UNIVERSITY OF CAMBRIDGE

MAT2

MATHEMATICAL TRIPOS Part II

Thursday, 12 June, 2025 9:00am to 12:00pm

PAPER 3

Before you begin read these instructions carefully.

The examination paper is divided into two sections. Each question in Section II carries twice the number of marks of each question in Section I. Section II questions also carry an alpha or beta quality mark and Section I questions carry a beta quality mark.

Candidates may obtain credit from attempts on **at most six questions** from Section I and from any number of questions from Section II.

Write on **one side** of the paper only and begin each answer on a separate sheet.

Write legibly; otherwise you place yourself at a grave disadvantage.

At the end of the examination:

Separate your answers to each question.

Complete a gold cover sheet **for each question** that you have attempted, and place it at the front of your answer to that question.

Complete a green main cover sheet listing **all the questions** that you have attempted.

Every cover sheet must also show your Blind Grade Number and desk number.

Tie up your answers and cover sheets into **a single bundle**, with the main cover sheet on the top, and then the cover sheet and answer for each question, in the numerical order of the questions.

STATIONERY REQUIREMENTS
Gold cover sheets
Green main cover sheet
Script paper
Rough paper

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

SECTION I

1G Number Theory

(a) Let $f(x,y)$, $g(x,y)$ be binary quadratic forms. Define what it means for f and g to be *equivalent*.

(b) A binary quadratic form $f(x,y) = ax^2 + bxy + cy^2$ is said to be primitive if $\gcd(a,b,c) = 1$ (i.e. there is no prime number p dividing each of a, b, and c). Show that if f and g are equivalent, then f is primitive if and only if g is primitive.

(c) Compute the number of equivalence classes of primitive, positive definite binary quadratic forms of discriminant -80.

2I Topics in Analysis

State Runge's theorem about the uniform approximation of holomorphic functions by polynomials.

Explain how to explicitly construct a sequence of polynomials converging uniformly to $1/z$ on the semicircle $\{z : |z| = 1,\ \operatorname{Re} z \leqslant 0\}$.

Show that there exists a sequence of polynomials $P_n(z)$ such that

$$P_n(z) \to \begin{cases} 1 & \text{if } |z| < 1 \text{ and } \operatorname{Re} z > 0, \\ 0 & \text{if } |z| < 1 \text{ and } \operatorname{Re} z = 0, \\ -1 & \text{if } |z| < 1 \text{ and } \operatorname{Re} z < 0 \end{cases}$$

pointwise as $n \to \infty$.

3K Coding & Cryptography

Explain what is meant by a *Bose-Ray Chaudhuri-Hocquenghem (BCH) code with design distance* δ. Prove that, for such a code, the minimum distance between codewords is at least δ. [Results about the Vandermonde determinant may be quoted without proof provided they are clearly stated.]

How many errors will the code detect? How many errors will it correct? Justify your answers.

4F Automata & Formal Languages

(a) The following register machines M and N given explicitly by their programs compute characteristic functions, i.e., $f_{M,1} = \chi_A$ and $f_{N,1} = \chi_B$. Determine A and B. Justify your answer.

$$\begin{array}{lll}
 & & M \\
q_{\mathrm{S}} & \mapsto & ?_{\varepsilon}(0, q_1, q_0) \\
q_0 & \mapsto & -(0, q_2, q_0) \\
q_1 & \mapsto & +_{\mathbf{1}}(0, q_{\mathrm{H}}) \\
q_2 & \mapsto & +_{\mathbf{0}}(0, q_{\mathrm{H}}) \\
 & & \\
q_{\mathrm{H}} & \mapsto & ?_{\varepsilon}(0, q_{\mathrm{H}}, q_{\mathrm{H}})
\end{array}
\qquad
\begin{array}{lll}
 & & N \\
q_{\mathrm{S}} & \mapsto & ?_{\mathbf{0}}(0, q_3, q_0) \\
q_0 & \mapsto & -(0, q_2, q_0) \\
q_1 & \mapsto & +_{\mathbf{1}}(0, q_{\mathrm{H}}) \\
q_2 & \mapsto & +_{\mathbf{0}}(0, q_{\mathrm{H}}) \\
q_3 & \mapsto & -(0, q_1, q_3) \\
q_{\mathrm{H}} & \mapsto & ?_{\varepsilon}(0, q_{\mathrm{H}}, q_{\mathrm{H}})
\end{array}$$

(b) By modifying N or otherwise, give the explicit program of a register machine that computes the characteristic function of the set $\{w\mathbf{1}\,;\, w \in \mathbb{B}\}$. Justify your answer.

(c) Suppose you are given the program of a register machine that computes the characteristic function of a set X. Describe how to explicitly modify the given program in order to obtain the program of a register machine that computes the characteristic function of the complement $\mathbb{B}\backslash X$. Justify your answer.

5K Statistical Modelling

Define the *generalized linear model* in its most general form as introduced in the lectures, which should include a link function, a dispersion parameter, and known weights on the data points. Your answer should clearly describe the mathematical assumptions on different components of the model. Write down the log-likelihood function of this model (up to an additive constant).

What is the canonical link function for Poisson generalized linear models? Justify your answer.

6A Mathematical Biology

A population $n(t)$ is modelled by the Malthusian delay differential equation

$$\frac{dn}{dt}(t) = r\,n(t - \tau)\,,$$

where r and $\tau > 0$ are constants.

(a) Give a biological interpretation of the delay time τ.

(b) Suppose that $n(t) = 1$ for $t \in [-\tau, 0]$. Show that $n(t) = rt + 1$ for $t \in [0, \tau]$. Determine $n(t)$ for $t \in [\tau, 2\tau]$.

(c) Show that the delay differential equation admits a periodic solution when $r\tau = -\pi/2$. Why is this solution not appropriate for describing a population?

7E Further Complex Methods

The beta function is defined by

$$B(p,q) = \int_0^1 (1-t)^{p-1}\, t^{q-1}\, dt,$$

for $\mathrm{Re}\,(p) > 0$ and $\mathrm{Re}\,(q) > 0$.

(a) By writing $\Gamma(z)^2$ as a double integral, show that for $\mathrm{Re}\,(z) > 0$,

$$\Gamma(z)^2 = B(z,z)\Gamma(2z)\,,$$

where $\Gamma(z)$ denotes the gamma function. [*Hint: You may find the transformation* $(s,t) \to (r,u)$*, given by* $t = ru$*,* $s = r(1-u)$*, helpful.*]

(b) Deduce that $B(z,z) \sim 2/z$ as $z \to 0$ with $\mathrm{Re}\,(z) > 0$.

8B Classical Dynamics

Let $\mathcal{C}$ be a solid cone of height l with circular cross-section of radius R at the base. (The height is the distance between the base and the vertex along the axis of symmetry.) Denote the axis of symmetry by $\mathbf{e}_3$, which is directed from the vertex of $\mathcal{C}$ to its base. The cone has uniform density ρ, so that the total mass is $M = \frac{1}{3}\pi R^2 l\rho$.

The principal moments of inertia of $\mathcal{C}$ with respect to its centre of mass are

$$I_1^{\mathrm{CM}} = I_2^{\mathrm{CM}} = \frac{1}{80}\pi R^2 l(4R^2 + l^2)\rho \quad \text{and} \quad I_3^{\mathrm{CM}} = \frac{1}{10}R^4 l\rho\,.$$

Using standard Euler angles (ψ, θ, ϕ) to describe the orientation of $\mathcal{C}$, the angular velocity has components

$$\boldsymbol{\omega} = (\dot\psi + \cos\theta\,\dot\phi)\mathbf{e}_3 + (\cos\psi\sin\theta\,\dot\phi - \sin\psi\,\dot\theta)\mathbf{e}_2 + (\sin\psi\sin\theta\,\dot\phi + \cos\psi\,\dot\theta)\mathbf{e}_1$$

with respect to the principal axes $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$.

(a) Compute the centre of mass of $\mathcal{C}$. State the parallel axis theorem. Find the principal moments of inertia of $\mathcal{C}$ about its vertex.

(b) Let the vertex of $\mathcal{C}$ be fixed. Using the formulae for the angular velocity with respect to the principal axes given above, write down the Lagrangian for the dynamics of $\mathcal{C}$, taking the magnitude of downward gravitational acceleration g to be constant. Identify any ignorable (that is to say, cyclic) coordinates, and find the corresponding conserved quantities. Obtain the Hamiltonian for the system.

(c) Find a Hamiltonian system on a two-dimensional phase space by reducing the number of degrees of freedom by means of the ignorable coordinates you found in part (b). Show that the configuration in which $\mathbf{e}_3$ is oriented vertically upward is an equilibrium which is stable if the angular momentum about this axis is sufficiently large.

9E Cosmology

The equation governing the evolution of density-perturbation modes $\delta(\mathbf{k},\tau)$ in conformal time τ is

$$\delta''(\mathbf{k},\tau) + \mathcal{H}(\tau)\,\delta'(\mathbf{k},\tau) - \frac{3}{2}\Omega_M(\tau)\,\mathcal{H}(\tau)^2\delta(\mathbf{k},\tau) = 0\,, \qquad (\dagger)$$

where a prime denotes $\frac{\partial}{\partial\tau}$, $\mathcal{H}(\tau) = a'/a$, a is the scale factor, Ω_M is the density of non-relativistic matter relative to the total density, and $\mathbf{k}$ is the comoving wavevector.

In the following, we consider a flat, matter-dominated universe after equal matter-radiation ($\tau \geqslant \tau_{\mathrm{eq}}$), for which you may assume that $\Omega_M \approx 1$ and $a(\tau) = (\tau/\tau_0)^2$, where τ_0 is the conformal time today.

(a) By seeking a power-law solution of the form $\delta = \tau^\beta$, show that the general solution of equation ($\dagger$) for the matter-dominated era ($\tau_{\mathrm{eq}} \leqslant \tau \leqslant \tau_0$) takes the form

$$\delta(\mathbf{k},\tau) = A(\mathbf{k})\tau^2 + B(\mathbf{k})\tau^{-3}\,, \qquad (*)$$

where $A(\mathbf{k})$, $B(\mathbf{k})$ are arbitrary functions.

(b) Show that a mode with physical wavelength $\lambda(\tau) = 2\pi a(\tau)/k$, corresponding to the comoving wavenumber $k = |\mathbf{k}|$, crosses inside the cosmological horizon at time $\tau_H = 2\pi/(k\,c)$. Now consider a perturbation mode $\delta(\mathbf{k}_{\mathrm{eq}},\tau)$ with wavevector $\mathbf{k}_{\mathrm{eq}}$ that crosses inside the cosmological horizon at $\tau_H = \tau_{\mathrm{eq}}$, that is, at time of equal matter-radiation. Using equation ($*$), show that the linear growth of this perturbation mode is given today by

$$D_{\mathrm{eq}} \;:=\; \frac{\delta(\mathbf{k}_{\mathrm{eq}},\tau_0)}{\delta(\mathbf{k}_{\mathrm{eq}},\tau_{\mathrm{eq}})} = \frac{a(\tau_0)}{a(\tau_{\mathrm{eq}})}\,.$$

(c) Assume that for each wavevector $\mathbf{k}$, the amplitude of the corresponding mode at its horizon crossing time τ_H is given by $|\delta(\mathbf{k},\tau_H)| = \tau_H^2\,\hat{A}\,k^{1/2}$ with constant $\hat{A}$. Show that the power spectrum today takes the form

$$|\delta(\mathbf{k},\tau_0)|^2 = \frac{C}{k_{\mathrm{eq}}^4}k\,, \qquad k_{\mathrm{eq}} \leqslant k \leqslant k_0\,,$$

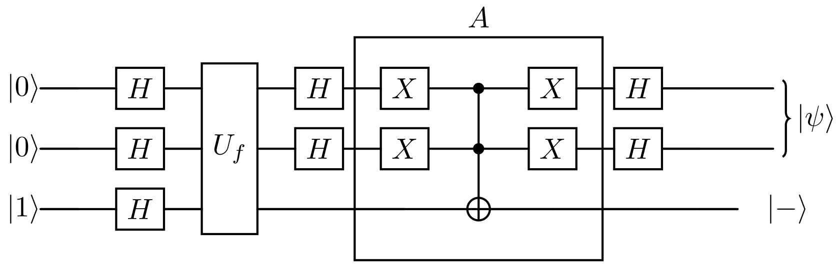
where the amplitude C should be specified in terms of $\hat{A}$ and D_{eq}. Here, k_0 is the wavenumber of a mode crossing inside the cosmological horizon at τ_0.

10C Quantum Information and Computation

Suppose you have a search space of dimension 4, with its elements encoded in binary $\{00, 01, 10, 11\}$. You are searching for the element $x_0 = 11$.

(a) Construct the circuit implementing the quantum oracle $U_f : |x\rangle |y\rangle \to |x\rangle |y \oplus f(x)\rangle$, for the function f, where $f(x) = 1$ if $x = x_0$, otherwise $f(x) = 0$.

(b) Consider the following quantum circuit:



(i) Prove that the boxed part, A, of the circuit implements the operator $I_0 = I - 2|00\rangle\langle 00|$, by showing that $A\,|x_1 x_2\rangle\,|-\rangle = I_0\,|x_1 x_2\rangle\,|-\rangle$ for $x_1, x_2 \in \{0, 1\}$. Hence, justify that the entire circuit implements the initial Hadamard transformations and a single Grover iteration $-Q = H^{\otimes 2} I_0 H^{\otimes 2} I_{x_0}$, where $I_{x_0} = I - 2\,|x_0\rangle\,\langle x_0|$.

(ii) Compute the output state $|\psi\rangle$.

(iii) What happens when we measure $|\psi\rangle$ in the computational basis?

(iv) How many times do we have to repeat $-Q$ to obtain x_0 in this example?

SECTION II

11G Number Theory

(a) Let $\theta \in \mathbb{R}$ be an irrational number with continued fraction expansion $\theta = [a_0, a_1, a_2, \ldots]$. Define the *convergents* p_n, q_n of θ. Show that if $\gamma > 0$, then there is a formula for each $n \geqslant 1$:

$$[a_0, a_1, a_2, \ldots, a_n, \gamma] = \frac{p_n\gamma + p_{n-1}}{q_n\gamma + q_{n-1}}.$$

(b) Compute the continued fraction expansion of $\theta = \sqrt{11}$.

(c) Let p_n, q_n be the convergents of $\theta = \sqrt{11}$. Show that if $n \geqslant 2$ is even, then

$$p_n^2 - 11q_n^2 = -2.$$

12F Automata & Formal Languages

(a) Let $D = (\Sigma, Q, \delta, q_0, F)$ be a deterministic automaton.

(i) Define what it means that a state $q \in Q$ is *inaccessible.*
(ii) Define what it means that two states $q, q' \in Q$ are *indistinguishable.*
(iii) Define what it means that the automaton D is *irreducible.*
(iv) State the relationship between irreducibility and the size of the smallest automaton for a regular language.

(b) Let $N = (\Sigma, Q, \Delta, q_0, F)$ be a non-deterministic automaton and $w = a_0 ... a_{n-1} \in \mathbb{W}$. We say that a sequence $(p_0, ..., p_n) \in Q^{n+1}$ is a witnessing sequence for w if for all $i < n$, we have $p_{i+1} \in \Delta(p_i, a_i)$. We say that it starts with q if $p_0 = q$ and that it ends with q' if $p_n = q'$.

(i) Let $q \in Q$ and $w \in \mathbb{W}$. Define $\widehat{\Delta}(q, w)$ and what it means that $w \in \mathcal{L}(N)$.
(ii) Describe the *subset construction* that takes a non-deterministic automaton N and constructs a deterministic automaton D such that $\mathcal{L}(D) = \mathcal{L}(N)$. [You should provide the construction of D but you do not need to prove that $\mathcal{L}(D) = \mathcal{L}(N)$.]
(iii) Prove that for $q, q' \in Q$ and $w \in \mathbb{W}$, we have that $q' \in \widehat{\Delta}(q, w)$ if and only if there is a witnessing sequence for w that starts with q and ends with q'.

(c) We say that a non-deterministic automaton $N = (\Sigma, Q, \Delta, q_0, F)$ is a Brzozowski automaton if

(Br$_1$) $F = \{q_*\}$ is a singleton;
(Br$_2$) for every $q \in Q$ there is a $w \in \mathbb{W}$ and a witnessing sequence for w starting from q and ending in q_*;
(Br$_3$) for every $w \in \mathbb{W}$ there is a unique $q \in Q$ such that there is a witnessing sequence for w starting from q and ending in q_*.

Let N be a Brzozowski automaton, D be the result of the subset construction applied to N, and D' be the automaton D with all inaccessible states removed. Show that D' is an irreducible automaton such that $\mathcal{L}(D') = \mathcal{L}(N)$.

[*In the entire question, you may use any results proved in the lectures, provided that you state them clearly.*]

13A Mathematical Biology

A discrete population $n = 0, 1, 2, \ldots$ undergoes a stochastic birth-death process, with birth rate $\alpha + \beta n$ and death rate $\gamma n(n-1)$, where α, β, and γ are all positive constants. Let $P_n(t)$ be the probability that the population is n at time t.

(a) Write down the master equation for $P_n(t)$.

(b) Determine a differential equation for the expectation value $\langle n(t) \rangle$.

(c) Assume that the probability distribution can be approximated by the Poisson distribution $P_n(t) = e^{-\lambda(t)} \lambda(t)^n / n!$ for some $\lambda(t)$. Compute $\langle n \rangle$ and $\langle n(n-1) \rangle$ in terms of $\lambda(t)$. Write down a differential equation for $\lambda(t)$ and determine the value of $\langle n(t) \rangle$ as $t \to \infty$.

(d) Treating $n(t)$ as continuous, and renaming it $x(t)$, the Fokker-Planck equation for the probability $P(x,t)$ takes the form

$$\frac{\partial P}{\partial t} = -\frac{\partial (uP)}{\partial x} + \frac{\partial^2 (DP)}{\partial x^2} \ .$$

What are the functions $u(x)$ and $D(x)$ for the birth-death process above? Show that $d\langle x(t) \rangle / dt$ is determined by the expectation value of $u(x)$ and that the resulting differential equation coincides with the equation for $\langle n(t) \rangle$ computed in part (b) above.

14E Cosmology

(a) Consider non-relativistic particles of mass m in equilibrium at temperature T with chemical potential μ. Assuming $k_{\rm B}T \ll mc^2$ and $\mu \ll mc^2$, show that both Bose-Einstein and Fermi-Dirac distributions reduce to the Maxwell-Boltzmann distribution

$$n = \left(\frac{4\pi g_s}{h^3}\right) \int_0^\infty dp\, p^2\, e^{-[E(p)-\mu]/(k_{\rm B}T)}\,.$$

Using $\int_0^\infty dp\, e^{-p^2/\sigma^2} = \frac{1}{2}\sigma\sqrt{\pi}$, show that for $E(p) = mc^2 + p^2/(2m)$,

$$n = g_s \left(\frac{2\pi m k_{\rm B} T}{h^2}\right)^{\frac{3}{2}} e^{(\mu - mc^2)/(k_{\rm B}T)}\,. \qquad (\dagger)$$

(b) The recombination of free electrons in the early universe is significantly affected by the abundance of helium-4 in the universe, which, in terms of the baryon density $n_{\rm B}$, is given by the parameter

$$Y_p = \frac{m_{\rm He}}{m_{\rm H}}\frac{n_{\rm He}}{n_{\rm B}} = 4\,\frac{n_{\rm He}}{n_{\rm B}} \;\approx\; \frac{1}{4}\,.$$

In the following we neglect doubly-ionized helium ($n_{\rm He^{++}} \approx 0$). Then the recombination of hydrogen and helium proceeds with ionization energies $I_{\rm H}$ and $I_{\rm He}$ according to

$$\begin{aligned} {\rm H}^+ + {\rm e}^- &\leftrightarrow {\rm H}^0 + \gamma\,, & I_{\rm H} &= (m_{\rm H^+} + m_{\rm e} - m_{\rm H^0})\,c^2 \;\approx\; 13.6\,{\rm eV}\,,\\ {\rm He}^+ + {\rm e}^- &\leftrightarrow {\rm He}^0 + \gamma\,, & I_{\rm He} &= (m_{\rm He^+} + m_{\rm e} - m_{\rm He^0})\,c^2 \;\approx\; 25.6\,{\rm eV}\,. \end{aligned}$$

(i) Using these equilibrium processes and equation ($\dagger$) together with $g_e = 2$, $g_{\rm H^+}/g_{\rm H^0} = \frac{1}{2}$ and $g_{\rm He^+}/g_{\rm He^0} = 1$, show that

$$\begin{aligned} \frac{n_{\rm e}\, n_{\rm H^+}}{n_{\rm H^0}} &= \left(\frac{2\pi\, m_{\rm e}\, k_{\rm B}T}{h^2}\right)^{3/2} e^{-I_{\rm H}/(k_{\rm B}T)}\,,\\ \frac{n_{\rm e}\, n_{\rm He^+}}{n_{\rm He^0}} &= 2\left(\frac{2\pi\, m_{\rm e}\, k_{\rm B}T}{h^2}\right)^{3/2} e^{-I_{\rm He}/(k_{\rm B}T)}\,. \end{aligned}$$

(ii) The hydrogen and helium ionization fractions are $X_{\rm H^+} := n_{\rm H^+}/n_{\rm H}$ and $X_{\rm He^+} := n_{\rm He^+}/n_{\rm He}$ (with $n_{\rm H} = n_{\rm H^0} + n_{\rm H^+}$, $n_{\rm He} = n_{\rm He^0} + n_{\rm He^+}$). Show that the free electron density is

$$\mathcal{F} \;\equiv\; \frac{n_{\rm e}}{n_{\rm B}} = \alpha\, X_{\rm H^+} + \beta\, X_{\rm He^+}\,,$$

where α and β should be specified in terms of Y_p.

(iii) Given the relation $n_B = \eta\, n_\gamma = \eta\left[16\pi\zeta(3)/(hc)^3\right](k_{\rm B}T)^3$ (with baryon-to-photon ratio η), use the fractional densities $X_{\rm H^+}$, $X_{\rm He^+}$ and $\mathcal{F}$ to obtain a closed set of two equations that describes recombination for both hydrogen and helium. Verify that taking the $Y_p \to 0$ limit yields the usual expression for Saha's equation with only hydrogen.

15C Quantum Information and Computation

(a) Given two equally likely states $|\alpha_0\rangle$ and $|\alpha_1\rangle$, show that the probability P_s of correctly distinguishing between the states using a quantum measurement is bounded as follows,

$$P_s \leqslant \frac{1}{2}\left(1 + \sqrt{1 - |\langle\alpha_0|\alpha_1\rangle|^2}\right),$$

and that the bound is tight. Consequently, show that $|\alpha_0\rangle$ and $|\alpha_1\rangle$ can be perfectly distinguished, that is, $P_s = 1$, if and only if they are orthogonal.

(b) Consider the task of distinguishing between two equally likely unitary gates U_1 and U_2. This is accomplished by choosing some state $|\psi\rangle$ and then distinguishing between the outputs $U_1|\psi\rangle$ and $U_2|\psi\rangle$ as in part (a). Let us define the numerical range of a unitary U as the following subset of the complex plane

$$N(U) := \{\langle\psi|U|\psi\rangle \;:\; ||\psi|| = 1\} \subseteq \mathbb{C}\,.$$

Show that U_1 and U_2 can be perfectly distinguished if and only if $0 \in N(U_2^\dagger U_1)$.

(c) Denote the spectrum (the set of all eigenvalues) of a unitary U by $\operatorname{spec} U$. Show that the spectrum of any unitary matrix U is contained in the unit circle in the complex plane: $\operatorname{spec} U \subseteq \{\lambda \in \mathbb{C} \;:\; |\lambda| = 1\}$.

(d) The numerical range of a unitary U is equal to the convex hull of its eigenvalues; that is, if $\operatorname{spec} U = \{\lambda_1, \lambda_2, \ldots, \lambda_n\}$, we have

$$N(U) = \operatorname{conv}(\operatorname{spec} U) = \left\{\sum_{i=1}^{n} p_i\lambda_i \;:\; p_i \geqslant 0, \sum_i p_i = 1\right\}.$$

Use this to draw a sketch of the numerical range of the unitary matrix

$$U = \begin{bmatrix} 1 & 0 & 0 \\ 0 & e^{i\pi/4} & 0 \\ 0 & 0 & e^{i3\pi/4} \end{bmatrix}.$$

(e) The spectral arc length $\theta(U) \in [0, 2\pi)$ of a unitary U is the length of the smallest arc (in radians) that contains all the eigenvalues of U on the unit circle. Show by means of two figures that for the unitary phase gate

$$U_\gamma = \begin{bmatrix} 1 & 0 \\ 0 & e^{i\gamma} \end{bmatrix},$$

where $\gamma \in [0, 2\pi)$, the spectral arc length is given by

$$\theta(U_\gamma) = \begin{cases} \gamma, & \text{if } \gamma < \pi \\ 2\pi - \gamma, & \text{if } \gamma \geqslant \pi\,. \end{cases}$$

(f) Using parts (b)–(e), justify that two equally likely unitary gates U_1 and U_2 can be perfectly distinguished if and only if $\theta(U_2^\dagger U_1) \geqslant \pi$.

16H Logic and Set Theory

Let x be a set. A choice function for subsets of x is a function $f\colon \mathbb{P}x\setminus\{\varnothing\} \to x$ such that $f(y) \in y$ for all non-empty subsets y of x.

(a) Let x be a set and f be a choice function for subsets of x. Use f and recursion to show that there is an ordinal α and a bijection between α and x. Use this to show that the axiom of choice implies the well-ordering principle.

(b) Show that the statement "for any two sets x and y, either there is an injection from x to y or an injection from y to x" implies the axiom of choice.

[You may use Hartogs's lemma without proof.]

(c) Define the notion of *initial ordinal* and define $\aleph_\alpha$. Show that an ordinal is an infinite initial ordinal if and only if it has cardinality $\aleph_\alpha$ for some ordinal α.

Working in ZFC, we write $\mathrm{card}(x)$ for the least ordinal α that is in bijection with x. Let I be a set and $\{\kappa_i : i \in I\}$ be initial ordinals. We define

$$\sum_{i\in I} \kappa_i := \mathrm{card}\left(\bigsqcup_{i\in I} \kappa_i\right) \text{ and } \prod_{i\in I} \kappa_i := \mathrm{card}\left(\prod_{i\in I} \kappa_i\right).$$

(d) Assume that $\{\kappa_i : i \in I\}$ and $\{\lambda_i : i \in I\}$ are initial ordinals such that for every $i \in I$, we have $\kappa_i < \lambda_i$. Show that $\sum_{i\in I} \kappa_i < \prod_{i\in I} \lambda_i$.

[*Hint: Construct an injection from the disjoint union to the product, and show that there is no such surjection.*]

(e) Using part (d) or otherwise, show that $\aleph_\omega < \aleph_\omega^{\aleph_0}$. Deduce that $2^{\aleph_0} \neq \aleph_\omega$.

[You may use standard properties of cardinal arithmetic without proof.]

17F Graph Theory

(a) What does it mean for a graph G to be *Eulerian*? If $|G| \geqslant 3$, state and prove a necessary and sufficient condition for G to be Eulerian.

Define the *line graph* $L(G)$ of G. Show that $L(G)$ is Eulerian if G is regular and connected.

(b) Let G be a connected planar graph with n vertices, e edges and f faces. Prove that $n - e + f = 2$.

The size of a face is the number of edges that form its boundary. Deduce that $e \leqslant g(n-2)/(g-2)$, where g is the smallest size of a face.

(c) Let G be a (not necessarily planar) graph with n vertices and e edges. Suppose that G is drawn in the plane, but with edges allowed to cross. (The edge xy cannot contain any vertex except x or y.) Let $t(G)$ be the number of pairs of edges which cross.

(i) Show that $t(G) \geqslant e - 3n + 6$.

(ii) Suppose now that $e \geqslant 4n$. Show that $t(G) \geqslant e^3/64n^2$. [*Hint: you may wish to consider a random subset of* $V(G)$ *containing each vertex of* G *independently with probability* $4n/e$.]

18J Galois Theory

(a) List the transitive subgroups of S_4.

(b) Let L/K be an extension of fields of characteristic not equal to 2. Suppose that $L = K(\sqrt{a}, \sqrt{b})$ for some $a, b \in K^*$ with $a, b, ab \notin (K^*)^2$. Show that L/K is Galois of degree 4, compute its Galois group, and draw the lattice of intermediate fields.

(c) Compute the minimal polynomials of $\alpha = \sqrt{3+\sqrt{3}}$, $\beta = \sqrt{3-\sqrt{3}}$, $\gamma = \sqrt{3+\sqrt{6}}$, and $\delta = \sqrt{3-\sqrt{6}}$ over $\mathbb{Q}$. Show that the hypotheses of part (b) are satisfied by $\mathbb{Q}(\alpha,\beta)/\mathbb{Q}(\sqrt{3})$ and $\mathbb{Q}(\gamma,\delta)/\mathbb{Q}(\sqrt{6})$.

(d) Deduce that $\mathrm{Gal}(\mathbb{Q}(\alpha,\beta)/\mathbb{Q}) \cong D_8$. Draw the lattice of subgroups of D_8, and the lattice of subfields of $\mathbb{Q}(\alpha,\beta)$, writing each field in the form $\mathbb{Q}(x_1,\ldots,x_m)$.

[*Hint: You may use that* $\alpha+\beta = \sqrt{2}\gamma$ *and* $\alpha - \beta = \sqrt{2}\delta$.]

19J Representation Theory

Let V_n be the vector space of homogeneous polynomials in x, y of degree n over the complex numbers.

(a) Define the *standard action* of $G = SU_2$ on V_n.

Write down the matrix by which an element of G acts on V_3, with respect to the standard basis $\{x^i y^j : i \geqslant 0, j \geqslant 0, i+j=3\}$ of V_3.

Define the *character* of a finite dimensional complex representation V of G and write down the character of V_n.

(b) Show every finite dimensional complex representation V of G is isomorphic to V^*.

(c) Show that for every irreducible finite dimensional complex representation V of G the action of G on $V \otimes V$ factors through $G/\{\pm I\}$. Is this true for complex representations which are not irreducible?

(d) Decompose $V_n \otimes V_n$ into irreducibles.

(e) For any finite dimensional complex representation V of G, compute the character of $\bigwedge^2 V$ in terms of the character of V.

(f) Decompose $\bigwedge^2 V_n$ into irreducibles.

[*You must justify or prove your answers. You may use any results from lectures, but you must quote them carefully. In part (d) you may not just quote the Clebsch–Gordon formula.*]

20F Algebraic Topology

(a) State a version of the Seifert–van Kampen theorem. Let (X, x_0) be a based topological space, and suppose that $\alpha : (S^1, *) \to (X, x_0)$ is a based map. Prove that there is an isomorphism

$$\pi_1(X \cup_\alpha D^2, x_0) \cong \pi_1(X, x_0)/\langle\langle[\alpha]\rangle\rangle.$$

Use this to construct a connected cell complex Y such that

$$\pi_1(Y, y_0) \cong \langle a, b \,|\, a^2 b^{-3} \rangle.$$

[You may assume a description of $\pi_1(S^1 \vee S^1, *)$ provided it is clearly stated.]

(b) What does it mean for $p : \widetilde{X} \to X$ to be a *covering space*? For the cell complex Y constructed in part (a), suppose we have a covering space $p : \widetilde{Y} \to Y$ such that $\widetilde{Y}$ is path-connected, and, for $\widetilde{y}_0 \in p^{-1}(y_0)$, we have that $p_*\pi_1(\widetilde{Y}, \widetilde{y}_0)$ is the normal subgroup of $\pi_1(Y, y_0)$ generated by a. Given any $y \in Y$, how many points are in $p^{-1}(y)$? Give an explicit description of $\widetilde{Y}$ as a cell-complex.

21I Linear Analysis

(a) State and prove the Ascoli-Arzelà theorem.

(b) Consider a sequence of differentiable functions $f_n : \mathbb{R} \to \mathbb{R}$ with

$$\sup_{n \geqslant 0} \sup_{x \in \mathbb{R}} (|f_n(x)| + |f_n'(x)|) < +\infty.$$

Show that there exist a subsequence $f_{\phi(n)}$ (where $\phi : \mathbb{N} \to \mathbb{N}$ is strictly increasing) and a continuous and bounded function $f : \mathbb{R} \to \mathbb{R}$ such that

$$\forall R > 0, \quad \lim_{n \to \infty} \sup_{|x| \leqslant R} |f_{\phi(n)}(x) - f(x)| = 0.$$

Can we conclude that $\lim_{n\to\infty} \sup_{x\in\mathbb{R}} |f_{\phi(n)}(x) - f(x)| = 0$? Justify your answer.

22H Analysis of Functions

Let dx denote the Lebesgue measure on $\mathbb{R}^n$ and $L^1(\mathbb{R}^n)$ be the space of measurable functions $f : \mathbb{R}^n \to \mathbb{R}$ such that $\int_{\mathbb{R}^n} |f(x)|\, dx < \infty$.

(a) Denote by $[f] = \{g \in L^1(\mathbb{R}^n) : f = g \text{ almost everywhere}\}$ the equivalence classes for the almost everywhere equality relation, and show that $\|[f]\|_1 = \int_{\mathbb{R}^n} |f(x)|\, dx$ defines a complete norm on $\mathcal{L}^1(\mathbb{R}^n) := \{[f] : f \in L^1(\mathbb{R}^n)\}$.

[You may use the Riesz-Fischer theorem without proof if clearly stated.]

(b) Let $(X, \|\cdot\|_X)$ be a Banach space such that: (i) the inclusion $X \subset \mathcal{L}^1(\mathbb{R}^n)$ holds, and (ii) the convergence in $\|\cdot\|_X$ implies convergence almost everywhere on $\mathbb{R}^n$ along a subsequence.

(i) Show that there exists a constant $C > 0$ such that $\|x\|_1 \leqslant C\|x\|_X$ for all $x \in X$.

(ii) Must X be complete for $\|\cdot\|_1$? Justify your answer.

[*You may use results from Linear Analysis without proof if correctly stated.*]

23G Riemann Surfaces

Given a Riemann surface R and a covering map $\pi : S \to R$, where S is a connected Hausdorff topological space, explain how S can be given the structure of a Riemann surface such that π is an analytic map.

What does it mean to say that S is *simply connected*? State the uniformisation theorem and write down the group of analytic automorphisms $\mathrm{Aut}(R)$ for each simply connected Riemann surface R.

If X is a topological space and G is a group of homeomorphisms of X, define what it means to say that this action of G on X is a *covering space action*.

If R is the Riemann surface $\mathbb{C}_\infty$ and H is a subgroup of $\mathrm{Aut}(R)$ whose action on R is a covering space action, show that the quotient R/H is a Hausdorff space.

Give an example of a Riemann surface R and a group G of homeomorphisms of R whose action is a covering space action but such that the quotient space R/G is not Hausdorff. Must R/G be Hausdorff if R is simply connected?

24J Algebraic Geometry

In this question, all algebraic varieties are defined over a field k of characteristic zero. Let $V \subset \mathbb{P}^n$ be a curve.

Define the *degree* $\deg(V)$ of $V \subset \mathbb{P}^n$, and prove that it is well-defined.

Suppose $n < m$ and let $\varphi : \mathbb{P}^n \to \mathbb{P}^m$ be the linear embedding

$$(x_0 : \cdots : x_n) \mapsto (x_0 : \cdots : x_n : 0 : \cdots : 0).$$

For a curve $V \subset \mathbb{P}^n$, show that the degree of V in $\mathbb{P}^n$ agrees with the degree of $\varphi(V)$ in $\mathbb{P}^m$.

Prove that the degree is not an isomorphism invariant by providing an example of isomorphic curves $V_1, V_2 \subset \mathbb{P}^n$ with $\deg(V_1) \neq \deg(V_2)$.

Let $S = (x_0x_2 - x_1^2, x_0x_3 - x_1x_2, x_1x_3 - x_2^2) \subset k[x_0, x_1, x_2, x_3]$ and define $V = Z(S)$ to be the zero locus of S in $\mathbb{P}^3$. By considering an affine piece or otherwise, show that V is a curve in $\mathbb{P}^3$, and compute its degree. Prove that there do *not* exist homogeneous polynomials $F_1, \ldots, F_r$ such that $V = Z(F_1, \ldots, F_r)$ and $\deg(V) = \prod_{i=1}^r \deg(Z(F_i))$.

Give an example of two irreducible curves in a projective space $\mathbb{P}^n$ which have the same degree but are not isomorphic. [You may use without proof the fact that a smooth projective curve in $\mathbb{P}^2$ of degree $d \geqslant 2$ has genus $g = (d-1)(d-2)/2$.]

25I Differential Geometry

(a) Given a surface S (2-manifold) in $\mathbb{R}^3$, define the *first fundamental form* and express it in a local parametrisation, then define the *Gauss map* and the *second fundamental form*, and express them in a local parametrisation. Define the *Gauss curvature* and the *mean curvature.*

(b) Let $s \mapsto (Y(s), Z(s))$ be a planar curve parametrised by arc-length in the yz-plane with $Y(s) > 0$ for all s. The surface S of revolution attained by rotating this curve about the z-axis is parametrised by $\phi(u,v) = (Y(u)\cos v, Y(u)\sin v, Z(u))$.

(i) Calculate the first fundamental form, the Gauss map and the second fundamental form in the parametrisation ϕ. Deduce that the Gauss curvature K is equal to $-Y''/Y$ and give an expression for the mean curvature H in terms of Y and Z.

(ii) Given a curve $\alpha : I \to S$ on a surface S parametrised by arc-length, define what it means for the curve to be a *geodesic*, the *Christoffel symbols*, and the *geodesic equations* in terms of the Christoffel symbols.

(iii) Given a surface of revolution S with the above parametrisation, $\alpha(t) = \phi(u(t), v(t))$ a curve on S, prove that if α is a geodesic then $[Y(u)]^2\,\dot{v}$ is constant.

26H Probability and Measure

Let $(\Omega, \mathcal{A}, \mathbb{P})$ be a probability space. We consider in this question real valued random variables. We recall $a \wedge b = \min\{a, b\}$.

(a) Show that $X_n \xrightarrow[n\to\infty]{(P)} X \iff \lim_{n\to\infty} \mathbb{E}(|X_n - X| \wedge 1) = 0$ and that convergence in probability implies almost sure convergence along a subsequence.

(b) Show that $X_n \xrightarrow[n\to\infty]{(P)} X$ does not imply almost sure convergence $X_n \xrightarrow[n\to\infty]{a.s.} X$ by considering a sequence $(X_n)_{n\geqslant 1}$ of independent real random variables with $\mathbb{P}(X_n = 0) = 1 - \frac{1}{n}$ and $\mathbb{P}(X_n = 1) = \frac{1}{n}$.

(c) Let $X_n \xrightarrow[n\to\infty]{(P)} X$ and suppose that for some $1 < r < +\infty$, $(X_n)_{n\geqslant 1}$ is bounded in L^r. Show that $\forall 1 \leqslant p < r$, $X_n \xrightarrow[n\to\infty]{L^p} X$.

27L Applied Probability

Let (ξ_i) be a sequence of i.i.d. non-negative random variables with ξ_1 having a probability density function and $\mathbb{E}\xi_1 = 1/\lambda < \infty$.

(a) Define the *renewal process* N_t formed by the sequence (ξ_i). Assuming the law of large numbers, show that $N_t/t \to \lambda$ almost surely as $t \to \infty$.

Let $L(t)$ denote the length of the renewal interval containing t.

(b) Define what it means for a random variable $\hat{\xi}_1$ to have the *size-biased distribution* corresponding to ξ_1. If ξ_1 has an exponential distribution, show that $L(t)$ converges in distribution to $\hat{\xi}_1$ as $t \to \infty$. [Your proof should not use the equilibrium theorem of general renewal processes.]

(c) For all $x, t > 0$, prove that $\mathbb{P}(L(t) \geqslant x) \geqslant \mathbb{P}(\xi_1 \geqslant x)$.

28L Principles of Statistics

(a) Consider a Bayesian model $X \mid \theta \sim \text{Pois}(\theta)$ where the parameter $\theta \in (0, \infty)$ has prior distribution π given by $\theta \sim \text{Gamma}(\alpha, \lambda)$ where $\alpha, \lambda > 0$. Show that the posterior distribution $\theta \mid X$ has a $\text{Gamma}(\alpha + X, \lambda + 1)$ distribution. [*Hint: A* $\text{Gamma}(\alpha, \lambda)$ *distribution has density function* $f(y) = \lambda^\alpha y^{\alpha-1} e^{-\lambda y}/\Gamma(\alpha)$ *for* $y > 0$.]

(b) Consider now a decision problem involving a statistical model $\{P_\theta : \theta \in \Theta\}$ and loss function $L : \Theta \times \Theta \to [0, \infty)$.

(i) What is meant by the *risk* of a decision rule $\delta : \mathcal{X} \to \Theta$? Given a prior distribution π on Θ, what does it mean for δ to be a π-*Bayes estimator*? What does it mean for δ to be *minimax*?

(ii) Suppose a decision rule δ has constant risk r, and there is a sequence $\pi_1, \pi_2, \ldots$ of priors on θ where, writing $r_j < \infty$ for the π_j-Bayes risk of the π_j-Bayes estimator, we have that $r = \lim_{j \to \infty} r_j$. Show that δ is minimax.

(iii) Suppose now that $\mathcal{X} = \mathbb{R}$ and $\Theta = (0, \infty)$. Consider the weighted quadratic loss $L(\delta(x), \theta) = \theta^{-1}(\theta - \delta(x))^2$ and a prior π for θ. Show that a π-Bayes rule δ_π is given by $\delta_\pi(x) = \big(\mathbb{E}(\theta^{-1} \mid X = x)\big)^{-1}$.

(c) Finally show that in the model $X \sim \text{Pois}(\theta)$ where $\theta \in \Theta = (0, \infty)$, the decision rule $\delta(X) = X$ is minimax under the loss given in part (b) (iii). [*Hint: If* $Y \sim \text{Gamma}(\alpha, \lambda)$ *for* $\alpha > 1$ *and* $\lambda > 0$, *then* $\mathbb{E}(Y^{-1}) = \lambda/(\alpha - 1)$.] [You may interchange expectations and limits, that is apply the dominated convergence theorem, in your answer without justification.]

29K Stochastic Financial Models

Consider the a discrete-time market model with interest rate r and one stock with time-n price S_n. Suppose $S_0 > 0$ is given and that $S_n = S_{n-1}\xi_n$ for all $n \geqslant 1$, where the stochastic process $(\xi_n)_{n\geqslant 1}$ generates the filtration. Suppose that there are constants $-1 < a < r < b$ such that the random variable ξ_n takes values in $\{1+a, 1+b\}$ and that $0 < \mathbb{P}(\xi_n = 1+b) < 1$ for every $n \geqslant 1$.

(a) Introduce a European call of maturity N and strike K. Use the fundamental theorem of asset pricing to show that the call has a unique time-0 no-arbitrage price $\mathrm{EC}(N, K)$ of the form

$$\mathrm{EC}(N,K) = \sum_{n=0}^{N} w(n,N)\Big(S_0(1+a)^n(1+b)^{N-n} - K\Big)^+$$

where the positive numbers $w(n, N)$ are to be determined in terms of the given constants.

(b) Let $\mathrm{EP}(N, K)$ be the unique time-0 no-arbitrage price of a European put of maturity N and strike K. Show that

$$\mathrm{EP}(N,K) = (1+r)^{-N}K - S_0 + \mathrm{EC}(N,K).$$

(c) Find positive numbers u and v such that

$$\mathrm{EC}(N+1,K) = u\ \mathrm{EC}\left(N, \frac{K}{1+a}\right) + v\ \mathrm{EC}\left(N, \frac{K}{1+b}\right)$$

for all N and K.

(d) A forward start call option is the right, but not the obligation, to buy one share of the stock at time N for the price λS_M, where $0 \leqslant M \leqslant N$ and λ are given constants. Let $\mathrm{FSC}(M, N, \lambda)$ be its unique time-0 no-arbitrage price. Determine a strike K such that the following holds:

$$\mathrm{FSC}(M,N,\lambda) = \mathrm{EC}(N-M,K).$$

30D Asymptotic Methods

(a) Consider the function

$$I(x) = \int_C f(z) e^{x\phi(z)} dz, \qquad (\dagger)$$

where C is a complex contour, x is real and positive, f and ϕ are complex-valued functions, ϕ possesses a simple saddle point z_0, and $f(z_0) \neq 0$. Suppose C can be deformed so that it passes through z_0 without changing the value of $I(x)$.

Show that the saddle point's leading order asymptotic contribution to I is

$$f(z_0)\sqrt{\frac{2\pi}{x|\phi''(z_0)|}}\, e^{x\phi(z_0)+i\alpha}, \qquad \text{as} \quad x \to \infty,$$

where α is the angle of the tangent to the steepest descent curve at $z = z_0$.

[You may quote, without proof, results from Laplace's method for real integrals.]

(b) The Legendre polynomials can be expressed by the Schläfli integral:

$$P_n(t) = \frac{1}{2^{n+1}\pi i} \oint_C \frac{(z^2-1)^n}{(z-t)^{n+1}} dz,$$

where n is a positive integer, $t = \cos\theta$, with $0 < \theta < \pi$, and C is any closed anti-clockwise contour encircling t.

(i) Express $P_n(t)$ in the form

$$P_n(t) = \frac{1}{2^{n+1}\pi i} \oint_C f(z,t) e^{n\phi(z,t)} dz,$$

for some functions f and ϕ, and show that the saddle points of ϕ are located at $z = z_\pm = e^{\pm i\theta}$.

(ii) Show that $\mathrm{Arg}[\phi''(z_\pm)] = \mp(\theta + \pi/2)$. Hence sketch an appropriate contour C that passes through z_+ and z_-, calculating its angle α at each saddle point.

(iii) Find the leading order asymptotic approximation of $P_n(t)$ as $n \to \infty$, in the form

$$P_n(t) \sim A \cos\left(n\theta + \tfrac{1}{2}\theta - \tfrac{1}{4}\pi\right),$$

where you should determine $A(\theta)$.

31A Dynamical Systems

(a) State and prove Dulac's Theorem. What is the divergence test?

(b) Consider the system

$$\dot{x} = \mu x - y - (x^3 + xy^2 - \lambda x)(x^2 + y^2),$$
$$\dot{y} = x + \mu y - (y^3 + x^2 y - \lambda y)(x^2 + y^2),$$

for a real parameter μ and constant $\lambda > 0$.

(i) Show that there is a fixed point at the origin and classify its stability.

(ii) Find how the number of periodic orbits varies with the value of μ and hence identify two bifurcation points.

[*Hint: use polar coordinates* (r, θ).]

(iii) Identify the type of bifurcation occurring at the larger value of μ and, without detailed computation, write down its normal form. Draw the steady and periodic solutions in a (μ, r) diagram.

(iv) Show that, as μ varies, the locations of the periodic orbits are consistent with the divergence test.

32D Integrable Systems

(a) Use the Gelfand–Levitan–Marchenko equation

$$K(x,y) + F(x+y) + \int_x^\infty K(x,z)F(z+y)dz = 0,$$

with $F(x) = \beta_0 \exp(8\chi^3 t - \chi x)$ to find the one-soliton solution

$$u(x,t) = -\frac{2\chi^2}{\cosh^2\left[\chi(x - 4\chi^2 t - \phi)\right]}$$

to the KdV equation

$$u_t - 6uu_x + u_{xxx} = 0,$$

where $u = u(x,t)$. Here β_0 and χ are constants, and ϕ is another constant that you should determine.

[*You may use any facts about the inverse scattering transform without proof.*]

(b) By considering the operators $A^\dagger A$ and $AA^\dagger$ where $A = \partial_x + \chi \tanh(\chi x)$, show that the Schrödinger operator $-\partial_x^2 + U$ with a potential

$$U(x) = u(x, t=0, \phi=0)$$

admits only one bound state. Find the corresponding energy level.

33C Principles of Quantum Mechanics

(a) Write down the Hamiltonian for a two-dimensional quantum harmonic oscillator with unit mass and frequency. Determine the energy eigenvectors $|E\rangle$ and eigenvalues E and discuss their degeneracy.

(b) Define the angular momentum operator L and explicitly compute its commutation relations with H. Hence constrain the form of $\langle E|L|E'\rangle$.

(c) Consider the operators $T_{ij} = a_i^\dagger a_j$ where $i, j \in \{x, y\}$, and a_x and a_y are the annihilation operators in the x and y directions respectively. Compute the commutator $[T_{ij}, T_{kl}]$ and hence $[T_{ij}, H]$.

(d) Now consider $T^a = \frac{1}{2}\sigma^a_{ij} T_{ij}$ for $a = 1, 2, 3$ where σ^a are the Pauli matrices satisfying $[\sigma^a, \sigma^b] = 2i\epsilon_{abc}\sigma^c$. Compute the commutators $[T^a, T^b]$. Relate L to the T^a. Using these results and what you know about the representations of the group SU(2), determine the possible angular momentum eigenvalues $L|E, \ell\rangle = \ell|E, \ell\rangle$ that an energy eigenvector $|E\rangle$ can have. Compare your result to the degeneracy discussed in part (a).

$$\left[\textit{Hint: the Pauli matrices are } \sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},\ \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix},\ \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.\ \right]$$

34B Applications of Quantum Mechanics

Consider a system of three atoms arranged on a circle, which can also be viewed as a one-dimensional crystal with atoms equally spaced by a distance a under periodic boundary conditions. The atoms are labelled 0, 1 and 2, and the wavefunction corresponding to an electron bound to the n-th atom is denoted by $\psi_n(x)$. The periodic boundary conditions imply that ψ_n is identified with ψ_{n+3}.

The atomic Hamiltonian is H_0 with $H_0|\psi_n\rangle = E_0|\psi_n\rangle$. The tunnelling between atomic sites is represented by a potential V, which has matrix elements

$$\begin{aligned} \langle\psi_n|V|\psi_n\rangle &= \alpha \qquad \forall\ n\ , \\ \langle\psi_n|V|\psi_{n'}\rangle &= -A \quad \text{for } n \neq n'\ , \end{aligned}$$

where α and A are real constants and $n, n' = 0, 1, 2$. The stationary state of the total Hamiltonian, $H = H_0 + V$, is denoted by Ψ and can be written as

$$|\Psi\rangle = \sum_{n=0}^{2} c_n|\psi_n\rangle\ ,$$

with $H|\Psi\rangle = E|\Psi\rangle$ and $c_n \in \mathbb{C}$ for $n = 0, 1, 2$.

(a) Assuming $\langle\psi_n|\psi_m\rangle = \delta_{nm}$, show that the coefficients (c_0, c_1, c_2) satisfy the linear equations

$$\begin{pmatrix} E_0+\alpha & -A & -A \\ -A & E_0+\alpha & -A \\ -A & -A & E_0+\alpha \end{pmatrix} \begin{pmatrix} c_0 \\ c_1 \\ c_2 \end{pmatrix} = E \begin{pmatrix} c_0 \\ c_1 \\ c_2 \end{pmatrix}\ .$$

Write out explicitly the energy eigenvalues of the system. Show that the possible solutions for (c_0, c_1, c_2) are

$$(1,1,1)\ , \qquad (1,\omega,\omega^2)\ , \qquad (1,\omega^2,\omega)\ ,$$

where ω is a cube root of unity. [*Hint:* $x^3 - 3A^2x - 2A^3 = 0$ *has a double root at* $x = -A$.]

(b) Interpret these solutions in terms of a wavenumber k and determine the possible values of k. Write the corresponding eigenvectors $|\Psi_k\rangle$ in terms of $|\psi_n\rangle$. What is the Brillouin zone for this system?

(c) Let $\psi_n(x) = \psi(x - na)$. Show that, for each value of k, $\Psi_k(x)$ can be written as

$$\Psi_k(x) = u_k(x)\, e^{ikx}\ ,$$

$$u_k(x) = \sum_{n=0}^{2} \psi(x - na)\, e^{-ik(x-na)}\ .$$

Check that $u_k(x + a) = u_k(x)$. State Bloch's theorem in one dimension and justify why it applies to this system.

35C Statistical Physics

This question concerns a Fermi gas of non-relativistic, non-interacting electrons confined to a two-dimensional planar sheet with total area A, at chemical potential $\mu > 0$. Note that the electron has $g_s = 2$ spin states.

(a) Calculate the density of states $g(E)$ of the 1-particle system (including the numerical coefficient). Write down the Fermi–Dirac distribution.

(b) Suppose the system is at absolute zero ($T = 0$). Show that the total energy of the system is $E_{\text{tot}} = \frac{1}{2} N E_F$, where E_F is the Fermi energy and N is the number of electrons. Calculate the degeneracy pressure p.

(c) Show that at finite temperature $T > 0$, the change in the number of electrons N relative to the $T = 0$ state is given by

$$\Delta N = X \int_{-E_F}^{\infty} dE' \left[\operatorname{sgn}(\beta E'/2) - \tanh(\beta E'/2) \right],$$

where $E' = E - E_F$, $\operatorname{sgn}(x) = x/|x|$ is the sign function, and X is a coefficient which you should determine. Explain why, at low temperatures, $\Delta N \approx 0$ to a very good approximation. Work out the corresponding formula for the change of the total energy ΔE_{tot}, and use its scaling with respect to β to show that

$$\Delta E_{\text{tot}} \propto T^2.$$

[You need not work out the constant of proportionality.]

36B Electrodynamics

Starting from a suitable general solution of Maxwell's equations, which you may state without derivation, find the total power $\mathcal{P}$ emitted through a large spherical surface of radius R by a localised source with time-dependent electric dipole moment $\mathbf{p}(t)$ in the dipole approximation,

$$\mathcal{P} \simeq \frac{\mu_0}{6\pi c}|\ddot{\mathbf{p}}(t - R/c)|^2 .$$

You should state clearly the conditions under which the approximation is valid.

A simple model of a pulsar consists of a solid uniform sphere of mass M and radius $\mathcal{R}$ spinning with angular frequency Ω around an axis $\hat{\mathbf{z}}$. The body has a time-dependent magnetic dipole moment $\mathbf{p}$ that is inclined at a constant angle α to the z-axis and rotates according to

$$\mathbf{p}(t) = p_0 \left[\sin\alpha \, \cos(\Omega t)\, \hat{\mathbf{x}} + \sin\alpha \, \sin(\Omega t)\, \hat{\mathbf{y}} + \cos\alpha \, \hat{\mathbf{z}}\right] ,$$

where p_0 is a real constant. Calculate the total power $\mathcal{P}$ emitted by the pulsar in the dipole approximation. Assuming that the angular frequency of rotation $\Omega(t)$ slowly varies with time so that energy is conserved, calculate the time taken for this system to lose half its initial rotational energy E_0 due to emission of radiation. [You may assume that the rotational energy of a solid uniform sphere of radius $\mathcal{R}$ and mass M is $E = I\Omega^2/2$ with moment of inertia $I = 2M\mathcal{R}^2/5$.]

37E General Relativity

The Schwarzschild metric, in units with $G = c = 1$, is given by

$$ds^2 \;=\; -\left(1 - \frac{2M}{r}\right) dt^2 \;+\; \left(1 - \frac{2M}{r}\right)^{-1} dr^2 \;+\; r^2 (\, d\theta^2 + \sin^2\theta\, d\phi^2\,)\,. \qquad (*)$$

(a) Show that for a light ray that is radial ($d\theta = d\phi = 0$) and ingoing ($dr/dt < 0$), for $r > 2M$ the quantity

$$v \;=\; t \,+\, r \,+\, 2M \log\left|\, \frac{r}{2M} - 1 \,\right|$$

is constant.

(b) Express the Schwarzschild metric $(*)$ in terms of coordinates r, v, θ, ϕ, with v defined as above for $r > 0$. What can be deduced about the nature of the metric at $r = 2M$?

(c) Determine all possible radial trajectories for light rays for $r > 0$. For these solutions, find dt_*/dr as a function of r, where $t_* = v - r$, and hence sketch the solutions in the r-t_* plane.

(d) Comment on the contrasting behaviour of light rays in the regions $r > 2M$ and $r < 2M$ and, by considering light cones at representative points, discuss the implications for the motion of massive particles.

(e) An astronaut Alice (A) sends radial light signals at proper time intervals $\Delta\tau_A$ to an observer Bob (B) who receives them at proper time intervals $\Delta\tau_B$. Alice and Bob are at rest in the coordinate system t, r, θ, ϕ with $r_A = 2M + \varepsilon$, where $0 < \varepsilon \ll M$, and $r_B \gg M$. Find an approximate expression for $\Delta\tau_B/\Delta\tau_A$ and comment on the significance of your result.

38D Fluid Dynamics

A steady, two-dimensional, laminar plume (narrow, quasi-vertical flow) rising from a point source of buoyancy in an otherwise stationary environment can be modelled using the boundary-layer equations

$$\rho\left(u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}\right)=-\frac{\partial p}{\partial x}+\mu\frac{\partial^2 u}{\partial y^2}+b(x)\delta(y),$$

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0,$$

where u and v are vertical and horizontal velocity components, respectively, with respect to Cartesian coordinates x vertical and y horizontal, and where δ is the Dirac delta function, so that $b(x)$ represents a buoyancy force confined to the vertical axis $y=0$. The symbols ρ, μ and p represent the fluid's density, dynamic viscosity and pressure respectively.

(a) Show that

$$\frac{d}{dx}\int_{-\infty}^{\infty}\rho u^2\,dy=b(x).$$

(b) Given that $b(x)=Bx^{-1/5}$, where B is constant, show that the width of the plume Δ and the vertical velocity u scale as

$$\Delta\sim\left(\frac{\mu^2}{\rho B}\right)^{1/3}x^{2/5},\qquad u\sim\left(\frac{B^2}{\rho\mu}\right)^{1/3}x^{1/5}.$$

(c) Introduce a stream function $\psi(x,y)$ and consider a similarity solution $\psi(x,y)=u\Delta f(\eta)$, where f only depends on the similarity variable $\eta=y/\Delta$. Show that $f(\eta)$ satisfies an ordinary differential equation of the form

$$f'''+c_1(f')^2+c_2ff''=\delta(\eta),$$

where primes denote differentiation with respect to η, for some constants c_1 and c_2 that you should determine.

[*Hint: For any constant a, $\delta(ay)=|a|^{-1}\,\delta(y)$.*]

39A Waves

Let $\phi(x,t)$ be a real-valued function that satisfies the equation

$$\frac{\partial^2\phi}{\partial t^2} - c^2\frac{\partial^2\phi}{\partial x^2} + A^2c^2\phi = 0,$$

where both A and c are positive constants.

(a) Consider wave solutions of frequency ω and wavenumber k.

(i) Find the dispersion relation for such waves.

(ii) Sketch both the phase velocity c_p and the group velocity c_g as functions of k.

(iii) Do wave crests move faster or slower than a wave packet?

(b) Suppose that $\phi(x,0)$ is real and that

$$\phi(x,0) = \int_{-\infty}^{\infty} a(k)e^{ikx}dk, \quad \frac{\partial}{\partial t}\phi(x,0) = 0,$$

where $a(k)$ is a given function.

(i) Use the method of stationary phase to obtain an approximation for $\phi(Vt,t)$ for fixed $0 \leqslant V < c$ and large t.

[*Hint: You will need the result* $\int_{-\infty}^{\infty} e^{-au^2}\,du = \sqrt{\pi/a}$ *for* $\mathrm{Re}(a) \geqslant 0, a \neq 0$.]

(ii) Now suppose the initial condition is even, so that $\phi(x,0) = \phi(-x,0)$. Consider the limit of large t and deduce an approximation for the sequence of times at which $F(t) = \phi(Vt,t)$ satisfies both $F(t) = 0$ and $F'(t) > 0$.

40D Numerical Analysis

Consider the following two Cauchy problems: the diffusion equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}, \qquad 0 \leqslant t \leqslant 1, \quad x \in \mathbb{R}, \tag{$\dagger$}$$

with initial condition $u(x,0) = u_0(x)$; and the wave equation

$$\frac{\partial^2 v}{\partial t^2} = \frac{\partial^2 v}{\partial x^2}, \qquad 0 \leqslant t \leqslant 1, \quad x \in \mathbb{R},$$

with initial conditions $v(x,0) = v_0(x)$ and $\frac{\partial v}{\partial t}(x,0) = v_1(x)$. Further consider the discretisation of the diffusion equation,

$$u_m^{n+1} - \frac{1}{2}\mu\left(u_{m+1}^{n+1} - 2u_m^{n+1} + u_{m-1}^{n+1}\right) = u_m^n + \frac{1}{2}\mu\left(u_{m+1}^n - 2u_m^n + u_{m-1}^n\right), \tag{$\star$}$$

and the discretisation of the wave equation,

$$v_m^{n+1} - 2\rho v_m^n + v_m^{n-1} = \mu\left(v_{m+1}^n - 2v_m^n + v_{m-1}^n\right), \tag{$\star\star$}$$

where $m \in \mathbb{Z}$, $n = 1, \ldots, N$, $\mu > 0$ is the Courant number, and $\rho \in [1,2]$ is a constant parameter. The notation f_m^n here denotes the function f evaluated at the n^{th} time step and located at a spatial grid point labelled by index m. In all parts of the question below regarding stability, consider the 2-norm $\|\cdot\|_2$.

(a) Derive an expression for the amplification factor in a Fourier analysis of stability applied to a finite-difference discretisation of a linear partial differential equation.

(b) Determine the values of μ that make the method in equation ($\star$) stable for the diffusion equation as described above.

(c) Determine the values of μ, as a function of ρ, that make the method in equation ($\star\star$) stable for the wave equation as described above.

(d) Suppose we replace the Cauchy problem ($x \in \mathbb{R}$) in equation ($\dagger$) with the finite domain $0 \leqslant x \leqslant 1$, on which we apply Dirichlet boundary conditions $u(0,t) = u(1,t) = 0$. Determine the values of μ that make the method in equation ($\star$) stable for this problem.

[*You may use basic spectral properties of Toeplitz symmetric tridiagonal (TST) matrices without proof.*]

END OF PAPER