UNIVERSITY OF CAMBRIDGE

MAT2

MATHEMATICAL TRIPOS Part II

Monday, 09 June, 2025 1:30pm to 4:30pm

PAPER 1

Before you begin read these instructions carefully.

The examination paper is divided into two sections. Each question in Section II carries twice the number of marks of each question in Section I. Section II questions also carry an alpha or beta quality mark and Section I questions carry a beta quality mark.

Candidates may obtain credit from attempts on **at most six questions** from Section I and from any number of questions from Section II.

Write on **one side** of the paper only and begin each answer on a separate sheet.

Write legibly; otherwise you place yourself at a grave disadvantage.

At the end of the examination:

Separate your answers to each question.

Complete a gold cover sheet **for each question** that you have attempted, and place it at the front of your answer to that question.

Complete a green main cover sheet listing **all the questions** that you have attempted.

Every cover sheet must also show your Blind Grade Number and desk number.

Tie up your answers and cover sheets into **a single bundle**, with the main cover sheet on the top, and then the cover sheet and answer for each question, in the numerical order of the questions.

STATIONERY REQUIREMENTS
Gold cover sheets
Green main cover sheet
Script paper
Rough paper

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

SECTION I

1G Number Theory

(a) Using Fermat factorisation, find a non-trivial factorisation of $N = 14351$.

(b) Let $N \geqslant 1$ be an odd, composite integer that is not a square, and let $k \geqslant 1$. We say that Fermat factorisation for N succeeds after k steps if the first value $r \geqslant \sqrt{N}$ such that $r^2 - N$ is a square is $r = \lfloor \sqrt{N} \rfloor + k$.

Suppose that $N = 3p$, where $p > 3$ is a prime number. Find the value of k such that Fermat factorisation for N succeeds after k steps.

2I Topics in Analysis

State Liouville's theorem on approximation of algebraic numbers by rationals.

Prove that the number $\displaystyle\sum_{n=0}^{\infty} \frac{1}{10^{n^n}}$ is transcendental.

Deduce that there are uncountably many transcendental numbers.

3K Coding & Cryptography

State and prove Kraft's inequality.

Describe Shannon-Fano Coding. Explain why it works and give an upper bound on its expected word length.

4F Automata & Formal Languages

Consider the following table with classes of formal languages in the rows and closure properties in the columns (where "union", "intersection", and "complement" stand for closed under union, intersection, and complement, respectively). Fill the twelve entries of the table with "Yes" and "No", depending on whether the class of formal languages in the row has the closure property given in the column or not. You do not need to give arguments for "Yes" answers. For each "No" answer, either provide a counterexample or an argument why the class is not closed under the operation.

	union	intersection	complement
regular	◦	◦	◦
context-free	◦	◦	◦
computable	◦	◦	◦
computably enumerable	◦	◦	◦

[*If you give a counterexample from the lectures, you do not have to prove that it is a counterexample, provided that you state it correctly.*]

5K Statistical Modelling

A random variable $Y > 0$ is said to follow the Weibull distribution with parameters $\lambda > 0$ and $k > 0$ if $X = (Y/\lambda)^k$ follows the exponential distribution with rate parameter 1 (so the probability density function of X is e^{-x}, $x > 0$). Let $Y_1, \ldots, Y_n$ be an independent and identically distributed sample from this Weibull distribution.

Show that the probability density function of Y is given by

$$f_\lambda(y) = \frac{k}{\lambda}\left(\frac{y}{\lambda}\right)^{k-1} e^{-(y/\lambda)^k},\ y > 0.$$

For the rest of this question, suppose k is fixed. Show that $\{f_\lambda \ ; \ \lambda > 0\}$ is a one-parameter exponential family, and write down its natural parameter and sufficient statistic.

Show that $\mathbb{E}(Y^k) = \lambda^k$, then find the maximum likelihood estimator of λ from $Y_1, \ldots, Y_n$.

6A Mathematical Biology

A large group of people, of fixed number, are debating a proposition. The group consists of those in favour of the proposition, $Y(t)$, those opposed, $N(t)$, and those who are undecided, $U(t)$. Debates can change people's minds. The outcome of a continuous debate taking place over time is modelled by the equations

$$\frac{dY}{dt} = -\beta Y N \ , \quad \frac{dN}{dt} = (\beta - \alpha) Y N + \zeta U \ , \quad \frac{dU}{dt} = \alpha Y N - \zeta U \ .$$

The constants α, β and ζ are positive.

(a) Briefly give an interpretation of α, β, and ζ.

(b) Show that those in favour of the proposition and those opposed cannot exist in equilibrium. Determine the stability of any equilibria.

(c) Using part (b), determine if there are values of α, β and ζ for which everyone eventually favours the proposition, irrespective of the initial conditions.

7E Further Complex Methods

Consider the improper integral

$$\int_{-\infty}^{\infty} \frac{1}{x^6 - 1} \, dx \, .$$

Explain what is meant by the *Cauchy principal value* of this integral, and evaluate it.

8B Classical Dynamics

This question concerns a linear, triatomic molecule, consisting of two outer atoms of mass m on either side of an inner atom of mass M. All three atoms lie on a vertical line, taken as the y-axis (directed upwards), at heights $y_1 > y_2 > y_3$. The atoms move under the influence of a uniform, downward gravitational acceleration of magnitude g, as well as forces arising from the potential energy

$$\tfrac{1}{2} \left[k(y_1 - y_2)^2 + k(y_2 - y_3)^2 \right] \ .$$

The constants m, M, k and g are positive.

(a) Write down the Lagrangian $L(y_1, \dot{y}_1, y_2, \dot{y}_2, y_3, \dot{y}_3)$ for the system. Give an expression for the centre of mass Y of the molecule, and determine its time evolution $Y(t)$ assuming $Y(0) = A$, where A is a constant.

(b) Introduce generalised coordinates

$$Q_s = y_1 + y_3 \qquad \text{and} \quad Q_a = y_1 - y_3$$

and use your answer to part (a) to eliminate y_2 and obtain a Lagrangian $\hat{L}$ in terms of Q_s and Q_a. Hence obtain a differential equation for Q_a.

9E Cosmology

Consider the motion of light rays in a homogeneous and isotropic expanding universe with scale factor $a(t)$. Light emitted by a distant galaxy at wavelength λ_e is observed on Earth to have wavelength λ_0. The galaxy redshift z is defined by

$$1 + z = \frac{\lambda_0}{\lambda_e}.$$

(a) Assuming that the galaxy remains at a fixed comoving distance, show that the redshift is related to the scale factor by

$$1 + z = \frac{a(t_0)}{a(t_e)},$$

where the light is emitted at time t_e and observed today at time t_0.

(b) Suppose the galaxy is located at comoving position x and let L be the amount of energy emitted by the galaxy in photons per unit time. Show that the total energy per unit time crossing a sphere centred on the galaxy and intercepting the Earth is

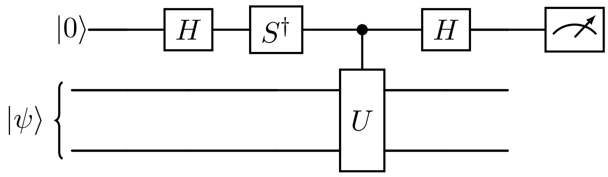
$$\frac{L}{(1+z)^n},$$

where n is an integer you should determine. Hence, show that the energy per unit time per unit area reaching the Earth is

$$\frac{L}{4\pi a^2(t_0)\, x^2\, (1+z)^n}.$$

10C Quantum Information and Computation

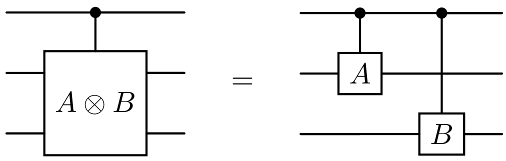
(a) Consider the following quantum circuit acting on a state $|0\rangle\,|\psi\rangle$



where $H = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$ and $S = \begin{bmatrix} 1 & 0 \\ 0 & i \end{bmatrix}$. Show that on measuring the first qubit in the computational basis, the probability of outcome 1 is

$$p(1) = \frac{1}{2}(1 - \mathrm{Im}\,\langle\psi|U|\psi\rangle)\,.$$

(b) Verify the following identity, where A and B are unitary matrices:



(c) Consider a 2-qubit initial state $|\psi\rangle$ and a matrix description $W = Z \otimes Z + X \otimes I$. By modifying any of the above circuits, draw new circuits to obtain $\langle\psi|W|\psi\rangle$ in terms of outcome probabilities of 1-qubit measurements. [*Hint: note that W is not a unitary matrix but is a linear combination of orthogonal matrices.*]

SECTION II

11K Coding & Cryptography

Describe the Huffman coding scheme and prove that Huffman codes are optimal.

A Huffman code is used to encode letters $a_1, \ldots, a_m$ with respective probabilities $p_1 \geqslant p_2 \geqslant \cdots \geqslant p_m$. Prove that, if $p_1 < 1/3$, all codewords have length at least 2. Prove that, if $p_1 > 2/5$, then there is a codeword of length 1.

Find a probability distribution for which both of the following codes are optimal.

(a) 0, 10, 110, 111

(b) 00, 01, 10, 11

12F Automata & Formal Languages

(a) Let $C, D \subseteq \mathbb{B}$. Give definitions of the following concepts:

(i) $C \leqslant_{\mathrm{m}} D$;

(ii) $C \equiv_{\mathrm{m}} D$; and

(iii) C is a *nontrivial index set*.

(b) The proof of Rice's theorem shows that nontrivial index sets I are not computable by either proving $\mathbf{K} \leqslant_{\mathrm{m}} I$ or $\mathbb{B} \backslash \mathbf{K} \leqslant_{\mathrm{m}} I$. State when the first or the second option holds according to the proof of Rice's theorem.

[*Define your notation; you do not need to prove your claim.*]

(c) For the nontrivial index sets

$$\mathbf{Emp} := \{w \in \mathbb{B}\,;\, \mathrm{W}_w = \varnothing\} \text{ and } \mathbf{Inf} := \{w \in \mathbb{B}\,;\, \mathrm{W}_w \text{ is infinite}\},$$

state in each case whether the first or the second option of (b) holds.

(d) Is the set $\{w \in \mathbb{B}\,;\, |w| \text{ is even}\}$ an index set? Justify your answer.

(e) Consider the nontrivial index set $\mathbf{Two} := \{w \in \mathbb{B}\,;\, |\mathrm{W}_w| \geqslant 2\}$ and show that $\mathbf{K} \equiv_{\mathrm{m}} \mathbf{Two}$.

(f) Consider the nontrivial index set

$$\mathbf{Cof} := \{w \in \mathbb{B}\,;\, \text{the complement of } \mathrm{W}_w \text{ is finite}\}$$

and show that both $\mathbf{K} \leqslant_{\mathrm{m}} \mathbf{Cof}$ and $\mathbb{B} \backslash \mathbf{K} \leqslant_{\mathrm{m}} \mathbf{Cof}$.

[*In the entire question, you may use any results proved in the lectures, provided that you state them precisely and correctly.*]

13K Statistical Modelling

Suppose $(X_1, Y_1), \ldots, (X_n, Y_n) \in \mathbb{R}^p \times \mathbb{R}$ are independent and identically distributed and the conditional distribution of Y_i given X_i is given by

$$Y_i \mid X_i \sim \mathrm{N}(X_i^T \beta, \sigma^2 v(X_i)),$$

where $v(x) > 0$ is a known function. We would like to use a sample of $(X_1, Y_1), \ldots, (X_n, Y_n)$ to estimate the unknown parameters $\beta \in \mathbb{R}^p$ and $\sigma^2 > 0$. Let $X \in \mathbb{R}^{n \times p}$ denote the matrix with the ith row being X_i and let $Y = (Y_1, \ldots, Y_n)^T \in \mathbb{R}^n$.

(a) Show that the maximum likelihood estimator of β is given by

$$\hat{\beta}(\Sigma) = (X^T \Sigma^{-1} X)^{-1} X^T \Sigma^{-1} Y,$$

where $\Sigma \in \mathbb{R}^{n \times n}$ is a diagonal matrix with the ith diagonal entry given by $v(X_i)$.

(b) Explain why the `R` code below returns the estimator in (a), where `X`, `Y`, and `Sigma` in the `R` environment store the value of X, Y, and Σ in the above model.

```
Y.tilde <- Y / sqrt(diag(Sigma))
X.tilde <- X / sqrt(diag(Sigma))
fit1 <- lm(Y.tilde ~ X.tilde - 1)
fit1$coefficients
```

Write down `R` code that returns the estimator $\hat{\beta}(I_n)$, where $I_n \in \mathbb{R}^{n \times n}$ denote the identity matrix.

(c) Sketch an argument that shows both $\hat{\beta}(\Sigma)$ and $\hat{\beta}(I_n)$ are consistent for estimating β and are asymptotically normal when $n \to \infty$. You may assume that the law of large numbers and the central limit theorem can be used for this model.

(d) Suppose $p = 1$. Find an expression for

$$\rho = \lim_{n \to \infty} \frac{\mathrm{Var}(\hat{\beta}(\Sigma))}{\mathrm{Var}(\hat{\beta}(I_n))}.$$

Your answer should depend on the distribution of X_1.

(e) Do you expect ρ to be $\geqslant 1$ or $\leqslant 1$? Explain why. Prove it using your expression of ρ in (d).

14E Further Complex Methods

The Dirichlet beta function is defined as

$$\beta(s) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^s} \tag{$\dagger$}$$

for $\mathrm{Re}\,(s) > 0$ and by analytic continuation to $\mathbb{C}$. The integral representation of equation ($\dagger$) for $\mathrm{Re}\,(s) > 0$ is given by

$$\beta(s) = \frac{1}{\Gamma(s)} \int_0^{\infty} \frac{t^{s-1}}{e^t + e^{-t}}\, dt\,,$$

where Γ is the gamma function.

(a) The Hankel representation is defined as

$$\beta(s) = \frac{\Gamma(1-s)}{2\pi i} \int_{-\infty}^{(0^+)} \frac{t^{s-1}}{e^t + e^{-t}}\, dt. \tag{$\ddagger$}$$

Draw a diagram to show the integration contour implied by the limits of the integral in equation ($\ddagger$). Show that this representation gives an analytic continuation of $\beta(s)$ as defined by equation ($\dagger$) to all $s \in \mathbb{C}$.

[You may assume that $\Gamma(s)\Gamma(1-s) = \pi \mathrm{cosec}\,(\pi s)$.]

(b) Use equation ($\ddagger$) to evaluate $\beta(0)$ and $\beta(-2)$. Show that if n is a non-negative integer then $\beta(-2n-1) = 0$.

(c) Consider the poles of the integrand of equation ($\ddagger$) on the imaginary axis, except for the pole at $t = 0$, if it exists. For what conditions on s does the sum of the residues at these poles converge? Assume that under these conditions it may be shown that the integral in equation ($\ddagger$) is equal to the sum of the residues multiplied by $-2\pi i$. Deduce the reflection formula

$$\beta(1-s) = \Gamma(s) \left(\frac{\pi}{2}\right)^{-s} \sin\left(\frac{s\pi}{2}\right) \beta(s),$$

explaining carefully why this formula is valid for all $s \in \mathbb{C}$.

15E Cosmology

(a) Consider the Friedmann-Lemaître-Robertson-Walker (FLRW) metric with co-moving curvature constant k (not normalised to unity),

$$ds^2 = -c^2dt^2 + a(t)^2 \left[\frac{dr^2}{1-k\,r^2} + r^2(d\theta^2 + \sin^2\theta d\phi^2)\right] .$$

(i) Briefly comment on the three geometries described by this metric and, in each case, calculate the proper distance between the points $r = 0$ and $r = \Delta r$ along curves with $dt = d\theta = d\phi = 0$.

(ii) For each geometry give new time and radial coordinates τ and χ that transform the metric to

$$ds^2 = a^2(\tau)\left[-c^2d\tau^2 + d\chi^2 + f^2(\chi)(d\theta^2 + \sin^2\theta d\phi^2)\right] ,$$

where the function $f(\chi)$ should be specified. Along which trajectories do radial light rays ($d\theta = d\phi = 0$) propagate in these coordinates?

(b) For an FLRW universe with vanishing cosmological constant the Friedmann and continuity equations are

$$\left(\frac{\dot{a}}{a}\right)^2 + \frac{kc^2}{a^2} = \frac{8\pi G}{3c^2}\,\rho\,, \qquad \dot{\rho} = -3\frac{\dot{a}}{a}\,(\rho + P)\ ,$$

where ρ is the energy density, P is the pressure and $\dot{a} = \frac{da}{dt}$. Consider an open universe ($k < 0$) filled with dark energy 'quintessence', which has an equation of state $P = -\frac{2}{3}\rho$. At $t = t_0$ we take $\rho(t_0) = \rho_0$ and $a(t_0) = 1$.

(i) Use the continuity equation to determine the rate at which the energy density falls as the universe expands and show that

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{\gamma}{a} + \frac{\beta}{a^2}\,,$$

where γ and β are positive parameters you should determine.

(ii) Solve the Friedmann equation for initial conditions $a(0) = 0$ to find the scale factor $a(t) = t(\sqrt{\beta} + \gamma t/4)$.

(iii) Calculate the age of the universe t_0 when $a(t_0) = 1$. Compare t_0 with the inverse Hubble parameter H_0^{-1} at t_0 in the limiting cases $\beta \gg \gamma$ and $\gamma \gg \beta$.

(iv) Our present universe is observed to be accelerating, through measurements of the deceleration parameter $q_0 := -\ddot{a}(t_0)\,a(t_0)/\dot{a}(t_0)^2 \approx -0.55$. Can the quintessence model outlined in this part of the question have $q_0 \leqslant -0.5$ for any parameter values?

16H Logic and Set Theory

In this question, an ordinal is a transitive set well-ordered by $\in$.

(a) Explain briefly why for every well-ordered set (a, r) there is a unique ordinal isomorphic to (a, r), the *order-type of* (a, r).

(b) Let $\alpha < \beta$ be ordinals, and let γ be the order-type of the interval

$$\beta \backslash \alpha = \{\delta \in \mathrm{ON} : \alpha \leqslant \delta < \beta\}.$$

Explain briefly why $\alpha + \gamma = \beta$.

(c) What is the order-type of the interval $\omega_1 \backslash \omega$? Justify your answer.

(d) Let α be a non-zero ordinal. Show that $\alpha = \omega^\delta \cdot n + \eta$ for ordinals δ, n and η such that $1 \leqslant n < \omega$ and $\eta < \omega^\delta$.

(e) Let δ be an ordinal, and assume that $\omega^\delta = X \cup Y$. Show that at least one of X and Y has order-type ω^δ.

(f) Let $\alpha = X \cup Y$ be a non-zero ordinal with both X and Y having order-type β. Using (d) and (e) or otherwise, show that $\alpha < \beta + \beta + \beta$. Is it always true that $\alpha < \beta + \beta$? Give a proof or counterexample.

17F Graph Theory

(a) State Menger's theorem for a graph G. Define the *connectivity* $\kappa(G)$ of G. State and deduce the vertex form of Menger's theorem from Menger's theorem.

Let $k \geqslant 2$. Show that every k-connected graph of order at least $2k$ contains a cycle of length at least $2k$.

(b) Suppose G is a graph with $|G| > 1$. Define the *edge connectivity* $\lambda(G)$ of G. Let $\delta(G)$ be the minimum degree of G. Prove that

$$\delta(G) \geqslant \lambda(G) \geqslant \kappa(G).$$

Let d, ℓ and k be any three positive integers with $d \geqslant \ell \geqslant k$. Show that there exists a graph G with $\delta(G) = d$, $\lambda(G) = \ell$ and $\kappa(G) = k$.

18J Galois Theory

Let K be a field containing a primitive cube root of unity ω. Consider the cubic polynomial $f(X) = X^3 + aX + b \in K[X]$ with roots $\alpha_1, \alpha_2, \alpha_3$ in a splitting field. Let $g(X) = (X - u^3)(X - v^3)$ where $u = \alpha_1 + \omega\alpha_2 + \omega^2\alpha_3$ and $v = \alpha_1 + \omega^2\alpha_2 + \omega\alpha_3$.

(a) Define the *discriminant* of a monic polynomial, and show that

$$\mathrm{Disc}(g) = -27\,\mathrm{Disc}(f).$$

Write uv and $u^3 + v^3$ as polynomials in a and b. Hence, or otherwise, compute a formula for $\mathrm{Disc}(f)$ in terms of a and b.

(b) Show that there is a formula in terms of radicals for the roots of a cubic polynomial.

(c) Compute the Galois groups of the following polynomials, stating carefully any results from the course that you use:

$$X^3 - 21X - 22, \qquad X^3 - 21X - 28, \qquad X^3 - 21X - 34.$$

19J Representation Theory

Let G be the (infinite) group generated by two elements r and t such that $trt^{-1} = r^{-1}$, $t^2 = 1$ and with all other relations a consequence of these.

(a) Let V be a finite dimensional complex representation of G. Show that if V is irreducible, then $\dim V \leqslant 2$.

(b) Find all one dimensional complex representations of G, and find all irreducible two dimensional complex representations of G up to isomorphism.

(c) For every positive integer $n \geqslant 3$, there is a surjective homomorphism $G \to D_{2n}$ to the dihedral group of order $2n$. Using this, we can regard a representation of D_{2n} as a representation of G. Which of the irreducible finite dimensional representations of G do *not* arise in this way?

(d) Show that the following 2×2 matrices can be used to construct a two dimensional representation of G containing a one dimensional subrepresentation that has no G-invariant complement:

$$A = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad B = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

20G Number Fields

(a) Let $f(X)$ be a monic polynomial with algebraic integer coefficients. Prove that the roots of f are algebraic integers. [You may use without proof the characterization of algebraic integers in terms of finitely generated modules, provided you state the result precisely.]

(b) Determine the ring of integers $\mathcal{O}_K$ in the field $K = \mathbb{Q}(\sqrt{17})$. Justify your answer.

(c) Let $\alpha = \sqrt{4+\sqrt{17}}$. By computing $\mathrm{N}_{K|\mathbb{Q}}(\alpha^2)$, or otherwise, show that $\alpha \notin K$.

(d) With α as in part (c), let $L = \mathbb{Q}(\alpha)$. Show for $\beta \in L$ that $\beta \in \mathcal{O}_L$ if and only if $\mathrm{N}_{L|K}(\beta) \in \mathcal{O}_K$ and $\mathrm{Tr}_{L|K}(\beta) \in \mathcal{O}_K$.

(e) Show that, if $a + b\alpha \in \mathcal{O}_L$ for some $a, b \in K$, then $2a \in \mathcal{O}_K$ and $2b \in \mathcal{O}_K$.

21F Algebraic Topology

State the Mayer-Vietoris theorem for a simplicial complex K which is the union of subcomplexes M and N.

Let K be a non-empty simplicial complex in $\mathbb{R}^m$, where we consider $\mathbb{R}^m$ as lying in $\mathbb{R}^{m+2}$ via the vectors $(x_1, \ldots, x_m, 0, 0)$. Let $c_1 = (0, \ldots, 0, 1, 0) \in \mathbb{R}^{m+2}$, $c_2 = (0, \ldots, 0, 0, 1) \in \mathbb{R}^{m+2}$ and $c_3 = (0, \ldots, 0, -1, -1) \in \mathbb{R}^{m+2}$. Let L be the collection of simplices in $\mathbb{R}^{m+2}$ given by

$$L := K \cup \{\langle v_0, v_1, \ldots, v_n, c_i\rangle \mid \langle v_0, v_1, \ldots, v_n\rangle \in K, i = 1, 2, 3\}.$$

Show that L is a simplicial complex. Find expressions for the simplicial homology groups of L in terms of the simplicial homology groups of K. [*You may use any results from lectures provided they are clearly stated.*]

22I Linear Analysis

(a) Let X be a Banach space.

(i) Define the *dual space* X^* of X, and show that it is a Banach space.

(ii) Find, with proof, the dual of l_p for each $1 < p < \infty$.

(iii) Describe, without proof, the duals of l_1 and c_0.

(b) Let c denote the space of convergent real sequences, with the supremum norm. Is c isomorphic to c_0? Justify your answer.

23H Analysis of Functions

[You may use results from Linear Analysis and Probability and Measure without proof provided they are clearly stated.]

(a) Let μ and ν be finite measures on a measurable space $(E, \mathcal{E})$ that are mutually absolutely continuous.

(i) Show carefully that there exists a μ-integrable function $w : E \to [0, \infty]$ such that $\nu(A) = \int_A w \, d\mu$ for every $A \in \mathcal{E}$.

(ii) For which values of $0 < p < \infty$ must $\int_E |w|^p \, d\mu$ be finite? Justify your answer.

(b) Let ν_n be a sequence of probability measures on $(E, \mathcal{E})$ for $n \geqslant 1$. Does there always exist a probability measure μ on $(E, \mathcal{E})$ such that all ν_n are absolutely continuous with respect to μ? Give a proof or counter-example.

24G Riemann Surfaces

Give the definition of a *Riemann surface.*

If R is a Riemann surface, show that any open connected subset of R is also a Riemann surface. Show also that if $z_1, \ldots, z_p \in R$ then $R \setminus \{z_1, \ldots, z_p\}$ is a Riemann surface. Can it happen that a Riemann surface R with a countably infinite set of points removed is still a Riemann surface?

Which of the following topological spaces can be given the structure of a Riemann surface? Justify your answers.

(i) The unit sphere $S^2 = \{(x, y, z) \mid x^2 + y^2 + z^2 = 1\}$ in $\mathbb{R}^3$.

(ii) The set X of points $\{(x, y, x/\sqrt{x^2 + y^2}, y/\sqrt{x^2 + y^2})\}$ in $\mathbb{R}^4$ where x and y are not both zero.

(iii) The set Y of points $\{(z, w) \in \mathbb{C} \times \mathbb{C} \mid zw - 2iw - iz - 2 = 0\}$.

25J Algebraic Geometry

Define what it means to be a *rational map* between irreducible projective varieties. Define what it means to be a *regular point* of a rational map between irreducible projective varieties.

Consider the rational map $\varphi : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ given by

$$(x : y : z) \mapsto (xy : xz : z^2).$$

Show that φ is not regular at the points $(0 : 1 : 0), (1 : 0 : 0)$ and is regular at every other point. Show that φ is a birational map which is an isomorphism on $\mathbb{P}^2 \setminus Z(xyz)$, the complement of the union of the coordinate hyperplanes.

Let $V \subset \mathbb{P}^2$ be the subvariety given by the vanishing of $x^2z^4 - x^3y^3 + z^6$. Show that V is irreducible, and that φ determines a birational equivalence between V and a non-singular plane cubic.

26I Differential Geometry

(a) Define the terms *critical point, critical value* and *regular value.* Let $A \in S_n(\mathbb{R})$ be a symmetric $n \times n$ matrix with real entries and $f : \mathbb{R}^n \to \mathbb{R}$ the map $f(X) = X^T AX$ for a column vector $X \in \mathbb{R}^n$. Show that f has only one critical value. Can it have more than one critical point? Justify your answer.

(b) Let $M_{2n}(\mathbb{R})$ be the set of $2n \times 2n$ matrices with real entries, and $\mathrm{Sp}_n(\mathbb{R}) \subset M_{2n}(\mathbb{R})$ the set of matrices A such that $A^T JA = J$ with $J := \left(\begin{smallmatrix} 0_n & \mathrm{Id}_n \\ -\mathrm{Id}_n & 0_n \end{smallmatrix}\right)$. Prove that $\mathrm{Sp}_n(\mathbb{R})$ is a submanifold of $M_{2n}(\mathbb{R})$ with dimension $2n^2 + n$.

[*You can use the pre-image theorem if properly stated.*]

(c) Let $\mathrm{Gr}_{1,3}(\mathbb{R})$ be the set of 3×3 symmetric matrices with real entries P so that $P^2 = P$ and $\mathrm{Trace}(P) = 1$.

(i) Prove that $\mathrm{Gr}_{1,3}(\mathbb{R})$ is a submanifold of $S_3(\mathbb{R})$ with dimension 2.

[*Hint: You might want to first prove that given $P \in \mathrm{Gr}_{1,3}(\mathbb{R})$, there exists $X \in \mathbb{R}^3$ such that the solutions to $P = YY^T$ with $|Y|^2 = 1$ for the Euclidean norm are exactly $Y \in \{X, -X\}$. Then use this fact to construct local parametrisations.*]

(ii) Does $\mathrm{Gr}_{1,3}(\mathbb{R})$ admit a global parametrisation by an open subset of $\mathbb{R}^2$?

27H Probability and Measure

Let $(X_j)_{j\geqslant 1}$ be a sequence of independent real random variables with uniform density $p_{X_j} = \frac{1}{2j}\mathbf{1}_{[-j,j]}$. Let $S_n = X_1 + \cdots + X_n$.

(a) State Lévy's theorem from the lectures.

(b) Show that the characteristic function of $n^{-3/2}S_n$ satisfies

$$\Phi_{n^{-3/2}S_n}(\xi) = \frac{n^{\frac{3n}{2}}}{\xi^n n!}\prod_{j=1}^{n}\sin\left(\frac{j\xi}{n^{3/2}}\right).$$

(c) Show that $n^{-3/2}S_n$ converges in law to a limit to be determined.

[*Hint: You can use the formula* $\sum_{j=1}^n j^2 = n(n+1)(2n+1)/6$.]

(d) Let σ_j^2 be the variance of X_j. Show that $\left(\sum_{j=1}^n \sigma_j^2\right)^{-1/2} S_n$ converges in law to a limit to be determined.

28L Applied Probability

Let $(X_t, t \geqslant 0)$ be a Poisson process on $\mathbb{R}^+$ with rate $\lambda > 0$.

(a) Assuming the infinitesimal definition of a Poisson process, find the distribution of X_t.

(b) Now condition on the event $X_t = n$ for some $t > 0$ and $n \in \mathbb{N}$. What is the probability that the last jump before t occurs before $3t/4$? What is the distribution of the number of jumps between $t/4$ and $3t/4$?

(c) Suppose $(X_t, t \geqslant 0)$ describes the arrival of particles into a system. Each particle then lives for a length of time that is independent of the arrival process and independent of the lives of other particles. The particle lifespans are exponentially distributed with mean $1/\mu$. Find the distribution of the number of particles alive at time t.

[*Clearly state all results you use. You may use that if* N *is a Poisson*(λ) *random variable, then* $\mathbb{E}(e^{\theta N}) = \exp(\lambda(e^{\theta} - 1))$.]

29L Principles of Statistics

(a) Suppose real-valued random variables $\widehat{\psi}_1, \widehat{\psi}_2, \ldots$ satisfy $\sqrt{n}(\widehat{\psi}_n - \psi) \xrightarrow{d} N(0, v)$ as $n \to \infty$ for $v > 0$ and deterministic $\psi \in \mathbb{R}$. Suppose $\phi : \mathbb{R} \to \mathbb{R}$ is continuously differentiable at ψ. Write down the asymptotic distribution of $\sqrt{n}(\phi(\widehat{\psi}_n) - \phi(\psi))$. [No proof is necessary.]

(b) Suppose we have data $X_1, \ldots, X_n \overset{\text{i.i.d.}}{\sim} \text{Exp}(\theta)$ with rate $\theta > 0$.

(i) Find the maximum likelihood estimator (MLE) $\widehat{\theta}_n$ for the rate θ.

(ii) Without appealing to the general theory for MLEs, obtain, with justification, an asymptotic confidence interval $\widehat{C}_n$ for θ centred on $\widehat{\theta}_n$ that satisfies $\mathbb{P}_\theta(\theta \in \widehat{C}_n) \to 1 - \alpha$ as $n \to \infty$ for a given $\alpha \in (0, 1)$.

(c) Now suppose that rather than observing the random variables X_i as in part (b), we instead only observe data $Y_1, \ldots, Y_n$ where $Y_i = \lfloor X_i \rfloor$ is the greatest integer less than or equal to X_i.

(i) Show that the MLE $\widetilde{\theta}_n$ for θ based on the data $Y_1, \ldots, Y_n$ is given by

$$\widetilde{\theta}_n = \log\left(\frac{1 + \bar{Y}}{\bar{Y}}\right).$$

(ii) Without appealing to the general theory for MLEs, obtain, with justification, the asymptotic distribution of

$$\sqrt{n}(\widetilde{\theta}_n - \theta).$$

[*Hint: Recall that if random variable Z follows a geometric distribution supported on* $\{0, 1, 2, \ldots\}$ *with success probability p, then* $\mathbb{E}Z = (1 - p)/p$ *and* $\text{Var}(Z) = (1 - p)/p^2$.]

30K Stochastic Financial Models

Let Z be a square-integrable random vector in $\mathbb{R}^n$ with $\mathbb{E}(Z) = b$ and $\text{Cov}(Z) = V$. Assume that the $n \times n$ matrix V is positive definite.

(a) Find the vector $\theta \in \mathbb{R}^n$ to maximise $\mathbb{E}(X) - \frac{1}{2}\text{Var}(X)$ subject to $X = \theta^\top Z$.

(b) Let θ_M be the maximiser from part (a). Consider the problem of maximising $F(\mathbb{E}(X), \text{Var}(X))$ subject to $X = \theta^\top Z$, where $F(\cdot, \sigma^2)$ is strictly increasing for all σ^2 and $F(m, \cdot)$ is strictly decreasing for all m. Assuming that a maximiser θ^* exists, show that it is of the form $\theta^* = \lambda\, \theta_\text{M}$ where λ is a non-negative real number.

(c) Set $X_\text{M} = \theta_\text{M}^\top Z$. For real constants α and β and for vectors $\varphi \in \mathbb{R}^n$, consider the expression

$$\mathbb{E}[(\alpha + \beta X_\text{M} - Y)^2]$$

where $Y = \varphi^\top Z$. For any fixed φ, find the values α and β that minimise this expression, and show that the minimum equals $\varphi^\top Q \varphi$ for a symmetric matrix Q that you should identify.

31L Mathematics of Machine Learning

Consider i.i.d. random variables $(X_1, Y_1), \ldots, (X_n, Y_n)$ taking values in $\mathcal{X} \times \{-1, 1\}$ and a convex surrogate loss $(x, y) \mapsto \phi(yh(x))$.

(a) Define the *empirical Rademacher complexity*, $\hat{\mathcal{R}}(\mathcal{H}(x_{1:n}))$, and the *Rademacher complexity*, $\mathcal{R}_n(\mathcal{H})$, for a class of functions $\mathcal{H}$ mapping $\mathcal{X}$ to $\mathbb{R}$. State the contraction lemma for the Rademacher complexity.

(b) Fix $s > 0$. Let $S \subseteq \mathbb{R}^{d\times d}$ be the set of symmetric, positive semidefinite matrices with eigenvalues $\lambda_1 \geqslant \lambda_2 \geqslant \ldots \geqslant \lambda_d$ satisfying $\sum_{i=1}^d \lambda_i \leqslant s$. Show that S is a convex set.

(c) Suppose that $\mathcal{X} = \{x \in \mathbb{R}^d : \|x\|_2 \leqslant C\}$, and let $\mathcal{H} = \{x \mapsto x^T M x : M \in S\}$. Prove that

$$\mathcal{R}_n(\mathcal{H}) \leqslant \frac{C^2 s}{\sqrt{n}}.$$

[*Hint: If* $A \in \mathbb{R}^{d\times d}$ *is a symmetric matrix with eigenvalues* $\alpha = (\alpha_1, \ldots, \alpha_d)$, *then* $Tr(A^T A) = \|\alpha\|_2^2$.]

(d) Let $\hat{h}$ minimise the empirical risk $\hat{R}_\phi(h)$ over $h \in \mathcal{H}$, where ϕ is the hinge loss. Let h^* be the minimiser of the risk $R_\phi(h)$ over $h \in \mathcal{H}$. Quoting any necessary result from the course, deduce that

$$\mathbb{E}R_\phi(\hat{h}) - R_\phi(h^*) \leqslant \frac{K}{\sqrt{n}},$$

for a constant K which you must specify.

32A Dynamical Systems

(a) Define a *Lyapunov function* for the system $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$ around a fixed point at the origin. State both the first and second Lyapunov Theorems and prove the first.

(b) Consider the system

$$\dot{x} = -x + 2x^2 + y^2 + xy^2,$$
$$\dot{y} = -y + 4x^2 + 2y^2 - 2x^2 y.$$

(i) Show that the fixed point at the origin is asymptotically stable.

(ii) Show that the basin of attraction of the origin includes the region

$$12x^2 + 6y^2 < 1.$$

(iii) Can the strict inequality in part (ii) be extended to include equality? Justify your answer, stating carefully any results you need.

33D Integrable Systems

(a) Let U, V and Φ be $n \times n$ matrices that depend on x and y and satisfy

$$\partial_x \Phi + U\Phi = 0, \quad \partial_y \Phi + V\Phi = 0.$$

Find a compatibility condition for this system of linear partial differential equations that involves only U and V.

(b) Let $n = 3$ and take

$$U = \begin{pmatrix} \partial_x u & 0 & \lambda \\ 1 & -\partial_x u & 0 \\ 0 & 1 & 0 \end{pmatrix}, \quad V = \begin{pmatrix} 0 & e^{-2u} & 0 \\ 0 & 0 & e^u \\ \lambda^{-1} e^u & 0 & 0 \end{pmatrix},$$

where λ is a constant parameter and $u = u(x, y)$. Show that in this case the compatibility conditions hold if u satisfies a partial differential equation of the form

$$\partial_x \partial_y u = F(u), \qquad (\dagger)$$

for some function $F(u)$ which should be determined.

(c) Find a one-parameter group of transformations G_α generated by the vector field $x\partial_x - \alpha y \partial_y$, where α is a constant, and determine the value of α for which G_α is a symmetry group of the partial differential equation ($\dagger$).

(d) For this value of α, find an ordinary differential equation characterising solutions to equation ($\dagger$) that are invariant under G_α.

34C Principles of Quantum Mechanics

(a) A Fermi oscillator has Hilbert space $\mathcal{H} = \mathbb{C}^2$ and Hamiltonian $H = B^\dagger B$, where $B^2 = 0$ and $B^\dagger B + BB^\dagger = 1$.

(i) Find the eigenvalues of H.

(ii) If $|1\rangle$ is a state obeying $H|1\rangle = |1\rangle$ and $\langle 1|1\rangle = 1$, find $B|1\rangle$ and $B^\dagger|1\rangle$.

(iii) Obtain a matrix representation of the operators B, $B^\dagger$ and H.

(b) Now consider a composite system comprised of two decoupled Fermi oscillators, with $H_a = B_a^\dagger B_a$ for $a = 1, 2$. The Hamiltonian of the composite system is $H_{\text{tot}} = E_1 B_1^\dagger B_1 + E_2 B_2^\dagger B_2$ where $E_{1,2}$ are non-negative real numbers.

(i) Determine the exact eigenvectors and eigenvalues of H_{tot}.

(ii) Write H_{tot}, $B_{1,2}$ and $B_{1,2}^\dagger$ as matrices in the basis of energy eigenvectors.

(iii) Assuming $E_1 \ll E_2$ and treating $E_1 B_1^\dagger B_1$ as a small perturbation to the unperturbed Hamiltonian $H_{\text{tot}}^{(0)} = E_2 B_2^\dagger B_2$, determine the eigenvectors and eigenvalues of H_{tot} to first order in perturbation theory. Discuss how your derivation relates to degenerate perturbation theory.

(c) Finally assume $E_1 = 0$ and $E_2 > 0$, and consider the new Hamiltonian

$$\tilde{H}_{\text{tot}} = E_2 B_2^\dagger B_2 + g(B_1 + B_1^\dagger)$$

where g is a real constant.

(i) Determine the exact eigenvectors and eigenvalues of $\tilde{H}_{\text{tot}}$.

(ii) Treating $g(B_1 + B_1^\dagger)$ as a small perturbation to the unperturbed Hamiltonian $\tilde{H}_{\text{tot}}^{(0)} = E_2 B_2^\dagger B_2$, determine the eigenvectors and eigenvalues of $\tilde{H}_{\text{tot}}$ to first order in perturbation theory. Discuss how your derivation relates to degenerate perturbation theory.

35B Applications of Quantum Mechanics

In this question you will study a one-dimensional particle of mass m governed by the Schrödinger equation,

$$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} + V(x)\psi(x) = E\,\psi(x)\,,$$

where $V(x)$ is a one-dimensional potential with $V(x) \to 0$ as $x \to \pm\infty$, and $E > 0$ is the energy of the particle.

(a) For a particle incident from the negative x-direction, show that

$$\psi(x) = e^{ikx} - \frac{im}{\hbar^2 k}\int_{-\infty}^{\infty} dx' e^{ik|x-x'|}V(x')\psi(x') \qquad (*)$$

solves the Schrödinger equation, where $k = \sqrt{2mE}/\hbar$.

(b) Consider the following potential

$$V(x) = -\lambda\left[\delta(x+a) + \delta(x) + \delta(x-a)\right]\,,$$

where $a > 0$, $\lambda > 0$ are constants and $\delta(x)$ is the Dirac delta function. For this potential, write down a solution to the Schrödinger equation using equation $(*)$. Write explicitly the set of three algebraic equations determining $\psi(a)$, $\psi(0)$, and $\psi(-a)$, and express these equations in matrix form.

By inspecting the asymptotic solution at $x \to +\infty$ and writing it as $S_{++}e^{ikx}$, find the scattering amplitude $S_{++}(k)$ in terms of $\psi(a)$, $\psi(0)$, and $\psi(-a)$.

Show that solutions to the algebraic equation

$$1 - \gamma - 2e^{-2ika}(1+\gamma) + e^{-4ika}(1+\gamma)^3 = 0$$

correspond to singularities of S_{++}, where $\gamma = ik\hbar^2/(\lambda m)$. By looking at limiting values of a, argue that there are solutions to this algebraic equation on the imaginary k axis. What is the interpretation of these singularities?

36C Statistical Physics

(a) What is the definition of a *partition function*? Explain why this quantity is useful to calculate.

(b) A spherical, hard planet with surface area A has an ideal-gas atmosphere consisting of N atoms, each with mass m and no internal degrees of freedom. The Hamiltonian for each atom is

$$H = \frac{p^2}{2m} + mgz,$$

where $z > 0$ is the height above the surface. Assume that the gravitational acceleration g is a constant and that the density of the gas becomes negligible at a height which is still small compared to the radius of the planet. The gas is in thermal equilibrium at temperature T. Calculate the following quantities for the atmosphere:

(i) The expected total energy $\langle E \rangle$ and the fractional fluctuations $\Delta E / \langle E \rangle$.

(ii) The average height $\langle z \rangle$ of an atom and the atmospheric pressure $p(z)$ considered as a function of height.

(iii) The entropy S, using the approximate form of Stirling's formula,

$$\ln N! \approx N \ln N - N.$$

Express your final answer in terms of the thermal wavelength $\lambda = \sqrt{\dfrac{2\pi\hbar^2}{mk_BT}}$, as well as the variables A, N, and $\langle z \rangle$. Comment on its relation to the Sackur-Tetrode equation for a gas in a box of volume V,

$$S = Nk\left(\ln\frac{V}{\lambda^3 N} + \frac{5}{2}\right).$$

37B Electrodynamics

A relativistic particle of mass m and charge q moves with four-velocity u^μ in the presence of a background electromagnetic field with field-strength tensor $F_{\mu\nu}$ according to the Lorentz force law,

$$\frac{du^\mu}{d\tau} = \frac{q}{m} F^\mu{}_\nu u^\nu .$$

Here τ is the proper time.

Assume that the electric and magnetic fields $\mathbf{E}$ and $\mathbf{B}$ are constant and homogenous and that the particle starts from rest at the origin $\mathbf{x} = \mathbf{0}$ at time $t = 0$ in some inertial frame. Find the subsequent trajectory of the particle, giving its spacetime position $(ct, \mathbf{x})$ explicitly as a function of τ, in the following special cases:

(i) $\mathbf{E} = (E, 0, 0)$ and $\mathbf{B} = \mathbf{0}$.

In this case consider a light signal directed along the positive x-axis emitted from the point $\mathbf{x} = (-h, 0, 0)$ at time $t = 0$, where $h > 0$. Find the time taken for the light signal to catch up with the particle and show that it never catches the particle if h exceeds a critical value that you should determine.

(ii) $\mathbf{E} = (E, 0, 0)$ and $\mathbf{B} = (0, 0, E/c)$.

In this case you should show that the particle trajectory lies on a cubic curve in the (x, y) plane that you should determine explicitly.

38E General Relativity

The metric for a spherically symmetric static spacetime has line element

$$ds^2 \;=\; -\left(1+\frac{r^2}{a^2}\right)dt^2 \;+\; \left(1+\frac{r^2}{a^2}\right)^{-1}dr^2 \;+\; r^2\Big(\,d\theta^2+\sin^2\theta\,d\phi^2\,\Big)\ ,$$

where $-\infty \leqslant t \leqslant \infty$, $r \geqslant 0$, $0 \leqslant \theta \leqslant \pi$, $0 \leqslant \phi \leqslant 2\pi$, a is a positive constant, and units are chosen with $c = 1$.

(a) Consider a time-like geodesic parametrised by proper time τ, with dots denoting differentiation with respect to τ. Find the Euler-Lagrange equation corresponding to the θ coordinate and explain why the geodesic may be assumed to lie in the equatorial plane $\theta = \pi/2$, without loss of generality. For such a geodesic, show that

$$\frac{1}{2}\,\dot{r}^2 \;+\; V(r) \;=\; \frac{1}{2}\left(\,E^2 - 1 - \frac{h^2}{a^2}\,\right)\ ,$$

where $E = \big(1 + r^2/a^2\big)\,\dot{t}$ and $h = r^2\dot{\phi}$ are constants of the motion and $V(r)$ is a function you should determine.

(b) Show that a massive particle fired from the origin, $r = 0$, attains a maximum value of the radial coordinate, $r = r_{\rm max}$, before returning to $r = 0$, and find the proper time this journey (from $r = 0$ to $r_{\rm max}$ and back) takes.

(c) Show that circular orbits with $r = r_0$ are possible for any $r_0 > 0$ and determine whether such orbits are stable. Show further that, on such an orbit, a clock measures coordinate time.

39D Fluid Dynamics

(a) State the principle of reversibility for Stokes flow.

(b) Consider a rigid cylinder falling downwards at zero Reynolds number near to a rigid, vertical wall. The cylinder's axis is horizontal and parallel to the wall (that is, it is perpendicular to both the direction in which the cylinder falls and the normal of the wall). Use part (a) to argue that the cylinder cannot migrate towards or away from the wall as it falls, but it may rotate around its axis.

(c) Suppose the cylinder in part (b) has radius a, falls with speed V and rotates with angular speed Ω. Its minimum distance from the wall is $h_0 \ll a$.

(i) Use geometrical arguments to show that the horizontal gap $h(x)$ between the wall and the cylinder satisfies $h(x) \approx h_0\left[1 + x^2/(2ah_0)\right]$ for $|x| \ll a$, where x is the vertical distance above the axis of the cylinder.

(ii) Use lubrication theory to determine the velocity and hence the vertical flux of fluid between the wall and the cylinder in terms of the vertical pressure gradient. Given that the pressure is equal to the uniform ambient pressure ahead of and behind the cylinder, determine the vertical flux in terms of V, Ω, a and h_0.

[*Hint: Use a frame of reference in which the cylinder has no vertical motion.*]

(iii) Given that the forces on the cylinder are dominated by those in the narrow gap between it and the wall, and that there is no torque applied to the cylinder, show that, in fact, the cylinder does not rotate.

[*Hint: You may quote the following integrals:*

$$\int_{-\infty}^{\infty} \frac{dt}{1+t^2} = \pi, \quad \int_{-\infty}^{\infty} \frac{dt}{(1+t^2)^2} = \frac{\pi}{2}, \quad \int_{-\infty}^{\infty} \frac{dt}{(1+t^2)^3} = \frac{3\pi}{8}.$$
]

40A Waves

Consider small and smooth perturbations of a compressible and homentropic fluid with reference density ρ_0, pressure p_0, and sound speed c_0.

(a) Using the linearized mass and momentum conservation equations, show that the velocity potential ϕ satisfies the wave equation.

(b) Hence derive the energy equation

$$\frac{\partial E}{\partial t} + \boldsymbol{\nabla} \cdot \mathbf{I} = 0,$$

and give expressions for the acoustic energy density E and the acoustic energy flux $\mathbf{I}$.

(c) The fluid occupies the half space $z > 0$, and is bounded by a flexible membrane of negligible thickness and mass at an undisturbed position $z = 0$. Small, smooth acoustic perturbations in the fluid with velocity potential $\phi(x, z, t)$ deflect the membrane to $z = \eta(z, t)$. The membrane is supported by springs that, in the deflected state, exert a restoring force $\mu\eta$ per unit area on the membrane, where μ is a constant.

(i) Show that waves proportional to $\exp[ik(x-ct)]$ and propagating freely along the membrane possess the dispersion relation

$$A^2 \left(\frac{c}{c_0}\right)^4 + \left(\frac{c}{c_0}\right)^2 - 1 = 0,$$

where A is a dimensionless parameter that you must determine.

(ii) Show that the wave's time-averaged acoustic energy flux perpendicular to the membrane $\langle I_z \rangle$ is zero, where you must carefully define the average $\langle \cdot \rangle$.

(iii) Derive approximate expressions for the phase speed c in the two limits $A \ll 1$ and $A \gg 1$, and briefly interpret the two limits.

41D Numerical Analysis

(a) Show that if $A \in \mathbb{R}^{n\times n}$ is symmetric, there exists a symmetric and tridiagonal matrix $H \in \mathbb{R}^{n\times n}$ that has the same eigenvalues as A, and that can be computed in finitely many arithmetic operations from the matrix elements of A.

(b) The standard QR algorithm (without shifts) is applied to a symmetric and tridiagonal matrix H. For $k = 0, 1, 2, ...$, let H_k be the k^{th} iteration of the QR algorithm and recall that $H_{k+1} = \overline{Q}_k^T H \overline{Q}_k$, where $\overline{Q}_k$ is orthogonal and $\overline{Q}_k \overline{R}_k$ is the QR factorization of H^{k+1} (that is, the $(k+1)^{\text{th}}$ power of H).

Suppose that the eigenvalues λ_i $(i = 1, ..., n)$ of H satisfy $|\lambda_1| < |\lambda_2| < \cdots < |\lambda_{n-1}| = |\lambda_n|$, and let the corresponding normalised eigenvectors of H be $\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_n$. Suppose also that the first two canonical basis vectors, $\mathbf{e}_1$ and $\mathbf{e}_2$, can be written as $\mathbf{e}_1 = \sum_{i=1}^n b_i \mathbf{w}_i$ and $\mathbf{e}_2 = \sum_{i=1}^n c_i \mathbf{w}_i$ where b_i and c_i $(i = 1, \ldots, n)$ are non-zero constants.

(i) Show that if $(H/\lambda_n)^k \mathbf{e}_1 \to \mathbf{v}_1$ and $(H/\lambda_n)^k \mathbf{e}_2 \to \mathbf{v}_2$ as $k \to \infty$, then $\mathbf{v}_1$ and $\mathbf{v}_2$ are linear combinations of $\mathbf{w}_{n-1}$ and $\mathbf{w}_n$.

(ii) Let $h_{3,2}^{(k)}$ be the matrix element of H_k at the 3rd row and 2nd column. Show that $h_{3,2}^{(k)} \to 0$ as $k \to \infty$.

END OF PAPER