UNIVERSITY OF CAMBRIDGE

MAT2

MATHEMATICAL TRIPOS Part II

Friday, 13 June, 2025 9:00am to 12:00pm

PAPER 4

Before you begin read these instructions carefully.

The examination paper is divided into two sections. Each question in Section II carries twice the number of marks of each question in Section I. Section II questions also carry an alpha or beta quality mark and Section I questions carry a beta quality mark.

Candidates may obtain credit from attempts on **at most six questions** from Section I and from any number of questions from Section II.

Write on **one side** of the paper only and begin each answer on a separate sheet.

Write legibly; otherwise you place yourself at a grave disadvantage.

At the end of the examination:

Separate your answers to each question.

Complete a gold cover sheet **for each question** that you have attempted, and place it at the front of your answer to that question.

Complete a green main cover sheet listing **all the questions** that you have attempted.

Every cover sheet must also show your Blind Grade Number and desk number.

Tie up your answers and cover sheets into **a single bundle**, with the main cover sheet on the top, and then the cover sheet and answer for each question, in the numerical order of the questions.

STATIONERY REQUIREMENTS
Gold cover sheets
Green main cover sheet
Script paper
Rough paper

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

SECTION I

1G Number Theory

Let p be an odd prime.

(a) Define the Legendre symbol $\left(\frac{a}{p}\right)$, and state the law of quadratic reciprocity that it satisfies.

(b) Give necessary and sufficient conditions on p mod 3 for the equation

$$X^2 + 11X + 31 = 0$$

to have a solution in $\mathbb{Z}/p\mathbb{Z}$.

(c) Give necessary and sufficient conditions on p mod 3 for the equation

$$X^3 = -1$$

to have (i) a solution in $\mathbb{Z}/p\mathbb{Z}$; and (ii) a unique solution in $\mathbb{Z}/p\mathbb{Z}$.

2I Topics in Analysis

Explain how to obtain a *continued fraction* expansion of a real number $x > 0$. Prove that the continued fraction for x terminates if and only if x is rational.

Determine the continued fraction of $\sqrt{3}$.

3K Coding & Cryptography

Define a linear feedback shift register (LFSR) and its associated feedback polynomial.

Suppose an LFSR has a feedback polynomial of degree d. Explain why the period produced by this LFSR cannot be longer than $2^d - 1$.

Explain why an LFSR that generates a maximal period must have an odd number of coefficients equal to 1 in its feedback polynomial. (That is, if the feedback polynomial is given by $x^d + a_{d-1}x^{d-1} + \cdots + a_0$, then an even number of the a_i should be equal to 1.)

The output sequence of an LFSR starts with 100000001. Give a minimal LFSR that generates this output, i.e. one whose feedback polynomial has least degree. Justify your answer.

4F Automata & Formal Languages

(a) Say what it means for a grammar to be *variable based.*

(b) Given two grammars $G = (\Sigma, V, P, S)$ and $G' = (\Sigma, V', P', S')$, give the definitions of the *concatenation grammar* H and the *regular concatenation grammar* H^{reg}, i.e., grammars H and H^{reg} such that

(i) if G and G' are variable based with disjoint sets of variables, then

$$\mathcal{L}(H) = \mathcal{L}(G)\mathcal{L}(G') \text{ and}$$

(ii) if G and G' are regular with disjoint sets of variables, then H^{reg} is regular and

$$\mathcal{L}(H^{reg}) = \mathcal{L}(G)\mathcal{L}(G').$$

[*You do not need to prove these statements, only to provide the definitions of the grammars.*]

(c) Find examples of regular grammars G and G' with disjoint sets of variables such that H is not a regular grammar. Justify your claim.

(d) Find examples of variable based grammars G and G' with disjoint sets of variables such that $\mathcal{L}(H^{reg}) \neq \mathcal{L}(G)\mathcal{L}(G')$. Justify your claim.

5K Statistical Modelling

Define *Akaike's Information Criterion* (AIC) for a general statistical model.

Consider the normal linear model $Y \sim \mathrm{N}(X\beta, \sigma^2 I_n)$ where $\beta \in \mathbb{R}^p$ is unknown, $\sigma^2 > 0$ is known, and $X \in \mathbb{R}^{n\times p}$ is non-random with full column rank. Show that the AIC in this model is equal to Mallows' C_p (up to constants)

$$C_p = \|Y - \hat{\mu}\|^2 + 2p\sigma^2,$$

where $\hat{\mu}$ is the fitted value of Y using ordinary least squares.

The mean squared prediction error (MSPE) of $\hat{\mu}$ is defined as

$$\mathrm{MSPE} = \mathbb{E}(\|Y^* - \hat{\mu}\|^2),$$

where Y^* is an independent and identically distributed copy of Y. Show that C_p is an unbiased estimator of the MSPE.

6A Mathematical Biology

Let x_n be the number of plants in season n. Each year, a proportion $r < 1$ of plants survives to season $n+1$. In addition, the number of seeds produced per plant that become plants the following year is $ke^{-\lambda x_n}$ with constants $k > 0$ and $\lambda > 0$.

(a) What values of k permit a non-vanishing equilibrium population?

(b) What additional requirement on k is needed for this equilibrium to be stable?

7E Further Complex Methods

The modified Bessel function $I_0(z)$, for $z \in \mathbb{C}$, is the unique solution of the differential equation

$$z\frac{d^2y}{dz^2} + \frac{dy}{dz} - zy = 0, \qquad (\dagger)$$

satisfying $y(0) = 1$.

Explain Laplace's method of seeking a solution $y(z)$ to equation ($\dagger$) of the form

$$y(z) = \int_C e^{zt} f(t)\, dt\,,$$

where the function $f(t)$ and the contour C are suitably chosen. Apply the method to show that

$$I_0(z) = \frac{1}{\pi}\int_{-1}^{1} \frac{e^{zs}}{(1-s^2)^{1/2}}\, ds.$$

8B Classical Dynamics

(a) State the *Jacobi identity* for the Poisson bracket $\{F, G\}$ on a phase space. Prove that if F and G are both conserved quantities for the flow generated by a Hamiltonian H, then $\{F, G\}$ is also conserved.

(b) Consider the following mappings $(q, p) \mapsto (Q(q, p), P(q, p))$, which depend on the parameter $\lambda \in \mathbb{R}$:

(i) $(Q, P) = (\lambda q, \lambda p)\,;$

(ii) $(Q, P) = (q, \lambda p)\,;$

(iii) $(Q, P) = (p, \lambda q)\,.$

For which values of λ are these mappings canonical? For each value of λ *either* show the mapping is *not* canonical *or*, if it is canonical, find a generating function for the mapping.

$\Big[$*Hint: the generating function will either be of type* $S = S(q, P; \lambda)$*, such that the mapping is equivalent to*

$$p = \frac{\partial S}{\partial q}\,, \qquad Q = \frac{\partial S}{\partial P}\,,$$

or of type $\Phi = \Phi(q, Q; \lambda)$ *with the mapping equivalent to*

$$p = \frac{\partial \Phi}{\partial q}\,, \qquad P = -\frac{\partial \Phi}{\partial Q}\,.\Big]$$

9E Cosmology

An inflationary Friedmann-Lemaître-Robertson-Walker universe is governed by the following slow-roll equations for the scale factor $a(t)$ and the scalar field $\phi(t)$,

$$H^2 = \frac{1}{3M_{\mathrm{Pl}}^2} V(\phi)\,, \qquad 3H\dot{\phi} = -V'(\phi)\,,$$

where a dot denotes d/dt, $H = \dot{a}/a$, $V'(\phi) = dV/d\phi$ and M_{Pl} is the Planck mass.

(a) Defining the slow-roll parameter

$$\epsilon(\phi) \;:=\; \frac{M_{\mathrm{Pl}}^2}{2}\left[\frac{V'(\phi)}{V(\phi)}\right]^2,$$

verify that the condition $\epsilon(\phi) \ll 1$ is consistent with the slow-roll condition $\dot{\phi}^2 \ll V$. Show that

$$\frac{H}{\dot{\phi}} = -\frac{1}{M_{\mathrm{Pl}}^2}\frac{V}{V'} = -\frac{1}{\sqrt{2}M_{\mathrm{Pl}}}\frac{1}{\sqrt{\epsilon}}\,.$$

(b) The amount of inflation is given by the number of e-folds by which the scale factor grows, $N = \log[a(t_f)/a(t_i)]$, where t_i and t_f are the start and end times of inflation. Denoting $\phi_i = \phi(t_i)$ and $\phi_f = \phi(t_f)$, show that in the slow-roll regime,

$$N = \int_{t_i}^{t_f} H\,dt \;\approx\; \frac{1}{\sqrt{2}M_{\mathrm{Pl}}}\int_{\phi_f}^{\phi_i}\frac{d\phi}{\sqrt{\epsilon(\phi)}}\,.$$

(c) Consider the potential $V(\phi) = V_0\,[1+\cos(\phi/f)]$, where V_0 and f are positive constants. Show that

$$\sqrt{\epsilon(\phi)} = \frac{M_{\mathrm{Pl}}}{\sqrt{2}f}\,\tan\frac{\phi}{2f}\,.$$

Hence, find that the number of e-foldings for this model is given by

$$N = \frac{2}{M_{\mathrm{Pl}}^2}\,f^2\left[\log\left(\sin\frac{\phi_i}{2f}\right) - \log\left(\sin\frac{\phi_f}{2f}\right)\right].$$

10C Quantum Information and Computation

(a) A *GHZ state* between three parties (Alice, Bob, and Charlie) is defined as

$$|\psi\rangle_{ABC} := \frac{1}{\sqrt{2}}(|000\rangle_{ABC} + |111\rangle_{ABC}) \,.$$

Write down a quantum circuit that generates this state from the initial state $|000\rangle_{ABC}$, justifying your answer.

(b) The aim below is to design a multi-party super-dense coding protocol between Alice, Bob, and Charlie, who are spatially separated and share the GHZ state $|\psi\rangle_{ABC}$ before communication begins.

Suppose that Alice wants to send two (classical) bits to Charlie and Bob wants to send one (classical) bit to Charlie. What operations should Alice and Bob perform on their respective qubits of $|\psi\rangle_{ABC}$ before sending their qubits to Charlie (over ideal qubit channels), and how does Charlie infer the classical bits sent by Alice and Bob? Be sure to consider all cases.

[The following orthonormal basis of 3 qubits will be useful,

$$\begin{aligned}
|\chi_1^{\pm}\rangle &:= \frac{1}{\sqrt{2}}(|000\rangle_{ABC} \pm |111\rangle_{ABC}) \\
|\chi_2^{\pm}\rangle &:= \frac{1}{\sqrt{2}}(|010\rangle_{ABC} \pm |101\rangle_{ABC}) \\
|\chi_3^{\pm}\rangle &:= \frac{1}{\sqrt{2}}(|001\rangle_{ABC} \pm |110\rangle_{ABC}) \\
|\chi_4^{\pm}\rangle &:= \frac{1}{\sqrt{2}}(|011\rangle_{ABC} \pm |100\rangle_{ABC}) \,.]
\end{aligned}$$

SECTION II

11G Number Theory

(a) Let p be a prime number, and let $N \in \mathbb{N}$. Define the *p-adic valuation* $v_p(N)$ of N.

(b) Show that if $k \in \mathbb{N}$, and p is a prime number such that $k+2 \leqslant p \leqslant 2k+1$, then $v_p(\binom{2k+1}{k+1}) = 1$.

(c) If $X \geqslant 1$ is a real number, define $P(X) = \prod_{p \leqslant X} p$, where the product is over prime numbers p less than or equal to X. Show that $P(X) \leqslant 4^X$ for all $X \geqslant 1$.

(d) If $X \geqslant 1$ is a real number, let $\pi(X)$ denote the number of prime numbers less than or equal to X. By considering $\pi(X) - \pi(\sqrt{X})$, or otherwise, show that there is a constant $c > 0$ such that $\pi(X) \leqslant cX/\log X$ for all $X \geqslant 2$.

12I Topics in Analysis

What is a *nowhere dense* set in a metric space? State and prove a version of the Baire category theorem. Deduce the following:

(i) There exists a continuous function $f : [0,1] \to \mathbb{R}$ that is not monotone on any interval of positive length. [You may assume that the space of continuous real valued functions on $[0,1]$ with the uniform norm is complete.]

(ii) If $F : \mathbb{R} \to \mathbb{R}$ is an infinitely differentiable function such that for each x there is an n (depending on x) such that $F^{(n)}(x) = 0$, then F is a polynomial.

13K Statistical Modelling

A three-year study was conducted at three sites on the survival status of patients suffering from cancer. The dataset also records whether or not the initial tumour was malignant. The data are tabulated in `R` as follows:

```
> cancer
  site malignant survive die total
1    A        no      40   7    47
2    A       yes      36  17    53
3    B        no      24   3    27
4    B       yes      35   6    41
5    C        no      15   4    19
6    C       yes       5   5    10
```

(a) Write down the mathematical model that is being fitted by the following `R` commands.

```
> fit1 <- glm(survive/total ~ site + malignant, family = binomial,
+             data = cancer, weights = total)
```

(b) In words or using mathematical equations, explain the (slightly abbreviated) output from the code below and describe how the numbers in the `Coefficients` table are computed. What are your conclusions based on the hypothesis tests in this table?

```
> summary(fit1)
Coefficients:
             Estimate Std. Error z value Pr(>|z|)
(Intercept)    1.6855     0.3431   4.913 8.98e-07 ***
siteB          0.8096     0.4344   1.864   0.0624 .
siteC         -0.5423     0.4825  -1.124   0.2610
malignantyes  -0.9048     0.3809  -2.375   0.0175 *
---
Signif. codes:  0 ‘***’ 0.001 ‘**’ 0.01 ‘*’ 0.05 ‘.’ 0.1 ‘ ’ 1
    Null deviance: 11.69300  on 5  degrees of freedom
Residual deviance:  0.85048  on 2  degrees of freedom
AIC: 29.003
```

[QUESTION CONTINUES ON THE NEXT PAGE]

(c) Consider a slightly different model fitted by the next `R` commands and the corresponding (abbreviated) `summary` output below. Explain why some of the p-values (under the column `Pr(>|z|)`) are the same in these two tables and others are different. Are you surprised that the p-value for `siteC` is significant in `summary(fit1)` (at level 0.05) but not significant in `summary(fit2)`? Explain your answer.

```
> cancer$site <- factor(cancer$site, levels = c("B", "A", "C"))
> fit2 <- glm(survive/total ~ site + malignant, family = binomial,
+             data = cancer, weights = total)
> summary(fit2)
Coefficients:
             Estimate Std. Error z value Pr(>|z|)
(Intercept)    2.4951     0.4610   5.413 6.21e-08 ***
siteA         -0.8096     0.4344  -1.864   0.0624 .
siteC         -1.3520     0.5613  -2.409   0.0160 *
malignantyes  -0.9048     0.3809  -2.375   0.0175 *
---
Signif. codes:  0 ‘***’ 0.001 ‘**’ 0.01 ‘*’ 0.05 ‘.’ 0.1 ‘ ’ 1
```

(d) Describe the hypothesis test performed in the `R` code below and your conclusion based on the results.

```
> fit3 <- glm(survive/total ~ malignant, family = binomial,
            data = cancer, weights = total)
> anova(fit3, fit1)
Analysis of Deviance Table

Model 1: survive/total ~ malignant
Model 2: survive/total ~ site + malignant
  Resid. Df Resid. Dev Df Deviance
1         4     7.4923
2         2     0.8505  2   6.6418

> qchisq(c(0.01, 0.05, 0.1, 0.9, 0.95, 0.99), 4)
[1]  0.2971095  0.7107230  1.0636232  7.7794403  9.4877290 13.2767041
> qchisq(c(0.01, 0.05, 0.1, 0.9, 0.95, 0.99), 2)
[1] 0.02010067 0.10258659 0.21072103 4.60517019 5.99146455 9.21034037
```

14A Mathematical Biology

Consider the reaction-diffusion system in one dimension with $x \in \mathbb{R}$,

$$\begin{aligned}\frac{\partial u}{\partial t} &= D\frac{\partial^2 u}{\partial x^2} + \alpha - (\beta+1)u + u^2 v,\\ \frac{\partial v}{\partial t} &= \frac{\partial^2 v}{\partial x^2} + \beta u - u^2 v,\end{aligned}$$

where the constants D, α, and β obey $D > 0$, $\beta > 1$, and $\alpha^2 > \beta - 1$. The variables u and v are both positive.

(a) First consider spatially homogeneous solutions. Determine the fixed point and sketch the nearby trajectories in the system's phase space.

(b) Now consider inhomogeneous solutions. Without calculation, explain why the system is stable for $D = 1$. Find the condition relating D, α, and β for the system to be unstable.

(c) Suppose we vary the diffusivity from $D = 1$ to the value at which the system first becomes unstable. What is the critical wavenumber $k_\star$ at which the instability first occurs?

15B Classical Dynamics

Let $I_1 < I_2 < I_3$ be the three principal moments of inertia of a rigid body that rotates freely with angular velocity $\boldsymbol{\omega}$ according to the Euler equations

$$\begin{aligned}I_1\dot{\omega}_1 &= (I_2 - I_3)\omega_2\omega_3\,,\\ I_2\dot{\omega}_2 &= (I_3 - I_1)\omega_3\omega_1\,,\\ I_3\dot{\omega}_3 &= (I_1 - I_2)\omega_1\omega_2\,,\end{aligned}$$

where the components ω_1, ω_2 and ω_3 of the angular velocity are taken with respect to the principal axes of inertia.

(a) Write down expressions for the energy E and the total angular momentum squared L^2, and prove that these are conserved using the Euler equations.

(b) Show that if $L^2 = 2EI_2$ there exist solutions in which the angular velocity is directed along the second principal axis, i.e., ω_1 and ω_3 are zero. What are the possible values for ω_2? Use linearisation to analyse the stability of these solutions.

(c) Still working under the condition $L^2 = 2EI_2$, use your expressions from part (a) to express ω_1 and ω_3 in terms of E and L^2, and hence obtain a first-order differential equation for ω_2. Integrate this equation and show that $\omega_2(t) = \mu\tanh(\lambda t)$ for some constants μ, λ which you should find. Briefly comment on the relation of this solution to your answer to part (b).

16H Logic and Set Theory

In this question, let V be a model of ZF set theory. Set-theoretic notation such as $\varnothing$, $\{x, y\}$, $\bigcup x$, and "x is finite" refers to the operations and properties in V. Let φ be a formula with one free variable. The φ-instance of the axiom-scheme of separation is the formula

$$(\forall x)(\exists s)(\forall z)(z \in s \Leftrightarrow (z \in x \wedge \varphi(z))).$$

For a set x, the set $\{z \in x : \varphi\}$ exists by the validity of the φ-instance of the axiom-scheme of separation in V.

(a) Define what it means to be a *class* and a *proper class* in V.

A class M is called transitive if whenever $x \in y$ and y is in M, then x is in M. A class M is called φ-closed if for all x in M, the set $\{z \in x : \varphi\}$ is also in M.

(b) Show that the collection of all finite sets in V is a proper class. Is it transitive? Is it φ-closed? Justify your claims.

(c) A set is called hereditarily finite if it is contained in a transitive and finite set. Is the collection of all hereditarity finite sets in V a proper class? Is it transitive? Is it φ-closed? Justify your claims.

[You are allowed to use properties of the von Neumann hierarchy and its rank function as proved in the lectures, provided that you state them correctly and precisely.]

(d) Let M be a transitive class in V such that for all x, y in M, we have that $\varnothing$, $\{x, y\}$, and $\bigcup x$ are in M. Show that the axioms of extensionality, empty set, pair-set and union are satisfied in M.

(e) Let M be a transitive class. Explain briefly why being φ-closed is not sufficient to prove the φ-instance of the axiom-scheme of separation in M.

[You do not have to provide an example of a class M where this fails.]

(f) Provide a map (without proof) $\varphi \mapsto \varphi^*$, where φ^* is also a formula, such that every transitive class M that is φ^*-closed satisfies the φ-instance of the axiom-scheme of separation. Note that this map can depend on M.

17F Graph Theory

(a) For $t \in \mathbb{N}$ define the *Ramsey number* $R(t)$. If $t \geqslant 2$, show that $R(t)$ exists and that $R(t) \leqslant 2^{2t}$.

(b) For any graph G, define $R(G)$ to be the least positive integer n such that in any red–blue colouring of the edges of the complete graph K_n, there must be a monochromatic copy of G. Explain briefly why $R(G)$ exists.

(i) Let $t \in \mathbb{N}$. Show that, whenever the edges of K_{2t} are red–blue coloured, there must be a monochromatic copy of the complete bipartite graph $K_{1,t}$.

(ii) Suppose that t is odd. Show that $R(K_{1,t}) = 2t$. If t is even, what is $R(K_{1,t})$? Justify your answer.

(iii) Let H be the graph on four vertices, obtained by adding an edge to a triangle. Compute $R(H)$, justifying your answer.

18J Galois Theory

(a) Define the *nth cyclotomic polynomial* $\Phi_n(X)$. Show that it has coefficients in $\mathbb{Z}$. Show further that if K is a subfield of $\mathbb{C}$ and $\zeta_n \in \mathbb{C}$ is a root of $\Phi_n(X)$ then the extension $K(\zeta_n)/K$ is Galois with abelian Galois group.

(b) What does it mean to say that a subfield $K \subset \mathbb{R}$ is *constructible*? Show that $\mathbb{Q}(\cos(2\pi/17))$ is constructible.

(c) Let K be a field with algebraic closure $\overline{K}$. For each $n \geqslant 1$ let $\zeta_n \in \overline{K}$ be a root of $\Phi_n(X)$. Let $K^{\text{cyc}} = \bigcup_{n \geqslant 1} K(\zeta_n)$. Decide whether $K^{\text{cyc}} = \overline{K}$ in each of the cases $K = \mathbb{R}$, $K = \mathbb{Q}$, and $K = \mathbb{F}_p$. Justify your answers.

19J Representation Theory

Let p be an odd prime, $G = SL_2(\mathbb{F}_p)$ be the special linear group over the field with p elements,

$$B = \left\{ \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} \middle| \, a, b \in \mathbb{F}_p, a \neq 0 \right\} < G$$

be the subgroup of upper triangular matrices and

$$U = \left\{ \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \middle| \, b \in \mathbb{F}_p \right\} < B$$

be the subgroup of uni-triangular matrices.

(a) Suppose that $\theta, \varphi \colon B \to \mathbb{C}^*$ are 1-dimensional complex representations of B.

(i) State Mackey's restriction formula and explain carefully what it says for $\mathrm{Res}_B^G \mathrm{Ind}_B^G \theta$.

(ii) Determine $\langle \mathrm{Ind}_B^G \theta, \mathrm{Ind}_B^G \varphi \rangle_G$ for all possible choices of θ and φ.

(b) Let $\chi : U \to \mathbb{C}^*$ be a non-trivial one dimensional representation of U.

(i) If $v \in \mathbb{F}_p$ show that

$$\chi(v\cdot) : \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \mapsto \chi\left(\begin{pmatrix} 1 & vx \\ 0 & 1 \end{pmatrix} \right)$$

is also a one dimensional representation of U.

(ii) Now consider any representation V of B. Show that if $\langle \mathrm{Res}_U^B V, \chi \rangle_U \neq 0$, then $\langle \mathrm{Res}_U^B V, \chi(v\cdot) \rangle_U \neq 0$ for at least $\frac{p-1}{2}$ elements v in $\mathbb{F}_p^*$.

(iii) Let T be the subgroup of B consisting of diagonal matrices and let θ be a one dimensional representation of T. Show that $\mathrm{Ind}_T^B \theta$ is a sum of three pairwise non-isomorphic irreducible representations of B of dimensions 1, $\frac{p-1}{2}$ and $\frac{p-1}{2}$.

20G Number Fields

(a) State Dirichlet's unit theorem.

(b) Define the *logarithmic embedding* (Log) and prove that its kernel contains only roots of unity.

(c) What can you say about the image of a fundamental system of units $u_1, \ldots, u_m$ under the logarithmic embedding? Here m is the rank of the unit group. [You do not need to prove your answer.]

(d) Let K be a number field with r real embeddings and s pairs of complex conjugate embeddings. Let $I = \langle \beta \rangle \subset \mathcal{O}_K$ be a principal ideal. Show that

$$\mathrm{Log}(\beta) = t(1, \ldots, 1, 2, \ldots, 2) + \sum_{i=1}^{m} \lambda_i \mathrm{Log}(u_i)$$

where $t, \lambda_1, \ldots, \lambda_m \in \mathbb{R}$ and the vector $(1, \ldots, 1, 2, \ldots, 2)$ has r 1's and s 2's. Compute t in terms of $\mathrm{N}(I)$ and $[K : \mathbb{Q}]$. [*Hint: Relate the sum of the coordinates of* $\mathrm{Log}(\beta)$ *to* $\mathrm{N}(\beta)$.] [Standard facts about norms may be quoted without proof.]

(e) Prove that, for every number field K, there is a constant $C < \infty$ such that, for every principal ideal $I \subset \mathcal{O}_K$, there is an element $\alpha \in I$ such that $I = \langle \alpha \rangle$ and $|\sigma(\alpha)| < C\mathrm{N}(I)^{1/[K:\mathbb{Q}]}$ for all embeddings $\sigma : K \to \mathbb{C}$.

21F Algebraic Topology

Let X be a triangulable space. State a formula for the rational homology groups $H_i(X; \mathbb{Q})$ given the ordinary homology groups $H_i(X)$. Define the *Euler characteristic* $\chi(X)$ of X. For K a simplicial complex triangulating X, state and prove a formula relating $\chi(X)$ to numbers of simplices in K.

Let Δ^3 denote the simplicial complex given by a standard 3-simplex together with all its faces. For each i, which standard abelian group is isomorphic to $H_i(\Delta^3)$?

Let $(\Delta^3)'$ be the barycentric subdivision of Δ^3, and let M be the 2-skeleton of $(\Delta^3)'$.

(i) Calculate the Euler characteristic of $|M|$.

(ii) Use this to compute the simplicial homology groups $H_i(M)$.

(iii) Suppose $f : |M| \to |M|$ is a homeomorphism. Must f have a fixed point? Briefly justify your answer.

22I Linear Analysis

(a) State the Riesz representation theorem. For a Hilbert space H and an operator $T \in L(H)$, define the *adjoint* T^* of T, proving that it exists and that $T^* \in L(H)$. If H has an orthonormal basis, describe (without proof) the relation between the matrices of T and T^* with respect to this basis.

(b) State the spectral theorem for compact Hermitian operators on l_2. Explain why it follows from this that every compact Hermitian operator on l_2 is a limit of finite rank Hermitian operators.

(c) Prove that every compact operator on l_2 is a limit of finite rank operators.

23H Analysis of Functions

(a) Let H be a (real) Hilbert space and x_n a sequence in H that converges weakly to x in H as $n \to \infty$.

(i) Prove that $\|x\|_H \leqslant \liminf_n \|x_n\|_H$.

(ii) Prove that $\|x_n\|_H \to \|x\|_H$ if and only if $\|x_n - x\|_H \to 0$.

(iii) Must x_n converge strongly to x along a subsequence? Justify your answer.

(b) Let $\Omega \subset \mathbb{R}^n$ be open and bounded. For $u \in H_0^1(\Omega)$ and $V \in L^\infty(\Omega)$, we define the functional

$$E(u) := \int_\Omega \left(|Du|^2 + Vu^2\right) dx.$$

(i) Show that for any sequence $u_n \in H_0^1(\Omega)$ that converges weakly to u in $H_0^1(\Omega)$, we must have $E(u) \leqslant \liminf_n E(u_n)$.

[You may use the Rellich-Kondrashov theorem without proof.]

(ii) Let $\lambda = \inf \mathcal{E}$ where

$$\mathcal{E} = \left\{E(u) \ : \ u \in H_0^1(\Omega),\ \|u\|_{L^2(\Omega)} = 1\right\}.$$

Show that there exists $w \in H_0^1(\Omega)$ such that $\|w\|_{L^2(\Omega)} = 1$ and $E(w) = \lambda$.

(iii) Prove that

$$\inf \left\{ \int_{\mathbb{R}} (Du)^2 \, dx \ : \ u \in H^1(\mathbb{R}),\ \|u\|_{L^2(\mathbb{R})} = 1 \right\} = 0.$$

Is this infimum attained? Justify your answer.

24J Algebraic Geometry

In this question, all algebraic varieties are defined over an algebraically closed field k of characteristic zero.

State the Riemann–Roch theorem, giving a brief explanation of each term.

A smooth projective curve X is covered by two affine pieces (with respect to different embeddings) which are affine plane curves with equations $y^2 = f(x)$ and $v^2 = g(u)$ respectively, with f a square-free polynomial of even degree $2n > 4$ and $u = 1/x, v = y/x^n$ in the function field of X.

Determine the polynomial $g(u)$.

Using a well-chosen rational differential, compute the canonical divisor K_X of X, and show that it has degree $2n - 4$.

Compute the genus of X.

Write down a basis for the space $L(K_X)$ of rational functions with poles bounded by K_X. Conclude that X cannot be embedded into $\mathbb{P}^2$.

25I Differential Geometry

(a) Given a surface $S \subset \mathbb{R}^3$, define the *exponential map* around a point, and state and prove the Gauss lemma (expressing the first fundamental form in the local parametrisation ϕ that maps the polar coordinates on the tangent space to the surface by the exponential map).

(b) Given a surface S and a smooth curve α on S parametrised by arc-length, define the *Gauss map* and the *geodesic curvature* of α in terms of a covariant derivative. What does it mean for α to have zero *geodesic curvature*?

(c) Give a statement of the global Gauss-Bonnet theorem with boundary terms.

(d) Let $S \subset \mathbb{R}^3$ be a compact connected oriented surface diffeomorphic to a sphere and with positive Gauss curvature everywhere. Prove that any two closed geodesic curves on S must intersect.

26H Probability and Measure

We consider a rope broken into two strands of lengths Y and Z. We let $X = Y + Z$ be the random length of the rope. We assume $\mathbb{E}[X^2] < +\infty$ and $Y = XU$ where U is a random variable independent of X with uniform law on $[0,1]$ (explicitly $p_U(u) = \mathbf{1}_{[0,1]}(u)$).

(a) Compute $\mathbb{E}[Y]$ and $\mathrm{Var}[Y]$ in terms of $\mathbb{E}[X]$ and $\mathrm{Var}[X]$.

(b) From now on, we assume that X has a continuous density $g \geqslant 0$, and that $h(x) = \int_x^{+\infty} \frac{g(t)}{t} dt$ is well defined and C^1 on $\mathbb{R}_+$. Compute the densities of (X,Y), (Y,Z), Y and Z.

(c) Give a necessary and sufficient condition on h for Y and Z to be independent, and compute the law of Y and Z in this case.

27L Applied Probability

(a) State the mapping theorem for a non-homogeneous spatial Poisson process on $\mathbb{R}^d$ with intensity function λ and a map $f : \mathbb{R}^d \to \mathbb{R}^s$. You should clearly state all the necessary conditions.

(b) Assume that the positions $(x,y,z) \in \mathbb{R}^3$ of stars in space are distributed according to a homogeneous spatial Poisson process Π with a constant intensity $\lambda > 0$.

(i) Let $f : \mathbb{R}^3 \to [0,\infty)$ be given by $f(x,y,z) = (x^2 + y^2 + z^2)^{3/2}$. Show that $f(\Pi)$ is again a homogeneous Poisson process on $[0,\infty)$. What is its intensity?

(ii) Let $R_1, R_2, \ldots$ be an increasing sequence of positive random variables such that R_k denotes the distance of the k-th closest star from the origin. Find the density function for the distribution of R_k. [*Hint: The sum of n independent Exp(1) random variables has a Gamma(n) distribution with density function $x^{n-1}e^{-x}/(n-1)!$ for $x > 0$.*]

28L Principles of Statistics

(a) Given a distribution function $F : \mathbb{R} \to [0,1]$, let $F^{-1} : [0,1] \to \mathbb{R}$ be the quantile function given by

$$F^{-1}(p) := \inf\{t : F(t) \geqslant p\}.$$

Show that if $U \sim U[0,1]$ then $F^{-1}(U) \sim F$. [*Hint: F is always right continuous, that is, $F(t + a_n) \downarrow F(t)$ for all $a_n \downarrow 0$.*]

(b) Describe the steps taken by the *Gibbs sampler* to generate approximate samples from a bivariate density $f_{XY} : \mathbb{R}^2 \to [0,\infty)$. Writing $(Y_1, X_1), (Y_2, X_2), \ldots$ for the Markov chain generated by the algorithm, show that f_{XY} is stationary for its transition kernel.

(c) Let n be an even number. Consider a Bayesian model

$$Z_1, \ldots, Z_n \mid \mu, \omega \overset{\text{i.i.d.}}{\sim} N(\mu, \omega^{-1}),$$

with improper prior density $\pi(\mu, \omega) = \lambda e^{-\lambda\omega}, \omega > 0$, i.e. an $\text{Exp}(\lambda)$ density for ω and a flat prior on μ. Explain how you can generate approximate samples from the posterior distribution $\Pi(\mu, \omega \mid Z_1, \ldots, Z_n)$ if you have ways of generating independent samples from $U[0,1]$ and $N(0,1)$. [You may assume $\Pi(\mu, \omega \mid Z_1, \ldots, Z_n)$ has a well-defined density.]

[*Hint: Recall that a* $\text{Gamma}(m, \lambda)$ *distribution has density* $f(y) \propto y^{m-1} e^{-\lambda y}$*. Moreover if* $m \in \mathbb{N}$ *and* $Z_1, \ldots, Z_m \overset{\text{i.i.d.}}{\sim} \text{Exp}(\lambda)$*, then* $\sum_{i=1}^m Z_i \sim \text{Gamma}(m, \lambda)$.]

29K Stochastic Financial Models

Let f be a smooth function that grows slowly enough so that all the integrands in this question are integrable.

(a) Let $Z \sim N(0,1)$. Show that

$$\frac{1}{2}\int_0^t \mathbb{E}[f''(\sqrt{s}Z)]ds = \mathbb{E}[f(\sqrt{t}Z)] - f(0)$$

for all $t \geqslant 0$.

(b) What does it mean to say a stochastic process is a *Brownian motion*?

(c) Let $(W_t)_{t\geqslant 0}$ be a Brownian motion and f be a function satisfying the assumptions of part (a). Define a process $(M_t)_{t\geqslant 0}$ by

$$M_t = f(W_t) - \frac{1}{2}\int_0^t f''(W_s)ds.$$

Show that $(M_t)_{t\geqslant 0}$ is a martingale with respect to the filtration generated by $(W_t)_{t\geqslant 0}$.

(d) Let $(W_t)_{t\geqslant 0}$ be a continuous process with $W_0 = 0$ such that for every $c \in \mathbb{R}$ the process $(M_t)_{\geqslant 0}$ is a martingale with respect to the filtration generated by $(W_t)_{t\geqslant 0}$, where

$$M_t = e^{cW_t} - \frac{c^2}{2}\int_0^t e^{cW_s}ds.$$

Show that $(W_t)_{t\geqslant 0}$ is a Brownian motion. [*Hint: You may wish to compute the conditional moment generating function of the increment* $W_t - W_s$.]

30L Mathematics of Machine Learning

Fix $s > 0$. Let $S \subseteq \mathbb{R}^{d\times d}$ be the convex set of symmetric, positive semidefinite matrices with eigenvalues $\lambda_1 \geqslant \lambda_2 \geqslant \ldots \geqslant \lambda_d \geqslant 0$ satisfying $\sum_{i=1}^d \lambda_i \leqslant s$. For any symmetric matrix $M \in \mathbb{R}^{d\times d}$, the projection $\pi(M)$ of M onto S is defined as the minimiser of $\mathrm{Tr}((M-Z)^T(M-Z))$ over $Z \in S$.

(a) Let $M \in \mathbb{R}^{d\times d}$ be symmetric. Show that if $\Pi \in \mathbb{R}^{d\times d}$ satisfies

$$\mathrm{Tr}((M-\Pi)^T(Z-\Pi)) \leqslant 0 \text{ for all } Z \in S,$$

then Π is the projection $\pi(M)$ of M onto S. [*Hint: The function* $(A,B) \mapsto \mathit{Tr}(A^TB) = \sum_{i,j} A_{ij}B_{ij}$ *is an inner-product.*]

(b) Now suppose that $M \notin S$ is positive semidefinite, with eigenvalues and eigenvectors (μ_i, v_i) for $i = 1, \ldots, d$. Using part (a), or otherwise, show that

$$\pi(M) = \sum_{i=1}^d \max(0, \mu_i - \rho) v_i v_i^T,$$

where $\rho > 0$ is such that $\sum_{i=1}^d \max(0, \mu_i - \rho) = s$. [*Hint: By von Neumann's trace inequality, if* $A, B \in \mathbb{R}^{d\times d}$ *are symmetric with eigenvalues* $\alpha_1 \geqslant \alpha_2 \geqslant \ldots \geqslant \alpha_d \geqslant 0$ *and* $\beta_1 \geqslant \beta_2 \geqslant \ldots, \geqslant \beta_d \geqslant 0$*, respectively, then* $|\mathit{Tr}(AB)| \leqslant \sum_{i=1}^d \alpha_i\beta_i$.]

(c) Consider i.i.d. random variables $(X_1, Y_1), \ldots, (X_n, Y_n)$ taking values in $\{x \in \mathbb{R}^d : \|x\|_2 \leqslant C\} \times \{-1, 1\}$. Let $M^{(1)} = 0 \in \mathbb{R}^{d\times d}$, and iteratively define, for a step size $\eta > 0$ and iteration $i = 1, \ldots, k-1$,

$$\begin{aligned} g_i &= -\frac{1}{n}\sum_{j=1}^n Y_j X_j X_j^T \frac{\exp(-Y_j X_j^T M^{(i)} X_j)}{1 + \exp(-Y_j X_j^T M^{(i)} X_j)}, \\ M^{(i+1)} &= \pi(M^{(i)} - \eta g_i). \end{aligned}$$

Let $\bar{M} = \frac{1}{k}\sum_{i=1}^k M^{(i)}$. The function $\bar{h} : x \mapsto x^T \bar{M} x$ approximates an empirical risk minimiser $\hat{h}$ over a certain hypothesis class $\mathcal{H}$ with a certain loss function ϕ. Give explicit forms for $\mathcal{H}$ and ϕ.

Carefully quoting any necessary result from the course, show that, for a choice of step size η which you must specify,

$$\hat{R}_\phi(\bar{h}) - \hat{R}_\phi(\hat{h}) \leqslant \frac{2sC^2}{\sqrt{k}}.$$

31D Asymptotic Methods

(a) Consider the function

$$U(x) = \int_0^\infty \frac{\mathrm{e}^{-xt}}{1+t} dt,$$

where x is real and positive. Use Watson's lemma to show $U(x) \sim \sum_{n=0}^{\infty} a_n x^{-n-1}$ as $x \to \infty$, where you should determine the coefficients a_n.

(b) Show that U is a solution to the differential equation

$$xy''(x) + (1-x)y'(x) - y(x) = 0, \tag{$\dagger$}$$

for real positive x.

(c) By a suitable transformation, rewrite equation ($\dagger$) as

$$v''(x) - \tfrac{1}{4}\left(1 + Ax^{-1} + Bx^{-2}\right) v(x) = 0, \tag{$\star$}$$

where A and B are constants you need to find. Determine that positive infinity is an irregular singular point of this equation.

(d) Consider Liouville-Green solutions to equation ($\star$) of the form $v(x) = \mathrm{e}^{S(x)}$ with $S'(x) \sim \sum_{n=0}^{\infty} b_n x^{-n}$ as $x \to \infty$. Calculate terms up to, and including, order x^{-1} in the expansion of S. Find the associated asymptotic expansion of y, and compare with your solution from part (a). What is the leading order asymptotic approximation of the other solution to equation ($\dagger$) as $x \to \infty$?

32A Dynamical Systems

Define what it means for a map $F : I \to I \subset \mathbb{R}$ to be chaotic according to Devaney.

Show that the sawtooth map,

$$F(x) = 2x \text{ [mod 1]},$$

satisfies this definition for $x \in [0, 1)$.

(a) In the following use binary representation to describe the action of F.

(i) Give a value for x that produces a chaotic sequence.

(ii) Show that there is only one fixed point of F.

(iii) Find all 2-cycles and 4-cycles of F and express them as fractions.

(b) Now consider $F^n(x)$, the map F applied n times.

(i) Show that $F^n(x) = 2^n x$ [mod 1].

(ii) Use part (b)(i) to determine the number of fixed points of F^n. Explain how this is consistent with your answers in part (a)(iii).

(iii) Hence show that the number of 2^k cycles of F is $2^{2^k} - 2^{2^{k-1}}$ when $k \geqslant 1$. [Cycles starting a different points count as different cycles.]

33C Principles of Quantum Mechanics

(a) Consider the commutators $[L_i, X_j]$ and $[L_i, P_j]$ between orbital angular momentum and the position and momentum operators. Write $\mathbf{L}$ in terms of $\mathbf{X}$ and $\mathbf{P}$ and use the canonical commutation relations between position and momentum to determine these commutators.

(b) Express $\mathbf{L} \cdot \mathbf{L}$ in the form $c_1 \mathbf{X} \cdot \mathbf{P} + c_2(\mathbf{X} \cdot \mathbf{X})(\mathbf{P} \cdot \mathbf{P}) + c_3(\mathbf{X} \cdot \mathbf{P})(\mathbf{X} \cdot \mathbf{P})$ where c_1, c_2, and c_3 are constants you should determine. Hence, by expressing $\mathbf{X}$ and $\mathbf{P}$ as operators acting on wavefunctions, determine the relation between $\mathbf{L} \cdot \mathbf{L}$ and the spherical Laplacian $\nabla^2_{S^2}$. [*Hint: recall that* $\nabla^2 = \partial_r^2 + (2/r)\partial_r + r^{-2}\nabla^2_{S^2}$.]

(c) Consider the hydrogen atom and neglect the spin of the electron. Give a basis that spans the degenerate energy subspace corresponding to the first excited state ($n = 2$ in the usual labelling). Focussing exclusively on this subspace, consider the following two scenarios.

(i) The Hamiltonian is perturbed by $\Delta H = g\mathbf{L} \cdot \mathbf{L}$ where g is a constant. Compute the new energy eigenstates and eigenvalues exactly.

(ii) The Hamiltonian is perturbed by $\Delta H = E(t)X_3 + B(t)P_3$, where $E(t)$ and $B(t)$ are time-dependent functions. Determine which matrix elements of ΔH must vanish using the result for the commutators $[L_3, X_3]$ and $[L_3, P_3]$ and the transformation of X_3 and P_3 under parity. Now define

$$z(t) := E(t)\langle 2,0,0|X_3|2,1,0\rangle + B(t)\langle 2,0,0|P_3|2,1,0\rangle \in \mathbb{C}$$

and assume $z(t) = c\, e^{i\omega t/\hbar}$ for some real constants c and ω. Find an exact solution of the Schrödinger equation given an initial state of the form $A_0|2,0,0\rangle + A_1|2,1,0\rangle$ where $A_0, A_1 \in \mathbb{C}$.

34B Applications of Quantum Mechanics

Consider a particle of mass m and electric charge e under the influence of a magnetic field $\mathbf{B}$ and a periodic potential. In these circumstances, the Hamiltonian is

$$H = \frac{1}{2m}\left[\mathbf{p} - e\mathbf{A}(\mathbf{x})\right]^2 + V(\mathbf{x}) \ .$$

Here $\mathbf{p} = -i\hbar\boldsymbol{\nabla}$ is the canonical momentum and $V(\mathbf{x})$ is a periodic potential dictated by the lattice Λ, that is, $V(\mathbf{x}) = V(\mathbf{x}+\mathbf{r})$ for all $\mathbf{r} \in \Lambda$. The magnetic field $\mathbf{B}$ is constant and we adopt the gauge

$$\mathbf{A}(\mathbf{x}) = \frac{1}{2}\mathbf{B} \times \mathbf{x} \ .$$

(a) Evaluate the commutators

$$[p_i + eA_i, p_j - eA_j] \qquad \text{and} \qquad [p_i + eA_i, p_j + eA_j] \ .$$

(b) The translation operator is defined as $T_{\mathbf{r}} = e^{i\mathbf{r}\cdot\mathbf{p}/\hbar}$. Show that $T_{\mathbf{r}}$ does not commute with H in the presence of a magnetic field.

In the presence of a magnetic field, it is useful to introduce the magnetic translation operator, defined as

$$\mathcal{T}_{\mathbf{r}} = \exp\left\{\frac{i}{\hbar}\mathbf{r}\cdot[\mathbf{p} + e\mathbf{A}(\mathbf{x})]\right\} \ .$$

Show how $\mathcal{T}_{\mathbf{r}}$ acts on functions, and show that $\mathcal{T}_{\mathbf{r}}$ commutes with the Hamiltonian.

(c) Show that

$$\mathcal{T}_{\mathbf{r}}\mathcal{T}_{\mathbf{r}'} = \exp\left[\frac{ie}{\hbar}(\mathbf{r}\times\mathbf{r}')\cdot\mathbf{B}\right]\mathcal{T}_{\mathbf{r}'}\mathcal{T}_{\mathbf{r}} \ .$$

The magnetic flux through a cell of the lattice is defined as $\Phi = \mathbf{B}\cdot(\mathbf{r}\times\mathbf{r}')$. Under what condition on Φ do the magnetic translations form an abelian group?

[*Hint: you may use, without proof, that* $e^{\mathbf{M}}e^{\mathbf{N}} = e^{\mathbf{M}+\mathbf{N}+\frac{1}{2}[\mathbf{M},\mathbf{N}]}$ *if* $\mathbf{M}$ *and* $\mathbf{N}$ *commute with* $[\mathbf{M},\mathbf{N}]$.]

35C Statistical Physics

(a) Define the *latent heat* L between the gas and liquid phases of a substance. Starting with the Gibbs free energy, derive the Clausius–Clapeyron relation for the first-order phase transition in the (T, p) plane

$$\frac{dp}{dT} = \frac{L}{T(V_{\text{gas}} - V_{\text{liq}})} .$$

What happens to L at the critical point?

(b) Consider a chain of N spin-1 atoms in an external magnetic field B, each with Hamiltonian

$$H = -\mu B s_z ,$$

where μ is a constant, and $s_z \in \{-1, 0, 1\}$. Suppose that the spin chain is in a canonical ensemble with inverse temperature β. Calculate the free energy F and the heat capacity C. What is the high-temperature limit of $-\beta F$ and C?

(c) Consider the same system as in part (b), but now suppose that the sign of the external magnetic field B is instantly reversed by an experimenter. What happens to each of β, F, and C?

Explain why the resulting system cannot be in thermal equilibrium with any gas. If the system is coupled to a gas, in which direction will heat flow? Your answers should make reference to the appropriate law(s) of thermodynamics.

36B Electrodynamics

(a) State Maxwell's equations for the fields $\mathbf{E}$, $\mathbf{H}$, $\mathbf{D} = \epsilon\,\mathbf{E}$ and $\mathbf{B} = \mu\,\mathbf{H}$ in a linear dielectric medium with electric and magnetic polarisation constants ϵ and μ. You may assume the absence of free charges and currents. Show explicitly that Maxwell's equations admit plane-wave solutions propagating with speed $v = 1/\sqrt{\epsilon\mu}$ and determine the magnetic polarisation vector $\mathbf{B}_0$ in terms of the corresponding electric polarisation vector $\mathbf{E}_0$ and the wave vector $\mathbf{k}$.

(b) Consider two such media, having distinct values $\epsilon_+ > \epsilon_-$ of the electric polarisation constant, filling the regions $x > 0$ and $x < 0$ respectively. The two media are assumed to share a common value of μ. Write down, with brief justification, boundary conditions for the components of the fields tangent and normal to the interface plane $x = 0$. You should state clearly which field components are continuous and which are discontinuous.

(c) Suppose an electromagnetic wave is incident from the region $x < 0$ resulting in a transmitted wave in the region $x > 0$ and also a reflected wave for $x < 0$. The angles of incidence, reflection and transmission are denoted θ_I, θ_R and θ_T respectively. By constructing a corresponding solution of Maxwell's equations, derive the law of reflection $\theta_I = \theta_R$ and Snell's law of refraction, $n_- \sin\theta_I = n_+ \sin\theta_T$, where $n_\pm$ are the indices of refraction of the two media in question.

(d) The incident, reflected and transmitted waves have polarisation vectors $\mathbf{E}_I$, $\mathbf{E}_R$ and $\mathbf{E}_T$ respectively. In the case that these vectors are all *normal* to the plane of incidence (the plane spanned by the incident and reflected wave vectors), determine the ratio $|\mathbf{E}_R|/|\mathbf{E}_I|$ as a function of θ_I and show that it is always non-zero for $0 \leqslant \theta_I \leqslant \pi/2$.

37E General Relativity

The metric $g_{\alpha\beta}$ for a four-dimensional spacetime satisfies the Einstein equations

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = \kappa T_{\mu\nu} ,$$

where $R_{\mu\nu}$ and R are the Ricci tensor and Ricci scalar, $T_{\mu\nu}$ is the energy-momentum tensor, and κ is a constant. We use units where $c = 1$ and assume throughout that $g_{\alpha\beta} = \eta_{\alpha\beta} + h_{\alpha\beta}$ where $\eta_{\alpha\beta} = \mathrm{diag}(-1, 1, 1, 1)$ is the Minkowski metric in Cartesian coordinates and $h_{\alpha\beta}$ and its derivatives are small. Then the Riemann tensor is given by

$$R_{\mu\nu\alpha\beta} = \frac{1}{2} \left(h_{\mu\beta,\,\nu\alpha} - h_{\nu\beta,\,\mu\alpha} - h_{\mu\alpha,\,\nu\beta} + h_{\nu\alpha,\,\mu\beta} \right) ,$$

to first order in small quantities.

(a) Let $x^0 = t$ and $x^i \, (i = 1, 2, 3)$ denote the Cartesian coordinates. Assume both that (i) $T_{00} = \rho$ is the mass density, where $\kappa\rho$ is small, and all other components of the energy-momentum tensor are negligible, and (ii) the metric is *almost static*, meaning that derivatives of $h_{\alpha\beta}$ with respect to t are negligible. Find R_{00}, working to first order in all small quantities, and hence show that

$$-\nabla^2 h_{00} = \kappa\rho \qquad \text{where} \qquad \nabla^2 = \delta^{ij} \partial_i \partial_j \, . \tag{$\dagger$}$$

(b) A massive particle moves non-relativistically in the spacetime of part (a), with $v^i = dx^i/dt$ small. Starting from the geodesic equations, show that

$$\frac{dv^i}{dt} = \frac{1}{2} \delta^{ij} \partial_j h_{00} \, , \tag{$\star$}$$

working to first order in both v^i and $h_{\alpha\beta}$. [You may quote the formula for the Levi-Civita connection $\Gamma^{\ \mu}_{\alpha\ \beta}$.]

By comparing equations ($\dagger$) and ($\star$) with the corresponding Newtonian equations, express h_{00} and κ in terms of the Newtonian gravitational potential Φ and Newton's constant G, where $\nabla^2 \Phi = 4\pi G \rho$.

(c) Consider a point mass M at the origin $r = 0$, where $r^2 = \delta_{ij} x^i x^j$, in an otherwise vacuum spacetime. Write down the Newtonian potential Φ for this point mass. Suppose that

$$h_{ij} = f(r) \, x_i x_j \, ,$$

for some function $f(r)$, where indices $i, j = 1, 2, 3$ are raised and lowered using δ_{ij}. By considering the Ricci scalar, or otherwise, find a differential equation for $f(r)$ and obtain the general solution for $r > 0$.

[You may use, without proof, the identities

$$\partial_i \partial_i [\, r^2 f(r) \,] = r^2 f'' + 6 r f' + 6 f \quad \text{and} \quad \partial_i \partial_j [\, x_i x_j f(r) \,] = r^2 f'' + 8 r f' + 12 f \, ,$$

where the summation convention applies to repeated indices of type $i, j = 1, 2, 3$.]

38D Fluid Dynamics

(a) Consider the incompressible flow of a Newtonian fluid with constant dynamic viscosity μ and density ρ subject to a conservative body force $\mathbf{f} = -\boldsymbol{\nabla}\psi$. Derive the equation for the rate of change of kinetic energy of the fluid in a domain $\mathcal{D}$ with boundary $\partial\mathcal{D}$ in the form

$$\frac{d}{dt}\int_{\mathcal{D}} \tfrac{1}{2}\rho|\mathbf{u}|^2\, dV + \int_{\partial\mathcal{D}} \tfrac{1}{2}\rho|\mathbf{u}|^2\mathbf{u}\cdot\mathbf{n}\, dS = \int_{\mathcal{D}} \mathbf{u}\cdot\mathbf{f}\, dV + \int_{\partial\mathcal{D}} \mathbf{u}\cdot\boldsymbol{\sigma}\cdot\mathbf{n}\, dS - 2\mu\int_{\mathcal{D}} \mathbf{e}:\mathbf{e}\, dV,$$

where $\mathbf{n}$ is the unit normal vector directed out of the boundary $\partial\mathcal{D}$, $\boldsymbol{\sigma}$ is the stress tensor and $\mathbf{e}$ is the rate-of-strain tensor. Give the physical interpretation of each term in this integral equation.

(b) Small-amplitude, free-surface waves on deep water occupying $-\infty < z < \eta(x,t) = A\exp(ikx - i\omega t)$ can be described by a velocity potential

$$\phi = B\exp(kz)\exp(ikx - i\omega t),$$

where A, B and k are constants, g is the gravitational acceleration, $\omega^2 = gk$ and real parts of complex quantities may be assumed. Determine B in terms of A, ω and k.

Assume now that the amplitude A is slowly varying, so A can be treated as constant over one period of oscillation. Determine the mean rate of dissipation averaged over a period of oscillation. Given that the total mean energy is $\frac{1}{2}\rho g|A|^2$, determine the slow rate of decay of the wave amplitude.

[*Hint: The mean over a period of the product of the real part of periodic complex functions F and G is the real part of $\frac{1}{2}FG^*$.*]

39A Waves

A perfect (but unusual) gas occupies a tube that lies parallel to the x-axis. The gas is initially at rest, with density ρ_0, pressure p_0, and specific heat ratio $\gamma = 3$, and occupies the region $x > 0$. At times $t > 0$, a piston, initially at $x = 0$, is pushed into the gas at a constant speed u_1. A shock wave then propagates at a constant speed V into the undisturbed gas ahead of the piston. Downstream of the shock, i.e. in the region between the piston and the shock, the density is $\rho_1 > \rho_0$ and the pressure is $p_1 > p_0$.

(a) Transform into a frame where the shock is at rest and write down the appropriate expressions for conservation of mass, momentum and energy across the shock.

(b) Determine the ratio ρ_1/ρ_0 when the shock moves three times as fast as the piston, i.e. when $u_1 = V/3$.

(c) Determine the corresponding ratio p_1/p_0 when $u_1 = V/3$.

(d) Express V in terms of p_0 and ρ_0 when $u_1 = V/3$.

[*You may assume that the internal energy per unit mass of perfect gas is $p/[\rho(\gamma-1)]$.*]

40D Numerical Analysis

(a) State and prove the Householder-John theorem.

(b) Define the *Jacobi method* for solving a system $A\mathbf{x} = \mathbf{b}$, with $A \in \mathbb{R}^{n \times n}$ and $\mathbf{b} \in \mathbb{R}^n$. Show that if A is a symmetric, positive-definite, tridiagonal matrix,

$$A = \begin{bmatrix} a_1 & b_1 & 0 & \cdots & 0 \\ b_1 & a_2 & b_2 & \ddots & \vdots \\ 0 & b_2 & a_3 & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & b_{n-1} \\ 0 & \cdots & 0 & b_{n-1} & a_n \end{bmatrix},$$

then the Jacobi method converges.

[*You may use without proof general convergence results of iterative methods for linear systems based on the spectral radius, provided they are clearly stated.*]

END OF PAPER