UNIVERSITY OF CAMBRIDGE

MAT2

MATHEMATICAL TRIPOS Part II

Tuesday, 10 June, 2025 1:30pm to 4:30pm

PAPER 2

Before you begin read these instructions carefully.

The examination paper is divided into two sections. Each question in Section II carries twice the number of marks of each question in Section I. Section II questions also carry an alpha or beta quality mark and Section I questions carry a beta quality mark.

Candidates may obtain credit from attempts on **at most six questions** from Section I and from any number of questions from Section II.

Write on **one side** of the paper only and begin each answer on a separate sheet.

Write legibly; otherwise you place yourself at a grave disadvantage.

At the end of the examination:

Separate your answers to each question.

Complete a gold cover sheet **for each question** that you have attempted, and place it at the front of your answer to that question.

Complete a green main cover sheet listing **all the questions** that you have attempted.

Every cover sheet must also show your Blind Grade Number and desk number.

Tie up your answers and cover sheets into **a single bundle**, with the main cover sheet on the top, and then the cover sheet and answer for each question, in the numerical order of the questions.

STATIONERY REQUIREMENTS
Gold cover sheets
Green main cover sheet
Script paper
Rough paper

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

SECTION I

1G Number Theory

Let $N = 136$, and let $G = (\mathbb{Z}/N\mathbb{Z})^{\times}$ denote the multiplicative group of units modulo N.

(a) Compute the order of G.

(b) Compute the least integer $m \geqslant 1$ such that for any $g \in G$, $g^m \equiv 1 \bmod N$.

(c) Write down an element $g \in G$ of order m.

2I Topics in Analysis

State Chebyshev's equal ripple criterion.

Let T_n be the Chebyshev polynomial of degree n satisfying $T_n(\cos\theta) = \cos(n\theta)$ for all $\theta \in \mathbb{R}$. Determine in terms of T_n a minimizer for $\sup_{-1\leqslant t\leqslant 1} |t^n - q(t)|$ among all the polynomials q of degree less than n.
[You may assume without proof that the coefficient of $T_n(t)$ at t^n is 2^{n-1}.]

Let f be a polynomial of degree at most n and such that $|f(t)| < 1$ for $-1 \leqslant t \leqslant 1$. By considering the roots of $T_n - f$, or otherwise, show that

$$|f(t)| < \max\{1, |T_n(t)|\}, \text{ for all } t \in \mathbb{R}.$$

3K Coding & Cryptography

Suppose codewords 000 and 111 are sent with probabilities 1/5 and 4/5 respectively through a Binary Symmetric Channel with error probability $p = 1/4$. If we receive 001 how should we decode if we use (i) ideal observer, (ii) maximum likelihood, (iii) minimum distance decoding? Justify your answers.

In light of this give some positives and negatives of the three decoding methods.

4F Automata & Formal Languages

(a) Say what it means for a language L to satisfy the *context-free pumping lemma.*

(b) For each of the following languages over the alphabet $\{\mathbf{0}, \mathbf{1}\}$, either give a context-free grammar that produces the language or prove that the language is not context-free. We write w^{R} for the reverse word of w, i.e., if $w = a_0 ... a_{n-1}$ then $w^{\mathrm{R}} = a_{n-1} ... a_0$.

(i) The language $L := \{\mathbf{0}^n\mathbf{1}^n\mathbf{0}^n\,;\, n > 0\}$.

(ii) The language $L := \{ww^{\mathrm{R}}\,;\, w \in \mathbb{B}^+\}$ of even length palindromes.

(iii) The language $L := \{ww^{\mathrm{R}}\,;\, w \in \mathbb{B}^+$ such that the number of $\mathbf{0}$s in w is equal to the number of $\mathbf{1}$s in $w\}$.

(iv) The language $L := \{\mathbf{0}^n\mathbf{1}^m\mathbf{0}^{n+m}\,;\, n, m > 0\}$.

5K Statistical Modelling

The `potato` dataset contains the crop yields of 60 equally divided plots in a farm, each randomly planted with one of three genotypes of potato. There are two possible alleles (`A`, `a`) and three possible genotypes (`aa`, `Aa`, `AA`). The `count` column counts the number of `A` alleles in `genotype`. Consider the following `R` code with truncated output.

```
> potato[c(1, 20, 21, 40, 41, 60), ]
   genotype count     yield
1        aa     0  9.038067
20       aa     0 10.199812
21       Aa     1 10.421516
40       Aa     1 11.793761
41       AA     2 12.786507
60       AA     2 11.214858

> summary(model1 <- lm(yield ~ genotype - 1, potato))
Coefficients:
           Estimate Std. Error t value Pr(>|t|)
genotypeaa   9.8328     0.2062   47.68   <2e-16 ***
genotypeAa  11.0535     0.2062   53.60   <2e-16 ***
genotypeAA  11.8059     0.2062   57.24   <2e-16 ***

> summary(model2 <- lm(yield ~ I(count >= 1), potato))
Coefficients:
                  Estimate Std. Error t value Pr(>|t|)
(Intercept)         9.8328     0.2161  45.511  < 2e-16 ***
I(count >= 1)TRUE   1.5969     0.2646   6.035 1.19e-07 ***

> anova(model2, model1)
Analysis of Variance Table
Model 1: yield ~ I(count >= 1)
Model 2: yield ~ genotype - 1
  Res.Df    RSS Df Sum of Sq      F  Pr(>F)
1     58 54.148
2     57 48.487  1    5.6612 6.6551 0.01249 *
---
Signif. codes:  0 ‘***’ 0.001 ‘**’ 0.01 ‘*’ 0.05 ‘.’ 0.1 ‘ ’ 1
```

After introducing necessary mathematical notation, write down the statistical models fitted above with all assumptions that are used to calculate the p-values. Which hypothesis below is tested in the analysis of variance, and what can you conclude about it? Write down the `R` code to test the other hypothesis using analysis of variance.

1. Full dominance: the effect of genotype on crop yield only depends on whether the genotype contains `A` allele or not.
2. No dominance: the effect of genotype on crop yield only is linear in the number of `A` alleles.

6A Mathematical Biology

A population of healthy foxes, $S(\mathbf{x},t)$, is territorial and tends not to move. In contrast, rabid infected foxes, with population $I(\mathbf{x},t)$, change their behaviour and migrate. The dynamics of foxes is captured by the following, non-dimensionalised, equations:

$$\frac{dS}{dt} = -IS\ , \quad \frac{dI}{dt} = \nabla^2 I + IS - \mu I\ , \quad \frac{dR}{dt} = \mu I\ ,$$

where μ is a constant.

(a) Give a biological explanation for each term on the right-hand side of these equations. What is the meaning of the population $R(\mathbf{x},t)$?

(b) Consider the spatially homogeneous case. Define the *reproductive ratio* for this system. Explain the reasoning behind the statement that rabies spreads among foxes only if the reproductive ratio is greater than 1.

(c) Suppose foxes move only in one spatial dimension. By writing $S(x,t) = S(\xi)$ and $I(x,t) = I(\xi)$, where $\xi = x - ct$ and $c > 0$, write down the equations for $S(\xi)$ and $I(\xi)$ that govern a travelling wave in this system.

(d) Consider the situation in which $S(\xi) \to 1$ and $I(\xi) \to 0$ as $\xi \to \infty$. By linearising about the leading edge of a wavefront, determine the minimum velocity at which a wave of infection spreads.

7E Further Complex Methods

(a) The function $F(z)$ is defined for all $z \in \mathbb{C} \setminus \{\pm i\}$ by

$$F(z) = \int_0^z \frac{1}{1+t^2}\, dt\,, \qquad (\dagger)$$

where the path taken for the integral is unrestricted except that it does not pass through either of the points $\pm i$. Show that the function $F(z)$ is multivalued. What are the possible values of $F(1)$?

(b) A curve $\mathcal{B}$ joins the points $\pm i$ along the imaginary axis, slightly displaced to the left of 0. Consider the function $F_{\mathcal{B}}(z)$ defined for $z \in \mathbb{C} \setminus \mathcal{B}$ by the integral $(\dagger)$, but with the restriction that the path of integration does not cross $\mathcal{B}$. Show that $F_{\mathcal{B}}(z)$ is a single-valued function.

(c) Show that, for large $|z|$, $F_{\mathcal{B}}(z) = \frac{1}{2}\pi + O(|z|^{-1})$. Hence, calculate the integral

$$\lim_{R\to\infty} \oint_{\gamma_R} \frac{F_{\mathcal{B}}(t)}{t}\, dt\,,$$

where the contour γ_R is an anti-clockwise circle of radius R.

8B Classical Dynamics

(a) Explain very briefly how to introduce action-angle variables ϕ, I for a Hamiltonian system determined in the standard way by a Hamiltonian $H(q,p)$ defined for $(q,p) \in \mathbb{R}^2$. [You may assume that all orbits are bounded for your discussion.] Furthermore, briefly explain what is meant by the *principle of adiabatic invariance of the action.*

(b) Consider the case

$$H(q,p) = \frac{p^2}{2m} - \frac{1}{|q|},$$

where m is a positive constant. Explain why, for solutions with $H = E = -|E| < 0$, the magnitude of q must remain bounded. Find the smallest possible $q_{\max} = q_{\max}(|E|)$ such that the interval $[-q_{\max}, q_{\max}]$ contains all possible values of $q(t)$ for such a solution.

Calculate the action I in terms of $|E|$ and m. Assuming further that the adiabatic invariance principle holds for this system, if m varies slowly over a long time interval, doubling in magnitude, how does the energy change?

[*Hint: you may make use of the integral* $\int_0^1 \sqrt{1-x^2}dx = \frac{1}{4}\pi$.]

9E Cosmology

Prior to the synthesis of light elements ($k_{\rm B}T \gtrsim 1$ MeV), neutrons and protons are kept in equilibrium by the weak interactions

$$n + \nu_e \leftrightarrow p + e^- \,, \qquad p + \bar{\nu}_e \leftrightarrow n + e^+ \,.$$

The ratio of the weak interaction rate $\Gamma_W \propto T^5$, which maintains equilibrium, relative to the Hubble expansion rate $H \propto T^2$, is

$$\frac{\Gamma_W}{H} \approx \left(\frac{k_{\rm B}T}{\kappa}\right)^3 \qquad \text{where} \;\; \kappa = 0.7 \text{ MeV} \,. \tag{$\dagger$}$$

(a) Assuming that the chemical potentials for all leptons are small, $\mu_{e^-} \ll k_{\rm B}T$ etc., show that, in equilibrium, the neutron-to-proton ratio can be expressed as

$$\frac{n_n}{n_p} \approx e^{-Q/(k_{\rm B}T)} \,,$$

where $Q = (m_n - m_p)c^2 = 1.29$ MeV is the mass difference between a neutron and a proton.

(b) Using equation (†), briefly explain why the neutron-to-proton ratio effectively 'freezes out' once $k_{\rm B}T < 0.7$ MeV. At this time, the ratio is $n_n/n_p \approx 1/6$, but it decreases to a final value $n_n/n_p \approx 1/7$ when deuterium forms at $k_{\rm B}T \approx 0.07$ MeV. Briefly specify why.

(c) Briefly explain why eventually almost all neutrons are captured in helium-4, and estimate the resulting helium mass parameter $Y_p = \rho_{\rm He}/\rho_{\rm B}$, where $\rho_{\rm He}$ is the helium-4 density, $\rho_{\rm B} = m_p \, n_{\rm B}$, and $n_{\rm B}$ is the baryon number density.

(d) Consider an otherwise identical universe where the constant κ in equation (†) is much larger than 0.7 MeV. Describe how this would affect the 'freeze-out' described by equation (†) and the helium mass parameter Y_p. Briefly discuss potential implications for stellar lifetimes and the origin of life in this alternative universe.

10C Quantum Information and Computation

Let $|\psi\rangle$ denote a 2-qubit state. Let $\mathcal{A} = \{|a_0\rangle, |a_1\rangle\}$, $\mathcal{B} = \{|b_0\rangle, |b_1\rangle\}$ and $\mathcal{C} = \{|c_0\rangle, |c_1\rangle\}$ be three orthonormal bases of $\mathbb{C}^2$, where,

$$|a_0\rangle = |0\rangle\,, \qquad |b_0\rangle = \frac{1}{2}|0\rangle + \frac{\sqrt{3}}{2}|1\rangle\,, \qquad |c_0\rangle = \frac{1}{2}|0\rangle - \frac{\sqrt{3}}{2}|1\rangle\,,$$
$$|a_1\rangle = |1\rangle\,, \qquad |b_1\rangle = \frac{\sqrt{3}}{2}|0\rangle - \frac{1}{2}|1\rangle\,, \qquad |c_1\rangle = \frac{\sqrt{3}}{2}|0\rangle + \frac{1}{2}|1\rangle\,.$$

Suppose the first qubit of the state $|\psi\rangle$ is measured in the basis $\mathcal{A}$ and the second qubit is measured in the basis $\mathcal{B}$. Let $P_\psi(\mathcal{A},\mathcal{B})$ denote the probability that these two measurements either yield the outcome (a_0, b_0) or the outcome (a_1, b_1). Probabilities $P_\psi(\mathcal{B},\mathcal{C})$ and $P_\psi(\mathcal{C},\mathcal{A})$ are defined analogously.

(a) Give expressions for $P_\psi(\mathcal{A},\mathcal{B})$, $P_\psi(\mathcal{B},\mathcal{C})$ and $P_\psi(\mathcal{C},\mathcal{A})$.

(b) What transformations relate (i) the basis $\mathcal{A}$ to the basis $\mathcal{B}$, and (ii) the basis $\mathcal{A}$ to the basis $\mathcal{C}$? Show that

$$\frac{|a_0a_0\rangle + |a_1a_1\rangle}{\sqrt{2}} = \frac{|b_0b_0\rangle + |b_1b_1\rangle}{\sqrt{2}} = \frac{|c_0c_0\rangle + |c_1c_1\rangle}{\sqrt{2}}\,. \qquad (*)$$

(c) For $|\psi\rangle = |0\rangle \otimes |0\rangle$, show that

$$P_\psi(\mathcal{A},\mathcal{B}) + P_\psi(\mathcal{B},\mathcal{C}) + P_\psi(\mathcal{C},\mathcal{A}) \geqslant 1\,.$$

(d) Denote the state in equation $(*)$ by $|\phi^+\rangle$. For $|\psi\rangle = |\phi^+\rangle$, determine $P_\psi(\mathcal{A},\mathcal{B}) + P_\psi(\mathcal{B},\mathcal{C}) + P_\psi(\mathcal{C},\mathcal{A})$.

SECTION II

11I Topics in Analysis

(a) Let $T \subset \mathbb{R}^2$ be a triangle with $I,\ J,\ K$ the three sides of T and $\partial T = I \cup J \cup K$. Prove that the following two statements are equivalent:

(i) If A, B, C are closed subsets of $\mathbb{R}^2$ such that $I \subset A$, $J \subset B$, $K \subset C$ and $T \subset A \cup B \cup C$, then $A \cap B \cap C \neq \emptyset$.

(ii) There does not exist a continuous map $f : T \to \partial T$ such that $f(I) \subset I$, $f(J) \subset J$ and $f(K) \subset K$.

(b) State Brouwer's fixed point theorem in the plane. Prove, using Brouwer's fixed point theorem, that there exists a complex number z with $|z| \leqslant 1$ such that $z^6 - 2z^5 + 4z^2 + 9z + 2 = 0$.

(c) Let $I = [-1, 1]$ and let $\beta, \gamma : I \to I \times I$ be continuous paths such that $\beta(-1) = (a, -1)$, $\beta(1) = (b, 1)$ and $\gamma(-1) = (-1, c)$, $\gamma(1) = (1, d)$ with $a, b, c, d \in I$. By considering a suitable continuous map $I \times I \to I \times I$ prove that the paths β and γ intersect.

[If you use the Jordan curve theorem, you must prove it.]

12K Coding & Cryptography

Consider a cryptosystem $\langle M, K, C\rangle$. Let e, d be the respective encryption and decryption functions. Model the key and messages as random variables k, m taking values in K, M, respectively and such that $m = d(c,k) \in M$ and $c = e(m,k) \in C$.

Define, both in words and formally, the *unicity distance*, U, of a cryptosystem.

Prove that

$$U = \frac{\log |K|}{\log |\Sigma| - H}$$

where Σ is the alphabet of the ciphertext and $H = H(m)$. Make clear any assumptions you make.

Suppose $M = \{0, 1, 2\}$ is emitted from a memoryless source with probabilities

$$P(m = 0) = 1/2,\ P(m = 1) = p \text{ and } P(m = 2) = 1/2 - p$$

where $0 \leqslant p \leqslant 1/4$. Let the key $k = (k_0, k_1, k_2)$ be chosen uniformly from the set of binary 3-tuples i.e. $K = \{(k_0, k_1, k_2) : k_i \in \{0, 1\}\}$. A sequence of messages $m_1, m_2, \ldots, m_n$ is encrypted to a sequence of ciphertexts $c_1, c_2, \ldots, c_n$ by

$$c_i = m_i + k_{i \bmod 3} \pmod 3$$

for $1 \leqslant i \leqslant n$.

Show that, if the unicity distance of the cryptosystem is at least 20, then we must have $H(2p, 1 - 2p) \geqslant 0.87$ (you may take $\log_2(3) = 1.585$).

Given that $H(2p, 1 - 2p) = 0.87$ is satisfied when $p = 0.15$, find all values of $p \in [0, 1/2]$ that give a unicity distance of at least 20.

Now suppose $p = 0$. Propose a new cipher for this source which has infinite unicity distance.

13E Further Complex Methods

(a) Show that under the change of variable $z = \cos x$ the equation

$$\frac{d^2w}{dx^2} + n^2 w = 0$$

becomes

$$(1 - z^2)\frac{d^2w}{dz^2} - z\frac{dw}{dz} + n^2 w = 0. \qquad (\dagger)$$

(b) Show that equation (†) is a Papperitz equation corresponding to the Papperitz-symbol or P-symbol

$$P \begin{Bmatrix} 1 & -1 & \infty & \\ 0 & 0 & -n & z \\ \frac{1}{2} & \frac{1}{2} & n & \end{Bmatrix},$$

explaining carefully the meaning of the symbol and the different elements appearing in it.

(c) Recall that the notation $F\big(A, B; C; \zeta\big)$ is used to denote the solution of the equation corresponding to the P-symbol

$$P \begin{Bmatrix} 0 & 1 & \infty & \\ 0 & 0 & A & \zeta \\ 1 - C & C - A - B & B & \end{Bmatrix},$$

for which $F\big(A, B; C; 0\big) = 1$.

Show that two linearly independent solutions of equation (†) are

$$w_1(z) = F\left(n, -n; \tfrac{1}{2}; \tfrac{1}{2}(1 - z)\right)$$

and

$$w_2(z) = (1 - z)^{1/2} F\left(-n + \tfrac{1}{2}, n + \tfrac{1}{2}; \tfrac{3}{2}; \tfrac{1}{2}(1 - z)\right),$$

explaining clearly any results on transforming Papperitz equations and P-symbols that you use.

(d) Deduce that $F\left(-\frac{1}{2}, \frac{3}{2}; \frac{3}{2}; u\right) = (1 - u)^{1/2}$, clearly justifying your reasoning.

14B Classical Dynamics

This question concerns a double pendulum, consisting of a simple pendulum of mass M and length l pivoted at the origin with angle θ_1 with respect to the vertical, together with another simple pendulum of mass m, also of length l and angle θ_2 with respect to the vertical, pivoted at the first mass M. The whole system moves freely in a vertical plane under the influence of a downward uniform gravitational acceleration of magnitude g. All the constants m, M, l and g are positive and, in addition, define $\omega_0 = \sqrt{g/l}$.

(a) Show that the Lagrangian for the system is

$$L = \frac{1}{2}Ml^2\dot{\theta}_1^2 + \frac{1}{2}m\left[l^2\dot{\theta}_1^2 + l^2\dot{\theta}_2^2 + 2l^2\cos(\theta_1 - \theta_2)\dot{\theta}_1\dot{\theta}_2\right] + Mgl\cos\theta_1 + mg(l\cos\theta_2 + l\cos\theta_1)\ .$$

(b) Write down the equations of motion and expressions for any conserved quantities. Furthermore, show that $\theta_1 = 0 = \theta_2$ is an equilibrium point, and derive the linearized equations of motion for small oscillations

$$(\theta_1, \theta_2) = (0,0) + (z_1, z_2)\,, \qquad |z_1| + |z_2| = o(1)\,,$$

around it.

(c) Find the four normal modes and show that the equilibrium point is stable. Consider the case when $\mu = m/M \ll 1$. Show that the characteristic frequencies are $\pm\omega$ and $\pm\omega'$, for some positive ω and ω' with $\omega - \omega' = \alpha\sqrt{\mu} + O(\mu)$, where α is a constant you should find.

15C Quantum Information and Computation

Let N be an odd integer that is not equal to the power of a prime number. Let a be an integer coprime to N with $1 < a < N$.

(a) Define the *order* of $a \bmod N$.

(b) Let $f : \mathbb{Z} \to \mathbb{Z}_N$ be the modular exponentiation function that has period r equal to the order of $a \bmod N$. Write down an explicit form for f and show that it is one-to-one within each period.

(c) Suppose r from part (b) is even and $(a^{r/2} + 1)$ is not divisible by N. How can one use Euclid's algorithm to obtain a factor of N? Justify your answer.

(d) Continuing from part (c), let m be the smallest integer for which $2^m > N^2$, and let B and b be integers such that $2^m = Br + b$ with $B = \lfloor \frac{2^m}{r} \rfloor$. Consider the state

$$|\varphi_1\rangle = \frac{1}{\sqrt{A}} \sum_{j=0}^{A-1} |x_0 + jr\rangle \, ,$$

where $x_0 \in \{0, 1, \ldots, 2^m - 1\}$ and $A = \begin{cases} B+1 & \text{if } x_0 \leqslant b \\ B & \text{if } x_0 > b \end{cases}$.

(i) For a positive integer M, let QFT_M denote the quantum Fourier transform modulo M. Give the action of QFT_M on the state $|x\rangle$, where $x \in \mathbb{Z}_M$.

(ii) Show that

$$|\varphi_2\rangle := \mathrm{QFT}_{2^m} |\varphi_1\rangle = \sum_{u=0}^{2^m-1} g(u) \, |u\rangle \, ,$$

and give a closed-form expression for $g(u)$.

(iii) Suppose $|\varphi_2\rangle$ is measured in the basis $\{|u\rangle\}_{u=0}^{2^m-1}$ to obtain a value of c satisfying

$$\left| c - k\frac{2^m}{r} \right| < \frac{1}{2},$$

for some $k \in \{0, 1, 2, \ldots, r-1\}$ that is coprime to r. Prove that there is at most one fraction k/r with a denominator $r < N$ satisfying

$$\left| \frac{c}{2^m} - \frac{k}{r} \right| < \frac{1}{2N^2} \, . \tag{$*$}$$

(iv) Suppose $N = 21$, $a = 10$, and you get the measured outcome $c = 427$. Using equation $(*)$ and a suitable continued fraction expansion, find r.
[*Hint:* $\left|\frac{427}{512} - \frac{5}{6}\right| < \frac{1}{2(21)^2}$]

16H Logic and Set Theory

Let L be a first-order language.

If M is an L-structure, we say that $\vartheta\colon M \to M$ is an automorphism of M if it is a bijection, and for all $m_1, \ldots, m_n \in M$ and all operation symbols ω of L (with arity n)

$$\vartheta(\omega_M(m_1, \ldots, m_n)) = \omega_M(\vartheta(m_1), \ldots, \vartheta(m_n)),$$

and for every predicate symbol φ of L (with arity n)

$$\big\{(\vartheta(m_1), \ldots, \vartheta(m_n)) : (m_1, \ldots, m_n) \in \varphi_M\big\} = \varphi_M.$$

We write $\mathrm{Aut}(M)$ for the set of automorphism of M. The set $\mathrm{Aut}(M)$ forms a group under composition. [*You do not need to prove this.*]

(a) Define the following notions:

(i) φ is a *sentence* in L;

(ii) T is a *theory* in L;

(iii) M is a *model* of T;

(iv) T is *consistent.*

(b) By appealing to a suitable theorem from the lectures, show that T is consistent if and only if it has a model.

(c) State the compactness theorem of first-order predicate logic and prove it using part (b) or otherwise.

For the remainder of this question, fix a consistent theory T in L. Expand L to a new language L_f obtained from L by adding a unary operation symbol f. Note that any L_f-structure can be thought of as a pair (M, θ), where M is an L-structure and θ is the interpretation in M of the additional operation symbol f.

(d) Specify a consistent theory T_f in L_f such that $T_f \supset T$ and an L_f-structure (M, ϑ) is a model of T_f if and only if M is a model of T and $\vartheta \in \mathrm{Aut}(M)$. Justify your claim.

Let X be any set and let L_X be the expansion of L with the additional operation symbols $\{f_x : x \in X\}$. Fix any group G and consider the expansion L_G of L and form the following theory in L_G: use the theories T_{f_g} from (d) and let

$$T_G := \bigcup_{g \in G} T_{f_g} \cup \Big\{(\exists\, x)\neg(f_g x = f_h x) \,:\, g, h \in G,\, g \neq h\Big\}$$
$$\cup \Big\{(\forall\, x)(f_g f_h x = f_k x) \,:\, g, h, k \in G,\, gh = k\Big\}.$$

We call a group G *T-good* if there is a model M of T such that $\mathrm{Aut}(M)$ contains an isomorphic copy of G and *T-bad* if it is not T-good.

[QUESTION CONTINUES ON THE NEXT PAGE]

(e) Let G be a group. Show that the following are equivalent:

(i) G is T-bad;

(ii) T_G is inconsistent;

(iii) there is a finite $X \subseteq G$ such that $T_G \cap L_X$ is inconsistent.

(f) Show that there is a theory T^* in the language of groups such that for any group G, G is a model of T^* if and only if G is T-good.

17F Graph Theory

In this question, no form of Menger's theorem or of the max-flow min-cut theorem may be assumed without proof.

(a) Let G be a bipartite graph with vertex classes X and Y. What is a *matching* from X to Y? State and prove Hall's marriage theorem, giving a necessary and sufficient condition for G to contain a matching from X to Y.

We define the *matching number* of a graph G to be the maximum size of a set of independent edges in G.

(i) If G is a k-regular bipartite graph with $|G| = n$ (for some $k > 0$), show that G has matching number $n/2$.

(ii) If G is an arbitrary k-regular graph with $|G| = n$ (for some $k > 0$), show that G has a matching number at least $(\frac{k}{4k-2})n$.

(iii) For $k = 2$, write down an infinite family of graphs G for which equality holds in (ii).

(b) Define the *eigenvalues* of a graph G. Let G be bipartite with vertex classes X and Y. If 0 is not an eigenvalue of G show that G contains a matching from X to Y.

18J Galois Theory

Let L/K be a finite extension of fields. Define what it means for L/K to be *normal*, *separable*, or *Galois*. Let $\overline{K}$ be an algebraic closure of K.

(a) Write $L = K(\alpha_1, \ldots, \alpha_n)$ for some $\alpha_1, \ldots, \alpha_n \in L$. Show by induction on n that $1 \leqslant \# \mathrm{Hom}_K(L, \overline{K}) \leqslant [L : K]$ and that the upper bound is an equality if L/K is separable.

(b) Show that $\# \mathrm{Aut}(L/K) \leqslant \# \mathrm{Hom}_K(L, \overline{K})$ with equality if L/K is normal.

(c) Deduce that if L/K is normal and separable then L/K is Galois.

(d) Find a prime number p such that the extension $\mathbb{F}_p(X)/\mathbb{F}_p(X^5)$ is Galois. [You may assume that all splitting fields are normal.]

19J Representation Theory

(a) Let $\rho : G \to GL_n(\mathbb{C})$ be a representation of a finite group G. Prove that ρ is isomorphic to a representation $\rho' : G \to GL_n(\mathbb{C})$ with

$$\rho'(G) \leqslant U_n = \{A \in \mathrm{Mat}_n(\mathbb{C}) \mid A\overline{A^T} = I\}.$$

(b) Let V be a finite dimensional complex representation of a group G. A bilinear form

$$(-,-) : V \times V \to \mathbb{C}$$

on V is G-invariant if $(v, w) = (gv, gw)$ for all $v, w \in V$ and $g \in G$.

Suppose now that V is an irreducible representation of G.

(i) Show that any G-invariant bilinear form on V is either non-degenerate or zero, and that any two G-invariant bilinear forms are proportional.

(ii) Show that any non-zero G-invariant bilinear form satisfies $(w, v) = \lambda(v, w)$ for all $v, w \in V$, where $\lambda \in \{\pm 1\}$ does not depend on v and w.

(c) Let $H \leqslant G$ be a subgroup of a finite group G, and V a finite dimensional complex representation of H. Define the *induced representation* $\mathrm{Ind}_H^G(V)$, and compute its character in terms of the character of V.

Show that if W is a finite dimensional complex representation of G, then the representations $\mathrm{Ind}_H^G(W \otimes V)$ and $W \otimes \mathrm{Ind}_H^G(V)$ are isomorphic.

20G Number Fields

(a) Define the *class group* of a number field. [You do not need to prove that it is a group.]

(b) Prove that the class group of a number field is finite. [You may use without proof the fact that, for every number field K, there is a constant C such that every ideal $I \subset \mathcal{O}_K$ contains a non-zero element α with $|\mathrm{N}(\alpha)| \leqslant C\mathrm{N}(I)$.]

(c) Let K be a number field and $I \subset \mathcal{O}_K$ an ideal. Prove that there is a positive integer n such that I^n is a principal ideal.

(d) A proper ideal $I \subset \mathcal{O}_K$ is called a primary ideal, if for all $\alpha, \beta \in \mathcal{O}_K$ such that $\alpha\beta \in I$ but $\alpha \notin I$, there is a positive integer k such that $\beta^k \in I$. Prove that an ideal in $\mathcal{O}_K$ is primary if and only if it is a power of a prime ideal.

21F Algebraic Topology

Let (X, x_0) be a based topological space. Define the *fundamental group* $\pi_1(X, x_0)$, and show that the composition law is well-defined and satisfies the group axioms.

Let $U(2)$ be the group of unitary 2×2 matrices, with the subspace topology from $\mathbb{C}^{2\times 2}$. Let $I \in U(2)$ denote the identity matrix.

(a) Let $\gamma : [0, 1] \to U(2)$ be given by $\gamma(t) = \begin{pmatrix} e^{2\pi i t} & 0 \\ 0 & 1 \end{pmatrix}$. Show that for non-zero $k \in \mathbb{Z}$, $[\gamma]^k$ is never the identity in $\pi_1(U(2), I)$. [You may use without proof a description of $\pi_1(S^1, *)$ provided it is clearly stated.]

(b) Show that $\pi_1(U(2), I)$ is abelian. [You may use without proof the fact that matrix multiplication gives a continuous map $U(2) \times U(2) \to U(2)$.]

22I Linear Analysis

(a) State the inversion theorem. State and prove the closed graph theorem.

(b) Now let X and Y be Banach spaces and let $S \in L(X, Y)$ be injective. We write S^{-1} for the inverse map to S, so that S^{-1} is a linear map from the image of S to X.

(i) Give an example to show that S^{-1} need not be continuous.

(ii) If $T \in L(X, Y)$ has the property that the image of T is contained in the image of S, show that $S^{-1} \circ T$ is continuous.

(iii) Give a counterexample to show that (ii) need not remain true if we drop the assumption that X is complete.

23H Analysis of Functions

(a) State and prove the Rellich-Kondrashov compactness theorem for the embedding of $H^1_0(\Omega)$ into $L^2(\Omega)$, where Ω is a bounded open subset of $\mathbb{R}^d$.

[You may use the Banach-Alaoglu and Plancherel theorems without proof.]

(b) Is the embedding of $H^1(\mathbb{R})$ into $L^2(\mathbb{R})$ compact? Justify your answer.

(c) Consider a bounded sequence f_n in $H^1(\mathbb{R})$ such that: (i) there is $C > 0$ so that $|f_n(x)| \leqslant C(1+x^2)^{-1}$ for all $x \in \mathbb{R}$ and $n \geqslant 1$, and (ii) f_n converges weakly to zero in $L^2(\mathbb{R})$. Prove that f_n converges strongly to zero in $L^2(\mathbb{R})$.

24G Riemann Surfaces

State and prove the identity theorem for Riemann surfaces.

Define what it means for $h : U \to \mathbb{R}$ to be a *harmonic function*, where U is a non-empty open connected subset of $\mathbb{R}^2$. Show that $h \in C^\infty(U)$.

Define also a *harmonic function* $H : R \to \mathbb{R}$, where R is a Riemann surface. Show that this is independent of the atlas chosen for R.

Suppose we have two functions $f, g : \mathbb{C} \to \mathbb{C}$ such that the product $f \cdot g$, defined pointwise by $(f \cdot g)(z) = f(z)g(z)$, is identically zero on $\mathbb{C}$. Must one of f or g be identically zero on $\mathbb{C}$ if:

(i) Both f and g are continuous on $\mathbb{C}$?

(ii) Both f and g are continuous on $\mathbb{C}$ and never simultaneously zero?

Now suppose we have two functions $f, g : \mathbb{R}^2 \to \mathbb{R}$ such that $f \cdot g$ is identically zero on $\mathbb{R}^2$. Must one of f or g be identically zero on $\mathbb{R}^2$ if:

(iii) Both f and g are in $C^\infty(\mathbb{R}^2)$?

(iv) Both f and g are harmonic?

25J Algebraic Geometry

In this question, all varieties are over an algebraically closed field k of characteristic zero.

Let $X \subset \mathbb{A}^n$ be an affine algebraic variety defined over k. Define the *tangent space* $T_{X,P}$ of X at a point $P \in X$, and define the *dimension* of X in terms of the tangent spaces of X.

Suppose that $X = Z(f) \subset \mathbb{A}^n$ where f is a non-constant polynomial. Show from your definition that X has dimension $n-1$. [Any form of the Nullstellensatz may be used if you state it clearly.]

Now suppose that $n \geqslant 3$ and $X = Z(f) \subset \mathbb{A}^n$ where f is a non-constant irreducible polynomial of degree at least 2. Let $P \in X$ be a smooth point of X, and translate $T_{X,P}$ by P to view it as an embedded hyperplane with $P \in T_{X,P} \subset \mathbb{A}^n$. Show that $X \cap T_{X,P}$ is singular at P.

Now let $Y := \{\varphi : \mathbb{A}^2 \to \mathbb{A}^3 \mid \varphi \text{ is linear but not injective}\}$. Show that Y is the zero locus of an ideal I which is generated by three quadrics. Compute the dimension of Y and identify any singular points of Y. [You may assume without proof that I is a radical ideal.]

26I Differential Geometry

(a) Define what is a *regular curve* and its *arc-length*, and prove that a regular curve can always be parametrised by arc-length. When so, define its *torsion*, and prove that the torsion is zero for planar curves.

(b) Consider a regular simple planar closed curve $\alpha : I = [a,b] \to \mathbb{R}^2$ enclosing an open bounded convex set Ω. We consider a line $L \subset \mathbb{R}^2$ outside Ω. Without loss of generality we assume $L = \{y = 0\}$ and $\Omega \subset \{y > 0\}$ and denote by x_0 and x_1 the minimum and maximum x-coordinate of $\alpha(I)$. You may assume that there are two smooth functions $u_\pm : [x_0, x_1] \to \mathbb{R}$ so that

$$\Omega = \{(x,y) \,:\, x \in (x_0, x_1) \text{ and } y \in (u_-(x), u_+(x))\}$$

with $u_- \leqslant u_+$ and u_- convex and u_+ concave. Then we define the following symmetrised set

$$S_L(\Omega) := \left\{ (x,y) \,:\, x \in (x_0,x_1), \text{ and } y \in \left(-\frac{u_+(x) - u_-(x)}{2}, \frac{u_+(x) - u_-(x)}{2} \right) \right\}.$$

(i) Prove that the areas enclosed satisfy $\mathcal{A}(S_L(\Omega)) = \mathcal{A}(\Omega)$.

[*Hint: Decompose the area into a trapezoid and two caps and figure out how they are transformed by the symmetrisation.*]

(ii) Prove that the perimeter of $S_L(\Omega)$ is at most the perimeter of Ω with equality if and only if Ω has an axis of symmetry parallel to L.

[*Hint: Calculate the perimeters following the decomposition of the previous hint, and reduce the inequality to be proven to a Minkowski inequality.*]

(iii) Deduce that any convex perimeter-minimizing domain Ω with fixed area and whose boundary is a regular curve must admit axes of symmetry in all directions.

27H Probability and Measure

Given a function $f : \mathbb{R} \to \mathbb{R}$, its Fourier transform is $\hat{f}(\xi) = \int_{\mathbb{R}} e^{-ix\xi} f(x)\, dx$ for $\xi \in \mathbb{R}$.

(a) State and prove the monotone convergence theorem.

(b) Let $\theta_n(x) = \left(1 - \frac{|x|}{n}\right)_+$ where $x_+ = \max\{x, 0\}$. Compute $\hat{\theta}_n$.

(c) Prove that there exists a universal constant $\alpha > 0$ such that: for any $f \in L^1 \cap L^\infty$ whose Fourier transform satisfies $\hat{f}(\xi) \geqslant 0$ for all $\xi \in \mathbb{R}$, one has

$$\|\hat{f}\|_{L^1} \leqslant \alpha \|f\|_{L^\infty}.$$

28L Applied Probability

(a) Consider a right-continuous continuous-time Markov chain X on $\mathbb{Z}$ starting from 0 such that $q_{0,1} = q_{0,-1} = 1/2$ and

$$q_{i,i+1} = \frac{2q_i}{3}, \quad q_{i,i-1} = \frac{q_i}{3}, \quad q_{-i,-i-1} = \frac{2q_{-i}}{3}, \quad q_{-i,-i+1} = \frac{q_{-i}}{3} \quad \forall i \geqslant 1;$$

with $q_i = 3^{|i|}$ for $i \in \mathbb{Z}$.

Is X recurrent? Is X explosive? Does X have an invariant distribution? Justify your answers.

(b) Let $X \sim \text{Markov}(Q)$ be an irreducible right-continuous continuous-time Markov chain on a countable state space with generator Q. Are the following statements true? Prove or give a counterexample.

(i) If the jump chain Y is positive-recurrent, then X is positive-recurrent.

(ii) If X is positive-recurrent, then the jump chain Y is positive-recurrent.

(c) Consider an $M/M/1$ queue with arrival and service rates $\lambda > 0$ and $\mu > 0$ respectively. After service, each customer returns to the beginning of the queue with probability $p \in (0,1)$. Let $(L_t)_{t \geqslant 0}$ denote the queue length.

(i) For which parameters is L transient, and for which is it recurrent?

(ii) When is it positive recurrent?

(iii) Find the invariant distribution when it exists and the expected queue length at equilibrium.

(iv) What is the distribution of the departure process at equilibrium?

[Clearly state all results you use. You may assume the recurrence and transience properties of simple random walks on $\mathbb{Z}$.]

29L Principles of Statistics

(a) Consider a statistical model $X_1, \ldots, X_n \overset{\text{i.i.d.}}{\sim} f(\cdot, \theta)$, $\theta \in \Theta \subseteq \mathbb{R}^p$, that satisfies the usual regularity conditions.

(i) Define the *score function* $S_n(\theta)$ and *Fisher information matrix* $I_n(\theta)$.

(ii) Show that $\mathbb{E}_\theta(S_1(\theta)) = 0$. [You may interchange integration and differentiation without justification.]

(iii) Recall that the score test statistic T_n for the null hypothesis $H_0 : \theta \in \Theta_0$ is given by

$$T_n := \frac{1}{n} S_n(\widetilde{\theta})^\top I_1(\widetilde{\theta})^{-1} S_n(\widetilde{\theta}),$$

where $\widetilde{\theta}$ maximises the log-likelihood over $\theta \in \Theta_0$. Show that in the case of a simple null $H_0 : \theta = \theta_0$, we have $T_n \xrightarrow{d} \chi_p^2$ as $n \to \infty$.

(b) Now consider the model $X_1, \ldots, X_n \overset{\text{i.i.d.}}{\sim} N(\mu, \sigma^2)$ where $\mu \in \mathbb{R}$ and $\sigma^2 > 0$.

(i) Consider the composite null $H_0 : \mu \in \mathbb{R}$, $\sigma^2 = 1$. Show that the score test statistic T_n for H_0 is given by

$$T_n = \left(\frac{1}{\sqrt{2n}} \sum_{i=1}^n \{(X_i - \bar{X})^2 - 1\} \right)^2,$$

where $\bar{X}$ is the sample mean.

(ii) Determine, with proof, the asymptotic distribution of T_n as $n \to \infty$ under the null hypothesis H_0. [*Hint: If* $Z \sim \chi_1^2$ *then* $\mathrm{Var}(Z) = 2$.]

30K Stochastic Financial Models

Let $(Z_n)_{0\leqslant n\leqslant N}$ be a real-valued process adapted to the filtration $(\mathcal{F}_n)_{0\leqslant n\leqslant N}$ where $\mathcal{F}_0$ is trivial and $N<\infty$ is not random. Suppose $\mathbb{E}(|Z_n|)<\infty$ for all $0\leqslant n\leqslant N$. Let $(V_n)_{0\leqslant n\leqslant N}$ be a supermartingale such that $V_n\geqslant Z_n$ almost surely for all $0\leqslant n\leqslant N$.

(a) Show that $\mathbb{E}(Z_\tau)\leqslant V_0$ for any stopping time τ. [You may use any result from the course if carefully stated.]

(b) Let

$$A_n=\sum_{k=0}^{n-1}\left[V_k-\mathbb{E}(V_{k+1}|\mathcal{F}_k)\right]$$

for $1\leqslant n\leqslant N$. Show that $(A_n)_{1\leqslant n\leqslant N}$ is previsible and non-decreasing.

(c) Set $A_0=0$ and let $M_n=V_n+A_n$ for $0\leqslant n\leqslant N$. Show that $(M_n)_{0\leqslant n\leqslant N}$ is a martingale.

(d) Now assume $V_N=Z_N$ and

$$V_n=\max\{Z_n,\mathbb{E}(V_{n+1}|\mathcal{F}_n)\},$$

and set $A_{N+1}=\infty$. Show that $\min\{V_n-Z_n,A_{n+1}-A_n\}=0$ for all $0\leqslant n\leqslant N$.

(e) Let $\tau^*=\min\{0\leqslant n\leqslant N: A_{n+1}>0\}$. Show that τ^* is a stopping time such that $\mathbb{E}(Z_{\tau^*})=V_0$.

31L Mathematics of Machine Learning

(a) Define the *shattering coefficient* $s(\mathcal{H}, n)$ and the *VC dimension* $VC(\mathcal{H})$ for a hypothesis class $\mathcal{H}$. State the Sauer-Shelah lemma.

(b) In each of the following cases, find $VC(\mathcal{H}_i)$,

(i) $\mathcal{H}_1 = \{\mathbb{1}_{[a,b]} : a, b \in \mathbb{R}\}$.

(ii) $\mathcal{H}_2 = \{\delta(2\mathbb{1}_{[a,b]} - 1) : a, b \in \mathbb{R}, \delta \in \{-1, 1\}\}$.

(c) Let $\mathcal{H}_3 = \{x \mapsto \operatorname{sgn}(x^T M x) : M \in \mathbb{R}^{d\times d}\}$. Show that

$$VC(\mathcal{H}_3) \leqslant \binom{d+1}{2}.$$

[You may use any theorems from lectures if they are precisely stated.]

(d) Consider

$$\mathcal{F} = \{\sum_{j=1}^{J} \beta_j h_j(x) : J < \infty, h_j \in \mathcal{H}_3, \beta_j > 0 \text{ for } j = 1, \ldots, J, \|\beta\|_1 \leqslant 1\}.$$

Prove that

$$\hat{\mathcal{R}}(\mathcal{F}(x_{1:n})) \leqslant \sqrt{\frac{(d^2 + d)\log(n+1)}{n}}.$$

[You may use any theorems from lectures about convex analysis and sub-Gaussian random variables if they are precisely stated.]

32D Asymptotic Methods

The incomplete gamma function $\gamma(x,y)$ is defined by

$$\gamma(x,y) = \int_y^\infty t^{x-1} \mathrm{e}^{-t}\, dt,$$

for real positive x and y.

(a) Using integration by parts, show that for fixed finite x,

$$\gamma(x,y) \sim y^{x-1}\mathrm{e}^{-y} \sum_{n=0}^{\infty} a_n(x)\, y^{-n}, \quad \text{as } y \to \infty,$$

where you should determine the coefficients $a_n(x)$.

(b) Give the leading-order term in the asymptotic approximation of $\gamma(x,y)$ for fixed finite y and as $x \to \infty$.

(c) Suppose that $x \to \infty$ and $y \to \infty$ with $y/x = \lambda$, where $\lambda > 1$ is a constant. Calculate the first two terms of the asymptotic expansion of $\gamma(x,y)$, in the form

$$\gamma(x,y) \sim y^{x-1}\mathrm{e}^{-y}\left[\, f(\lambda) + x^{-1} g(\lambda)\,\right],$$

where $f(\lambda)$ and $g(\lambda)$ are functions that you should determine.

[*Recall that* $\int_0^\infty t^n e^{-\alpha t} dt = \alpha^{-n-1} n!$, *for* n *a positive integer and* $\alpha > 0$.]

33A Dynamical Systems

State the Centre Manifold Theorem for the dynamical system $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, \mu)$ in $\mathbb{R}^n$ where μ is a real parameter. What is the key step in generating an extended centre manifold?

Consider the system

$$\dot{x} = x(\mu - y^2 - 2x^2),$$
$$\dot{y} = y(1 - x^2 - y^2),$$

where $x, y \geqslant 0$ and $\mu > 0$.

(a) Show that the fixed point $(0, 1)$ has a bifurcation at $\mu = 1$. Find a fixed point on the x-axis and determine the value of $\mu = \mu_c > 1$ at which it has a bifurcation.

(b) By finding the first approximation to the extended centre manifold, construct the normal form near the bifurcation point $(0, 1)$ when $\mu \approx 1$. Hence identify the type of bifurcation. By appealing to a symmetry of the system, explain why this bifurcation is expected.

(c) Show that there is another fixed point with $x > 0$, $y > 0$ and show how its structure is consistent with your normal form in part (b).

(d) Draw a sketch of the values of x at the fixed points as functions of μ indicating the bifurcation points and the regions where each branch is stable. [Detailed calculations are not required.]

34D Integrable Systems

(a) Define a *completely integrable system* on a $2n$-dimensional phase space M, and state the Arnold–Liouville theorem.

(b) Consider $M = \mathbb{R}^{2n}$ with coordinates $(p_i, q_i), i = 1, \ldots, n$, and the standard Poisson structure. Let

$$H = \frac{1}{2}\Big({p_1}^2 + \cdots + {p_n}^2 + {W_1}^2{q_1}^2 + \cdots + {W_n}^2{q_n}^2 + a_1q_1 + \cdots + a_nq_n\Big),$$

where $W_1, \ldots, W_n, a_1, \ldots, a_n$ are constants with $W_k \neq 0$ for all k.

(i) Find n independent functions $F_1, F_2, \ldots, F_n$ in involution with $\sum_{i=1}^{n} F_i = H$, and demonstrate that Hamilton's equations with Hamiltonian H are completely integrable.

(ii) Find the action variables.

35C Principles of Quantum Mechanics

(a) State the commutation relations for the spin operator $\mathbf{S}$ and describe the associated irreducible representations $\{|s,\sigma\rangle\}$, where s and σ are quantum numbers you should specify. Determine the Hermitian conjugate and the trace of $\mathbf{S}$.

(b) Henceforth consider only the Hilbert space of a particle of spin 3/2 and use the basis $\{|\sigma\rangle\}$ of eigenvectors of S_z. Using the relation

$$S_{\pm}|\sigma\rangle = \sqrt{s(s+1)-\sigma(\sigma\pm 1)}\hbar|\sigma\pm 1\rangle\,, \tag{1}$$

write down S_x and S_y as matrices.

(c) Let $\mathbf{n} = (\cos\varphi, \sin\varphi, 0)$ be a vector in $\mathbb{R}^3$, using the standard basis. Derive the states $|\mathbf{n}, 3/2\rangle$ for which the spin along the direction $\mathbf{n}$ is always measured to be $\frac{3}{2}\hbar$.

(d) Let $H = -\gamma\mathbf{B}\cdot\mathbf{S}$ be the Hamiltonian of the system, where γ is a constant and $\mathbf{B} = B\hat{\mathbf{z}}$ is an external magnetic field. Compute the state of the system at time t assuming that it started at time $t=0$ from (i) $|\hat{\mathbf{z}}, 3/2\rangle$, and (ii) $|\hat{\mathbf{x}}, 3/2\rangle$. Briefly interpret the results.

36B Applications of Quantum Mechanics

(a) A particle moving in an attractive potential $V_1(\mathbf{x})$ has a ground-state energy E_1, while a particle moving in an attractive potential $V_2(\mathbf{x})$ has a ground-state energy E_2. Using the variational method, show that $E_1 \geqslant E_2$ if $V_1(\mathbf{x}) \geqslant V_2(\mathbf{x})$ for all $\mathbf{x} \in \mathbb{R}^3$.

[*Hint: use the wavefunction of the particle in* $V_1(\mathbf{x})$ *as a trial function for* $V_2(\mathbf{x})$.]

(b) Consider a one-dimensional Hamiltonian $H = T + V$, with kinetic energy T and the attractive potential

$$V(x) = -\frac{\alpha}{|x|^n} \ ,$$

where α and n are positive constants. The exact ground state of the Hamiltonian H is $\psi_0(x)$. By considering the trial function $\psi(x) = \psi_0(\lambda x)$, use the variational method to show that there are no bound states for $n > 2$.

(c) Consider a two-level quantum system, where the Hamiltonian H_0 admits two eigenstates: $|\psi_1\rangle$ with energy E_1, and $|\psi_2\rangle$ with energy E_2. You may assume that the states are orthogonal, normalised, and non-degenerate, and that $E_1 < E_2$.

Consider the perturbation H_{p}, with matrix elements

$$\langle\psi_1|H_{\text{p}}|\psi_1\rangle = \langle\psi_2|H_{\text{p}}|\psi_2\rangle = 0 \ ,$$

and

$$\langle\psi_1|H_{\text{p}}|\psi_2\rangle = \langle\psi_2|H_{\text{p}}|\psi_1\rangle = h \ ,$$

and h constant. Find the exact eigenvalues of the Hamiltonian $H = H_0 + H_{\text{p}}$.

Estimate the ground-state energy of H using the variational method, where the trial function is

$$|\psi_\beta\rangle = \sin\beta|\psi_1\rangle + \cos\beta|\psi_2\rangle \ ,$$

and β is an adjustable parameter. How does your answer compare to the exact ground state?

37C Statistical Physics

(a) Starting with the first law of thermodynamics for the energy E, derive a formula for the variation of the enthalpy H. Define the temperature T and volume V as derivatives of H. From this, deduce the Maxwell relation

$$\left.\frac{\partial T}{\partial p}\right|_S = \left.\frac{\partial V}{\partial S}\right|_p ,$$

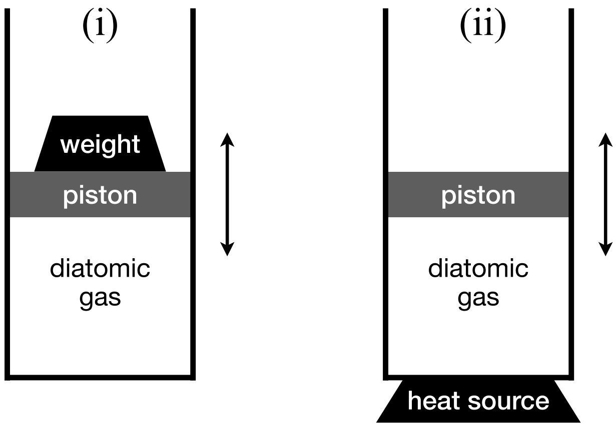
where p is the pressure and S is the entropy.

(b) Determine the enthalpy H of a diatomic ideal gas in terms of N and T, where the temperature T lies in a range for which vibrations of the molecule freeze out but rotations do not.

(c) A freely moving piston is inserted in a cylindrical container. The chamber below the piston contains the diatomic gas from part (b), which is at an initial temperature T_0 and thermally insulated from the outside environment. This piston is initially at a pressure p_0 due to the outside atmosphere. Consider the following two processes and calculate the change of temperature, ΔT, in each case. [See figures depicting the two processes below.]

(i) A weight is placed onto the piston, thereby increasing the pressure to p_1 on the piston. [You may assume this process is adiabatic.]

(ii) The pressure is held fixed while the base of the cylinder is heated such that an amount of heat Q is gradually added to the gas in the cylinder. As a result, the gas does some amount of work W on the piston, pushing it upward. [*Hint: use enthalpy.*]



38E General Relativity

(a) Consider a spacetime with metric $g_{\mu\nu}$. Write down the covariant derivative $\nabla_\alpha g_{\mu\nu}$ in terms of the connection $\Gamma_{\alpha\ \beta}^{\ \gamma}$, which is assumed to satisfy $\Gamma_{\alpha\ \beta}^{\ \gamma} = \Gamma_{\beta\ \alpha}^{\ \gamma}$. Determine the unique choice for the connection that ensures $\nabla_\alpha g_{\mu\nu} = 0$. In the remainder of this question we use this connection.

(b) If $R^\mu{}_{\nu\alpha\beta}$ is the Riemann curvature tensor, then

$$\nabla_\alpha \nabla_\beta U_\mu \,-\, \nabla_\beta \nabla_\alpha U_\mu \;=\; -R^\nu{}_{\mu\alpha\beta} U_\nu \qquad (*)$$

for any covariant vector field U_μ. By setting $U_\mu = \partial_\mu \phi$ in equation $(*)$, where ϕ is a scalar field, show that

$$R^\mu{}_{\alpha\beta\gamma} + R^\mu{}_{\beta\gamma\alpha} + R^\mu{}_{\gamma\alpha\beta} \;=\; 0\,.$$

State clearly any property of a vector field of the form $\partial_\mu \phi$ that your argument depends on.

(c) From equation $(*)$, derive an analogous expression for $\nabla_\alpha \nabla_\beta W_{\mu\nu} - \nabla_\beta \nabla_\alpha W_{\mu\nu}$, where $W_{\mu\nu}$ is a general covariant tensor field of rank 2, stating clearly any assumptions you make. By making a suitable choice for $W_{\mu\nu}$, deduce that

$$R_{\mu\alpha\beta\gamma} = -R_{\alpha\mu\beta\gamma}\,.$$

(d) Define the Ricci tensor $R_{\alpha\beta}$ and show that it is symmetric. For certain spacetimes,

$$R_{\mu\alpha\beta\gamma} = K(\, g_{\mu\beta} g_{\alpha\gamma} - g_{\mu\gamma} g_{\alpha\beta}\,)\,,$$

where K is a scalar. Compute the Ricci tensor and Ricci scalar and deduce that K is constant, assuming that the dimension n of the spacetime is four. How would this conclusion change for other values $n > 1$? [Identities involving covariant derivatives of the Ricci tensor may be used without proof but should be clearly stated.]

39D Fluid Dynamics

Write down the Stokes equations governing the flow of incompressible viscous fluid at zero Reynolds number, and show that the pressure and vorticity are harmonic.

A rigid sphere of radius a moves with velocity $\mathbf{U}$ through fluid of dynamic viscosity μ that is stationary far from the sphere. Write down the boundary conditions that should be applied to the normal and tangential components of the fluid velocity $\mathbf{u}$ on the surface of the sphere, explaining each in physical terms.

The velocity and pressure fields at a point $\mathbf{x}$ in the fluid can be written as

$$\mathbf{u} = \left(\frac{3}{4}\frac{a}{r} + \frac{1}{4}\frac{a^3}{r^3}\right)\mathbf{U} + \left(\frac{3}{4}\frac{a}{r^3} - \frac{3}{4}\frac{a^3}{r^5}\right)(\mathbf{U}\cdot\mathbf{x})\mathbf{x},$$

$$p = \frac{3}{2}\mu a \frac{\mathbf{U}\cdot\mathbf{x}}{r^3},$$

where the origin lies at the centre of the sphere and $r = |\mathbf{x}|$.

Using suffix notation, or otherwise, calculate the velocity gradient $(\boldsymbol{\nabla}\mathbf{u})_{ij} = \partial u_j/\partial x_i$. Hence:

(i) determine an expression for the vorticity;

(ii) calculate $\boldsymbol{\nabla}(1/r)$ and $\nabla^2(1/r)$, and use your answers to argue directly that the pressure and vorticity are harmonic;

(iii) prove that the flow is incompressible;

(iv) determine the stress $(\boldsymbol{\sigma}\cdot\mathbf{n})_i = \sigma_{ij}n_j$ on the surface of the sphere, where $\mathbf{n}$ is the outward unit normal to the sphere;

(v) determine the force exerted by the fluid on the sphere.

40A Waves

Consider the linearized Cauchy momentum equation, which governs small and smooth displacements $\mathbf{u}(\mathbf{x}, t)$ in a uniform, linear, isotropic and elastic solid of density ρ,

$$\rho \frac{\partial^2 \mathbf{u}}{\partial t^2} = (\lambda + \mu)\boldsymbol{\nabla}(\boldsymbol{\nabla} \cdot \mathbf{u}) + \mu \nabla^2 \mathbf{u}, \qquad (\dagger)$$

where λ and μ are the Lamé moduli.

(a) Show that this equation supports two distinct classes of wave motion: P-waves for the dilatation $\theta = \boldsymbol{\nabla} \cdot \mathbf{u}$ with phase speed c_P; and S-waves for the rotation $\boldsymbol{\omega} = \boldsymbol{\nabla} \times \mathbf{u}$ with phase speed c_S. You should express c_P and c_S explicitly in terms of the Lamé moduli.

(b) Consider plane-wave solutions to equation (†) of the form $\mathbf{u} = \mathbf{f}(\widehat{\mathbf{k}} \cdot \mathbf{x} - ct)$, where $\widehat{\mathbf{k}}$ is a unit vector. By direct substitution into equation (†), determine the form that $\mathbf{f}$ must take for P-waves and for S-waves, and express the dilatation and rotation in terms of these forms for each class of waves.

[*Hint: You may find the vector identity* $\nabla^2 \mathbf{q} = \boldsymbol{\nabla}(\boldsymbol{\nabla} \cdot \mathbf{q}) - \boldsymbol{\nabla} \times (\boldsymbol{\nabla} \times \mathbf{q})$ *useful.*]

(c) A planar interface at $z = 0$ separates two elastic solids of different densities and elastic moduli. A harmonic P-wave with wavevector $\mathbf{k}$ lying in the (x, z) plane is incident from $z < 0$ at an oblique angle. Show in a diagram the directions of all the reflected and transmitted waves, labelled with their polarisations, assuming that none of these waves are evanescent. State the boundary conditions on the components of the displacement and the stress that would, in principle, determine the amplitudes.

(d) Now consider a harmonic P-wave of unit amplitude with $\mathbf{k} = k(\sin\phi, 0, \cos\phi)$ incident on the planar interface $z = 0$ between two elastic and inviscid liquids with wave speed c_P and modulus λ in $z < 0$ and wave speed $\hat{c}_P = 2c_P$ and modulus $\hat{\lambda}$ in $z > 0$. Obtain solutions for the reflected and transmitted waves. Show that the magnitudes of these two waves are equal if

$$\sin^2\phi = \frac{3Z^2 - 4Z\hat{Z} + \hat{Z}^2}{\hat{Z}(\hat{Z} - 4Z)},$$

where $Z = \lambda / c_P$ and $\hat{Z} = \hat{\lambda} / \hat{c}_P$.

41D Numerical Analysis

Let N be an integer power of 2. The discrete Fourier transform (DFT) $\mathcal{F}_{2N} : \mathbb{C}^{2N} \to \mathbb{C}^{2N}$ is defined by

$$\mathbf{Y} = \mathcal{F}_{2N}\mathbf{y}, \text{ where } Y_k = \sum_{n=0}^{2N-1} y_n \exp\left(-\frac{\pi i}{N} nk\right), \quad 0 \leqslant k \leqslant 2N-1, \qquad (\dagger)$$

while the discrete cosine transform (DCT) $\mathcal{C}_N : \mathbb{R}^N \to \mathbb{R}^N$ and the discrete sine transform (DST) $\mathcal{S}_N : \mathbb{R}^N \to \mathbb{R}^N$ are defined by

$$\mathbf{Z} = \mathcal{C}_N\mathbf{x}, \text{ where } Z_k = \sum_{n=0}^{N-1} x_n \cos\left[\frac{\pi}{N}\left(n+\frac{1}{2}\right)k\right], \quad 0 \leqslant k \leqslant N-1,$$

$$\tilde{\mathbf{Z}} = \mathcal{S}_N\mathbf{x}, \text{ where } \tilde{Z}_k = \sum_{n=0}^{N-1} x_n \sin\left[\frac{\pi}{N}\left(n+\frac{1}{2}\right)(k+1)\right], \quad 0 \leqslant k \leqslant N-1,$$

for N even.

(a) Show that there exists an algorithm that computes the DFT of a vector of length $2N$ for which the number of multiplications required is $\mathcal{O}(N \log N)$.

(b) Let $\mathbf{x} \in \mathbb{R}^N$ and $\mathbf{y} \in \mathbb{R}^{2N}$ be related by $y_n = x_n$ for $0 \leqslant n \leqslant N-1$ and $y_n = x_{2N-n-1}$ for $N \leqslant n \leqslant 2N-1$. With $\mathbf{Y}$ defined as in equation (†), show that

$$\frac{1}{2}\exp\left(-\frac{\pi i}{2N}k\right) Y_k = \sum_{n=0}^{N-1} x_n \cos\left[\frac{\pi}{N}\left(n+\frac{1}{2}\right)k\right], \quad 0 \leqslant k \leqslant 2N-1.$$

(c) Use parts (a) and (b) to show that there exists an algorithm to compute the DCT of a vector of length N using $\mathcal{O}(N \log N)$ multiplications.

(d) Let $\mathbf{x} \in \mathbb{R}^N$ and $\boldsymbol{\xi} \in \mathbb{R}^N$ be related by $\xi_n = (-1)^n x_n$ for $0 \leqslant n \leqslant N-1$. By considering the DCT of $\boldsymbol{\xi}$, or otherwise, show that there exists an algorithm to compute the DST of a vector of length N using $\mathcal{O}(N \log N)$ multiplications.

END OF PAPER