\}$$

Find the matrix A' that represents Φ with respect to the bases

$$B' = \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\} \qquad C' = \left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \end{pmatrix} \right\}$$

Paper 1, Section II
5A Vectors and Matrices

(a) Define the *scalar product* of two vectors in $\mathbb{R}^n$ and the *norm* of a vector in $\mathbb{R}^n$.

(b) State and prove the *Cauchy–Schwarz inequality* for vectors in $\mathbb{R}^n$.

(c) Consider a plane P in $\mathbb{R}^3$ containing points $\mathbf{a} = (1,2,0)$, $\mathbf{b} = (1,1,0)$ and $\mathbf{c} = (0,1,1)$. Write an equation for P in the form $\mathbf{r} \cdot \hat{\mathbf{n}} = \beta$ where $\hat{\mathbf{n}}$ is a unit vector and $\beta \in \mathbb{R}$. [You should determine $\hat{\mathbf{n}}$ and β.]

(d) Find the shortest distance between the plane P and the point $\mathbf{d} = (2,0,3)$.

(e) Consider a sphere with equation $|\mathbf{r} - \mathbf{f}| = 2\sqrt{2}$, where $\mathbf{f} = (g,0,0)$ and g is a real constant. Find a condition on g for the sphere to intersect with the plane P. Describe this intersection geometrically, specifying how it depends on the value of g.

Paper 1, Section II

6C Vectors and Matrices

(a) Let A be a 3×3 matrix satisfying

$$A^3 - 4A^2 + 5A - 2I = 0,$$

where I denotes the identity matrix.

(i) Show that A is invertible.

(ii) Express A^{-1} as a polynomial in A.

(iii) Show that every power A^k ($k \geqslant 3$) can be written as a linear combination of I, A and A^2.

(b) Let A and B be similar $n \times n$ matrices so that $B = S^{-1}AS$ for some invertible matrix S.

(i) Show that $\ker B = S^{-1}(\ker A)$ and deduce that $\dim(\ker A) = \dim(\ker B)$.

(ii) State the rank-nullity theorem and use it to prove that $\text{rank}(A) = \text{rank}(B)$

(iii) Show that

$$\text{rank}(A^k) = \text{rank}(B^k) \quad \text{for all } k \geqslant 1.$$

(c) State the Cayley–Hamilton theorem for an $n \times n$ matrix B.
Let B satisfy $B^k = 0$ for some $k > n$. Use the Cayley–Hamilton theorem to obtain an expression for B^n in terms of $I, B, \ldots, B^{n-1}$ and deduce that $B^n = 0$.

Paper 1, Section II
7B Vectors and Matrices

Let H be an $n \times n$ Hermitian matrix with distinct eigenvalues μ_i, $i = 1, \ldots, n$.

(a) Show that $\mu_i \in \mathbb{R}$ and that the eigenvectors are orthogonal. Explain how to construct a unitary matrix U such that $H = U^\dagger \Lambda U$ where Λ is diagonal.

(b) Define the characteristic polynomial $\chi_H(\lambda)$ of H and give an expression for it in terms of the eigenvalues μ_i.

(c) The complex matrix M has components $M_{ij} = \delta_{ij} - x_i y_j^*$ where $x_i, y_i \in \mathbb{C}$ are components of normalised vectors $\mathbf{x}^* \cdot \mathbf{x} = \mathbf{y}^* \cdot \mathbf{y} = 1$. By considering the eigenvalues of M, or otherwise, show that

$$\det M = 1 - \mathbf{y}^* \cdot \mathbf{x} \, .$$

(d) The real matrix P has components $P_{ij} = \delta_{ij} - u_i u_j$ where $u_i \in \mathbb{R}$ are components of a normalised vector $\mathbf{u}$ with $\mathbf{u} \cdot \mathbf{u} = 1$. Show that, for $\lambda \neq \mu_i$,

$$\chi_{HP}(\lambda) = \chi_H(\lambda)\, f(\lambda) \quad \text{with} \quad f(\lambda) = 1 - \mathbf{u} \cdot (H - \lambda I)^{-1} H \mathbf{u} \, ,$$

where I denotes the identity matrix.

(e) Show that $\mathbf{u}$ is an eigenvector of HP with eigenvalue zero.

Denote the remaining eigenvalues of HP as σ_i, with $i = 1, \ldots, n-1$, ordered such that $\sigma_i < \sigma_{i+1}$. Take $H = \Lambda = \operatorname{diag}(\mu_1, \ldots, \mu_n)$ the diagonal matrix with eigenvalues ordered such that $\mu_i < \mu_{i+1}$.

Show that the eigenvalues σ_i of HP interlace the eigenvalues of H, such that

$$\mu_1 \leqslant \sigma_1 \leqslant \mu_2 \leqslant \sigma_2 \leqslant \ldots \leqslant \sigma_{n-1} \leqslant \mu_n \, .$$

Paper 1, Section II
8B Vectors and Matrices

A collection of three 2×2 matrices σ_a, with $a = 1, 2, 3$, are defined by

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \ , \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \ , \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \ .$$

(a) Which of the σ_a are Hermitian? Show that

$$\sigma_a \sigma_b = \delta_{ab} I + i \varepsilon_{abc} \sigma_c \ ,$$

where I denotes the identity matrix.

(b) The exponential of a matrix M is defined by the Taylor expansion

$$e^M = \sum_{k=0}^{\infty} \frac{M^k}{k!} \ .$$

Let $n_a \in \mathbb{R}$, with $a = 1, 2, 3$, be the components of a unit three-dimensional vector $\mathbf{n}$, i.e. $n_a n_a = 1$. Introduce the 2×2 matrix $\mathbf{n} \cdot \boldsymbol{\sigma} = n_1 \sigma_1 + n_2 \sigma_2 + n_3 \sigma_3$.

Compute $R = e^{i\theta\, \mathbf{n} \cdot \boldsymbol{\sigma}}$ for $\theta \in \mathbb{R}$. Show that R is unitary.

(c) Let $x_a \in \mathbb{R}$ be the components of a general three-dimensional vector $\mathbf{x}$, and $A = \mathbf{x} \cdot \boldsymbol{\sigma}$. Compute the determinant and trace of A.

Let

$$A' = R A R^{\dagger} \ .$$

Show that A' is Hermitian, and that $\det A' = \det A$ and $\operatorname{Tr} A' = \operatorname{Tr} A$. Hence argue that $A' = \mathbf{x}' \cdot \boldsymbol{\sigma}$ for some $\mathbf{x}'$ with $\mathbf{x}' \cdot \mathbf{x}' = \mathbf{x} \cdot \mathbf{x}$.

(d) Compute the components x'_a of $\mathbf{x}'$ in terms of x_a, n_a and θ.

[*Note: Index summation convention is used throughout this question.*]

END OF PAPER