UNIVERSITY OF CAMBRIDGE

MAT1

MATHEMATICAL TRIPOS Part IB

Friday, 13 June, 2025 1:30pm to 4:30pm

PAPER 4

Before you begin read these instructions carefully

The examination paper is divided into two sections. Each question in Section II carries twice the number of marks of each question in Section I. Section II questions also carry an alpha or beta quality mark and Section I questions carry a beta quality mark.

Candidates may obtain credit from attempts on **at most four** questions from Section I and **at most six** questions from Section II.

Write on **one side** of the paper only and begin each answer on a separate sheet.

Write legibly; otherwise you place yourself at a grave disadvantage.

At the end of the examination:

Separate your answers to each question.

Complete a gold cover sheet **for each question** that you have attempted, and place it at the front of your answer to that question.

Complete a green main cover sheet listing **all the questions** that you have attempted.

Every cover sheet must also show your Blind Grade Number and desk number.

Tie up your answers and cover sheets into **a single bundle**, with the main cover sheet on the top, and then the cover sheet and answer for each question, in the numerical order of the questions.

STATIONERY REQUIREMENTS
Gold cover sheets
Green main cover sheet
Treasury tag

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

SECTION I

1F Linear Algebra

Let V be a vector space and $\alpha : V \to V$ a linear map. What is the *dual space* V^*? If $\mathcal{B}$ is a finite basis of V, define what is meant by the *dual basis* $\mathcal{B}^*$ of V^* and prove that $\mathcal{B}^*$ is indeed a basis.

[*No result about dimensions of dual spaces may be assumed.*]

Let $V = P_2$ be the space of real polynomials of degree at most 2 and consider the linear maps from P_2 to $\mathbb{R}$

$$f_0(p) = p(0), \qquad f_1(p) = \int_0^1 p(t)dt, \qquad f_2(p) = \int_{-1}^0 p(t)dt.$$

Show that f_0, f_1, f_2 form a basis of P_2^* by exhibiting the basis of P_2 to which it is dual.

[*You may assume that* $\{1, t, t^2\}$ *is a basis of* P_2.]

2G Analysis and Topology

Let X be a complete, non-empty, metric space and $T : X \to X$. What does it mean to say that T is a contraction? State and prove the contraction mapping theorem.

Suppose that the n^{th} iterate of T (i.e. T applied repeatedly n times), T^n, is a contraction for some $n > 0$. Must T have a fixed point? If so, must it be unique?

3E Complex Analysis

State and prove the local maximum modulus principle. You may assume the mean value property for holomorphic functions provided it is clearly stated.

Let $D = \{z \in \mathbb{C} : |z| < 1\}$, and suppose $f : D \to D$ is a holomorphic function satisfying $f(0) = 0$. Show that if $\mathrm{Re}\, f(z) \leqslant \mathrm{Im}\, f(z)$ for all $z \in D$ then f must be constant. [*You may find it helpful to consider* $e^{af(z)}$, *where* a *is a constant to be chosen.*]

4C Quantum Mechanics

A quantum particle of mass m is confined to move inside the rectangular box

$$\{(x, y, z) : 0 \leqslant x \leqslant a\,, \quad 0 \leqslant y \leqslant b\,, \quad 0 \leqslant z \leqslant c\,\}.$$

Derive the energy eigenvalues and eigenfunctions under the assumption that $a < b < c$. (You need not normalize the eigenfunctions.)

What is the degeneracy of the ground state, i.e., the dimension of the eigenspace corresponding to the lowest energy eigenvalue, and similarly for the next to lowest energy eigenvalue (the first excited state)?

How do your conclusions change if $a < b = c$?

5B Electromagnetism

Beginning with the Maxwell equations in vacuum, derive a wave equation for the electric field $\mathbf{E}$ and show the plane wave of the following form is a solution:

$$\mathbf{E}(\mathbf{x},t) = Re\left(\mathbf{E}_0\, e^{i(\mathbf{k}\cdot\mathbf{x}-\omega t)}\right)\,, \quad \text{with } \ \mathbf{k}\cdot\mathbf{E}_0 = 0\,,$$

where $\mathbf{k}$ and $\mathbf{E}_0$ are constant vectors. Give an expression relating ω and $\mathbf{k}$. Find the corresponding plane wave solution for the magnetic field $\mathbf{B}$.

Consider the specific solution

$$\mathbf{E} = E_0\left(0, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)\cos(kx-\omega t)\,,$$

for which you should state the wavevector direction and the polarisation vector. Calculate the corresponding Poynting vector $\mathbf{S} = (1/\mu_0)\mathbf{E}\times\mathbf{B}$ and its time-average. Briefly explain its meaning.

6A Numerical Analysis

Consider the quadrature formula

$$\int_0^1 f(x)x\,dx \approx \sum_{i=0}^{1} a_i f(x_i), \quad x_i \in [0,1], \qquad f \in C[0,1], \qquad (*)$$

which is exact for polynomials of degree 1.

(a) For $i = 0, 1$, find expressions for the weights a_i in terms of the nodes x_0, x_1.

(b) Define what it means for $(*)$ to be a *Gaussian quadrature*, and determine the numerical values of the nodes x_0, x_1 in that case.

7H Markov Chains

Consider a Markov chain $(X_n)_{n\geqslant 0}$ on the state space $\{1,2,3,4\}$ with transition matrix

$$P = \begin{pmatrix} 1/2 & 0 & 1/2 & 0 \\ 0 & 1/3 & 0 & 2/3 \\ 1/3 & 0 & 1/3 & 1/3 \\ 0 & 1/4 & 0 & 3/4 \end{pmatrix}$$

(a) List the communicating classes of the chain. For each class say whether it is open or closed.

(b) Find $\lim_{n\to\infty} P^n$.

SECTION II

8F Linear Algebra

Let Q be a quadratic form on a real vector space. What is the *symmetric bilinear form* associated to Q and why does it exist? Define what it means for a symmetric bilinear form to be *positive semi-definite* and *positive definite*.

State and prove Sylvester's law of inertia, stating clearly any auxiliary result on the diagonalization of real quadratic forms that you require.

Let ϕ be a non-degenerate, symmetric bilinear form on a $2n$-dimensional real vector space V and suppose that $\phi(v,v) = 0$ for all v in a k-dimensional subspace E of V. Show that $k \leqslant n$.

Given two real quadratic forms $f(x) = \sum_{i,j=1}^{n} a_{ij}x_i x_j$ and $g(x) = \sum_{i,j=1}^{n} b_{ij}x_i x_j$, let (f,g) denote the quadratic form

$$(f,g)(x) = \sum_{i,j=1}^{n} a_{ij}b_{ij}x_i x_j.$$

Let a quadratic form $l^2(x) = (\sum_{i=1}^{n} l_i x_i)^2$ be the square of a real linear function. Determine the rank and signature of the quadratic forms l^2 and (l^2, s^2), where $s(x) = \sum_{i=1}^{n} s_i x_i$ is another real linear function.

Deduce that if f and g are positive semi-definite quadratic forms, then so is (f,g).

9E Groups, Rings and Modules

(a) A module M for a ring R is called *irreducible* if the only submodules $N \subseteq M$ are 0 and M.

(i) Show that a module M is irreducible if and only if for all $m \in M$, $m \neq 0$, the map $R \to M$, $r \mapsto rm$ is surjective.

(ii) Let $I = \{r \in R \mid rm = 0 \text{ for all } m \in M\}$. Show that if M is irreducible, R/I is a field.

(b) Let V be a finite dimensional vector space over a field k, and $\varphi : V \to V$ a k-linear map.

A subspace $W \leqslant V$ is *indecomposable* if $\varphi(W) \subseteq W$, and W can not be written as a direct sum $W' \oplus W''$, with $W' \neq 0, W'' \neq 0$, $\varphi(W') \subseteq W'$, $\varphi(W'') \subseteq W''$

(i) State the primary decomposition theorem for modules over a Euclidean domain, and explain how it gives a decomposition

$$V = \oplus V_\alpha,$$

where each summand is indecomposable.

Describe the *minimal polynomial* and the *characteristic polynomial* of φ in terms of this decomposition.

(ii) Now suppose $k = \mathbb{R}$. List the prime ideals in $\mathbb{R}[x]$.

For each prime ideal I in $\mathbb{R}[x]$ and $n > 0$ write an explicit matrix A that represents the action of x on $\mathbb{R}[x]/I^n$.

(iii) Let $B = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -2 \\ 0 & 0 & 0 & 0 & 1/2 & 0 \end{pmatrix}$.

Give the explicit normal form for $B : \mathbb{R}^6 \to \mathbb{R}^6$ that you have described in part (ii).

10G Analysis and Topology

State what it means for a function $f : \mathbb{R}^n \to \mathbb{R}^m$ to be differentiable at $x \in \mathbb{R}^n$, and define the differential $Df|_x$. You need not establish the uniqueness of the differential.

State the inverse function theorem.

Let $\mathcal{M} = \mathrm{Mat}(n \times n)$ be the space of real square matrices with n rows and n columns, which can be identified with $\mathbb{R}^{n^2}$. Consider the function $F : \mathcal{M} \to \mathcal{M}$ given by

$$F(A) = A^T A.$$

Briefly explain why F is differentiable at A for all $A \in \mathcal{M}$ and determine $DF|_A$. What is $\mathrm{Ker}\, DF|_I$?

Let $\mathcal{O} = \{R \in \mathcal{M} : R^T R = I\}$ and $T = \{B \in \mathcal{M} : B^T + B = 0\}$, each inheriting their topology as a subspace of $\mathcal{M}$.

By considering the map $A \mapsto F(A) + A - A^T - I$, or otherwise, show that there exist open sets $U, V \subset \mathcal{M}$ with $I \in U$ and $0 \in V$, together with a continuously differentiable bijection $\Phi : U \to V$, with continuously differentiable inverse, satisfying $\Phi(U \cap \mathcal{O}) = V \cap T$.

Deduce that every point in $\mathcal{O}$ has an open neighbourhood which is homeomorphic to an open set in T.

11E Geometry

(a) Define the *disc model* (D, g_{disc}) and the *upper half-plane model* $(\mathfrak{h}, g_{\mathfrak{h}})$ for the hyperbolic plane, and show that they are isometric.

If $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{R})$, then g induces an isometry of $\mathfrak{h}$. What is the matrix of the corresponding isometry of D?

(b) Define what is meant by a *hyperbolic triangle* in the hyperbolic plane, its *vertices*, and *ideal vertices*.

Let Δ be a hyperbolic triangle with only ideal vertices. What are its internal angles? Compute the area of Δ.

Compute the area of a hyperbolic triangle with internal angles α, β, γ.

For fixed $\alpha,\ \beta,\ \gamma < \pi$, show that $SL_2(\mathbb{R})$ acts transitively on the set of triangles with these internal angles.

12A Complex Methods
Recall the Heaviside function

$$H(t) := \begin{cases} 1, & t \geqslant 0 \\ 0, & t < 0 \end{cases}, \quad t \in \mathbb{R},$$

and recall that a function $h : \mathbb{R}_+ = [0, \infty) \to \mathbb{R}$ is said to be $T > 0$ periodic if $h(t+T) = h(t)$ for all $t \in \mathbb{R}_+$.

(a) Let $f : \mathbb{R}_+ \to \mathbb{R}$ be a bounded continuous function. Show that for any real number $\alpha \geqslant 0$,

$$\mathcal{L}\{f(t-\alpha)H(t-\alpha)\}(s) = e^{-\alpha s}F(s), \quad s > 0,$$

where $\mathcal{L}\{f(t)\}(s) = F(s)$ and $\mathcal{L}$ is the Laplace transform.

(b) Let $g : \mathbb{R}_+ \to \mathbb{R}$ be continuous and $T > 0$ periodic, and define

$$g_T(t) = \begin{cases} g(t), & 0 \leqslant t \leqslant T \\ 0, & t > T \end{cases}.$$

Show that

$$\mathcal{L}\{g(t)\}(s) = \frac{\mathcal{L}\{g_T(t)\}(s)}{1 - e^{-sT}}, \quad s > 0.$$

(c) Find the Laplace transform of the periodic function $h : \mathbb{R}_+ \to \mathbb{R}$ defined by

$$h(t) = \begin{cases} \sin(t), & 0 \leqslant t < \pi \\ 0, & \pi \leqslant t \leqslant 2\pi \end{cases}, \qquad h(t + 2\pi) = h(t), \quad t \geqslant 0.$$

13C Variational Principles

The equation of motion for a bead of mass m moving without friction on a cycloidal shaped wire is the Euler-Lagrange equation for the functional

$$S[\phi] = \int_0^T \Big(ma^2(1-\cos\phi)\dot{\phi}^2 - mga(1+\cos\phi) \Big)\, dt\,, \qquad T>0\,.$$

Write down the Euler-Lagrange equation for this functional, and show it implies that $u=\cos(\frac{\phi}{2})$ satisfies

$$\ddot{u} + \omega^2 u = 0, \qquad (*)$$

where ω^2 is a positive number which you should find. [*You should take* m, g, a *to be positive constants.*]

Using the change of dependent variable $\phi \to u = \cos(\frac{\phi}{2})$, define a new functional $\hat{S}[u] = \int_0^T f(u,\dot{u})\, dt$ such that $\hat{S}[u] = S[\phi]$; give a formula for f and give the Euler-Lagrange equation for $\hat{S}$. How is this equation related to $(*)$?

Give the second variation functional $\delta^2\hat{S}(\eta)$, where the variation functions η vanish at the endpoints $t=0$ and $t=T$. Consider the solution $u(t) = A\cos\omega t$ of $(*)$ with fixed endpoint conditions

$$u(0) = A, \qquad u(T) = A\cos\omega T,$$

on the interval $0 \leqslant t \leqslant T$. By considering the orthonormal collection of functions

$$e_n(t) = \sqrt{\frac{2}{T}}\sin\frac{n\pi t}{T}\,,$$

find a number t_0 such that $A\cos\omega t$ is a local minimizer of $\hat{S}$ if $T < t_0$ but not for $T > t_0$.

[*Hint: you may assume all variations to be of the form* $\eta = \sum_{n=1}^{\infty} c_n e_n(t)$, *and rearrange and interchange sums with derivatives as needed. Observe that* $\ddot{e}_n = -(n\pi/T)^2 e_n$.]

14D Methods

The function $u(x,t)$ satisfies

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} \quad \text{on} \quad -\infty < x < \infty$$

with

$$u(x,0) = \exp(-x^2), \qquad \frac{\partial u}{\partial t}(x,0) = 0,$$

where x is a space coordinate and t is time.

Define the spatial Fourier transform

$$\tilde{u}(k,t) = \int_{-\infty}^{\infty} u(x,t) e^{-ikx}\, dx,$$

and determine the differential equation and initial conditions satisfied by $\tilde{u}(k,t)$. By solving this differential equation, determine $\tilde{u}(k,t)$ explicitly. Thence, by calculating an appropriate integral, calculate $u(x,t)$. Interpret your solution physically.

15C Quantum Mechanics

(i) The angular momentum operators for a particle moving in three dimensional space are

$$L_a = -i\hbar\epsilon_{abc}x_b\frac{\partial}{\partial x_c}\,.$$

Show that if $f = f(r)$, where $r^2 = x_1^2 + x_2^2 + x_3^2$, is a smooth radial function, then if m, n are nonnegative integers $\chi_{m,n} = (x_1 + ix_2)^m x_3^n f(r)$ satisfies $L_3\chi_{m,n} = \lambda\chi_{m,n}$ for some λ depending on m, n which you should find. Find an analogous relation for $(x_1 - ix_2)^m x_3^n f(r)$.

(ii) The Hamiltonian for a particle moving in three spatial dimensions in a symmetric harmonic potential $V(x_1, x_2, x_3) = \frac{1}{2}m\omega^2(x_1^2 + x_2^2 + x_3^2) = \frac{1}{2}m\omega^2 r^2$ is

$$H\Psi = -\frac{\hbar^2}{2m}\left(\frac{\partial^2\Psi}{\partial x_1^2} + \frac{\partial^2\Psi}{\partial x_2^2} + \frac{\partial^2\Psi}{\partial x_3^2}\right) + V(x_1, x_2, x_3)\Psi\,, \qquad (x_1, x_2, x_3) \in \mathbb{R}^3\,.$$

Find the lowest eigenvalue of H and its corresponding eigenfunction Ψ_0. Next, find all the eigenfunctions and eigenvalues of H, and determine the degeneracy of each eigenvalue, i.e. the dimension of the corresponding eigenspace. (You are not required to normalize the eigenfunctions.)

Find χ such that $L_3\chi = 2\hbar\chi$ and $H\chi = \frac{7}{2}\hbar\omega\chi$.

[*Hint: in (ii) you may freely use the fact that the functions*

$$\psi_n(x) = h_n(x)e^{-\frac{1}{2}x^2}\,,$$

where h_n is the Hermite polynomial of degree n, constitute a complete orthogonal set and satisfy

$$-\frac{1}{2}\frac{\partial^2\psi_n}{\partial x^2} + \tfrac{1}{2}x^2\psi_n = (n + \tfrac{1}{2})\psi_n\,, \qquad \int \psi_m(x)\psi_n(x)dx = 0 \textit{ if } n \neq m\,.$$

Explicitly, the first three Hermite polynomials are given by

$$h_0(x) = 1\,, \quad h_1(x) = x \quad \textit{and} \quad h_2(x) = (2x^2 - 1)\,.\,]$$

16D Fluid Dynamics

An infinite range of hills has elevation $y = \eta(x) \equiv h \cos kx$ in Cartesian coordinates (x, y), where h and k are constants. High above the hills, the wind has uniform velocity $\mathbf{U} = (U, 0)$. Assume that the air flow above the hills is a laminar, potential flow $\mathbf{u} = \mathbf{U} + \nabla\phi$.

Without approximation, write down the equation and boundary conditions satisfied by ϕ. [*Note that the vector* $\mathbf{n} = (-\eta_x, 1)$ *is normal to the surface.*]

Now assume that both $hk \ll 1$ and $|\nabla\phi| \ll U$. Describe these approximations in physical terms.

Derive the linearised equation and boundary conditions satisfied by ϕ given these approximations, taking care to explain all the approximations that you make.

Solve the linearised equations for ϕ, and use your solution to determine the difference in pressure between the crests and troughs of the hills, assuming that the air has uniform density ρ. What is the dominant physical reason for the pressure difference in the limits (i) $kU^2/g \ll 1$ and (ii) $kU^2/g \gg 1$?

17H Statistics

Let $X_1, \ldots, X_m$ and $Y_1, \ldots, Y_n$ be independent random variables. Assume $X_i \sim N(\lambda, 1)$ and $Y_j \sim N(\mu, 1)$ for each $1 \leqslant i \leqslant m$ and $1 \leqslant j \leqslant n$, where the constants λ and μ are unknown. Let $\bar{X} = \frac{1}{m}\sum_{i=1}^{m} X_i$ and $\bar{Y} = \frac{1}{n}\sum_{j=1}^{n} Y_j$.

(a) Find the generalised likelihood ratio test of size α for $H_0^{(a)} : \lambda = 0 = \mu$ versus $H_1^{(a)} : \lambda, \mu$ unrestricted. Express your answer in terms of the cumulative distribution function F_k of the χ_k^2 distribution, for a suitable k.

(b) Find the generalised likelihood ratio test of size α for $H_0^{(b)} : \lambda = \mu$ versus $H_1^{(b)} : \lambda, \mu$ unrestricted. Express your answer in terms of the cumulative distribution function Φ of the $N(0, 1)$ distribution.

(c) Show, regardless of the true values of λ and μ, that there is a positive probability that the test from part (b) rejects $H_0^{(b)}$ but the test from part (a) does not reject $H_0^{(a)}$.

18H Optimisation

Given supplies $(s_i)_{1\leqslant i\leqslant m}$, demands $(d_j)_{1\leqslant j\leqslant n}$ and transport costs $(c_{ij})_{1\leqslant i\leqslant m,1\leqslant j\leqslant n}$, consider the problem of minimising

$$\sum_{i,j} c_{ij}x_{ij} \text{ subject to } \sum_j x_{ij} = s_i \text{ for all } i,$$
$$\sum_i x_{ij} = d_j \text{ for all } j,$$
$$x_{ij} \geqslant 0 \text{ for all } i, j.$$

Assume that all supplies and demands are non-negative, that $\sum_i s_i = \sum_j d_j$ and that the problem is not degenerate.

(a) Derive the dual problem. State the necessary and sufficient conditions for optimality of the primal problem in terms of an optimal solution of the dual problem.

(b) Suppose $(x_{ij})_{ij}$ is a basic feasible solution of the problem. How many ordered pairs (i, j) are such that $x_{ij} > 0$?

(c) Explain the transportation algorithm. Your answer should include a method for choosing an initial basic feasible solution as well as the details of the pivot step. Why does the algorithm terminate at the optimal solution?

(d) Suppose that both the supplies $(s_i)_i$ and demands $(d_j)_j$ are integer-valued. Show that there is an integer-valued optimal solution $(x_{ij})_{ij}$.

END OF PAPER